MPC: implications of a growth condition on exponentially controllable systems ⋆

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Abstract: We analyze stability and performance of unconstrained nonlinear model predictive control schemes. To this purpose, we introduce a growth condition which significantly improves known stability and performance estimates for exponentially controllable systems. Moreover, we present sufficient conditions implying the validity of this condition.

Keywords: Nonlinear model predictive control, receding horizon control, asymptotic stability, exponential controllability, suboptimality

1. INTRODUCTION

Model predictive control (MPC, often also termed receding horizon control) algorithms are used in order to solve optimal control problems on an infinite horizon, e.g., for stabilizing a nonlinear control system at the origin. To this end, a performance criterion is optimized over the predicted trajectories of the system, but – in contrast to the original problem – on a finite horizon. Then the first element of the computed sequence of control values is applied and the optimization horizon is shifted. This procedure is repeated iteratively. Hence, a sequence of finite horizon optimal control problems is solved in order to deal with the system on the infinite horizon.

MPC is especially attractive due to its ability to incorporate constraints. But – although the basic concept is easily understandable – the stability and performance analysis is far from being trivial and has attracted considerable attention during the last years, cf. Allgöwer et al. [2000], Mayne, Rawlings [2009]. Often stabilizing terminal constraints or costs are added in order to ensure stability, cf. Gilbert, Keerthy [1988], Allgöwer, Chen [1998]. However, estimating the obtained performance in comparison to an optimal control for the infinite horizon problem is rather difficult.

In this paper, which extends results from Grüne et al. [2010] we focus on unconstrained MPC. This class of MPC schemes is appealing for its numerical simplicity and its widespread use in industrial applications, cf. Badgwell, Qin [2003]. Our goal consists of providing checkable conditions in order to guarantee stability and, in addition, to estimate the performance of the resulting MPC closed loop.

For this purpose, we combine the controllability assumption from Grüne [2009] with a growth condition in order to to tighten the stability and performance bounds obtained in the mentioned literature. The main contribution of this paper is a detailed analysis of this additional requisite. We verify its validity for sampled–data systems governed by – finite or infinite dimensional – differential equations.

Moreover, we investigate the scenario of sufficiently fast sampling which is often required for practical applications, cf. Nešić, Teel [2004]. To this end, we perform a convergence analysis which yields that the direct application of Grüne [2009] leads to very pessimistic bounds. However, incorporating our growth condition counteracts problems which are related to fast sampling.

The paper is organized as follows: in Section 2 we give a precise formulation of the problem in consideration. Section 3 covers known results from Grüne [2009], Grüne, v. Lossow, and Worthmann [2009] including our growth condition which is investigated in detail in the following section which contains our main results. Section 5 concludes the paper by illustrating the stability criteria on a synchronous generator model from Bazanella et al. [2003].

2. PRELIMINARIES AND PROBLEM FORMULATION

We consider a nonlinear discrete time control system given by

$$x(n+1) = f(x(n), u(n)), \quad x(0) = x_0 \tag{1}$$

with $x(n) \in X$ and $u(n) \in U$ for $n \in \mathbb{N}_0$. Here the state space X and the control value space U are arbitrary metric spaces. We denote the space of control sequences $u : \mathbb{N}_0 \to U$ by $\mathcal{U}$ and the solution trajectory for given $u \in \mathcal{U}$ by $x_u(\cdot)$. Note that constraints can be incorporated by replacing X and U by appropriate subsets of the respective spaces. For simplicity of exposition, however, we will not address feasibility issues in this paper.

A typical class of such discrete time systems are sampled–data systems induced by a controlled ordinary differential equation

$$\dot{\varphi}(t) = g(\varphi(t), \tilde{u}(t)) \tag{2}$$

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with sampling period $T > 0$ and piecewise constant control function $\tilde{u}(\cdot) : \mathbb{R}_0^+ \to U$, i.e., $\tilde{u}(t) = constant$ for all $t \in [iT, (i+1)T)$, $i \in \mathbb{N}_0$. We define $f(x, u)$ by $\varphi(T; x, \tilde{u})$, i.e., the solution of (2) with $\tilde{u}(t) = u \in \mathbb{R}^m$ for $t \in [0, T)$ and initial state $x \in \mathbb{R}^n$. In order to guarantee existence and uniqueness of the solution $\varphi(\cdot; x, \tilde{u})$ we require Lipschitz–Continuity (with Lipschitz constant L_g) of $g(\cdot, \cdot)$ with respect to its first argument. Furthermore, we obtain the estimate

$$\|\Phi(t; x_0, \tilde{u})\| \leq e^{L_g t}\|x_0\| \tag{3}$$

from Gronwall's inequality for $\tilde{u}(t) = 0$ for $t \in [0, t)$ and, as a consequence, $\|x_u(n)\| \leq e^{L_g nT}\|x_0\|$ for the discrete time system with $u \equiv 0$, cf. Chicone [2006].

Our goal consists of stabilizing the system (1) at the origin in an optimal way. For this purpose, we define a running cost $l : X \times U \to \mathbb{R}_0^+$ which specifies a performance criterion. Since $l(\cdot, \cdot)$ has to characterize the equilibrium we require the existence of a control value $u^\star \in U$ such that $l(0, u^\star) = 0$ and $l(x, u) > 0$ for $x \neq 0$. Consequently, we aim at minimizing the infinite horizon cost $J_\infty(x_0, u) = \sum_{n=0}^{\infty} l(x_u(n), u(n))$. We denote the optimal value function for this problem by

$$V_\infty(x_0) := \inf_{u \in \mathcal{U}} J_\infty(x_0, u). \tag{4}$$

Since infinite horizon optimal control problems are in general computationally intractable, we use a receding horizon approach in order to compute an approximately optimal controller. To this end, we consider the finite horizon functional

$$J_N(x_0, u) = \sum_{n=0}^{N-1} l(x_u(n), u(n)) \tag{5}$$

with *optimization horizon* $N \in \mathbb{N}_{\geq 2}$ inducing the optimal value function

$$V_N(x_0) = \inf_{u \in \mathcal{U}} J_N(x_0, u). \tag{6}$$

By solving this finite horizon optimal control problem we obtain N control values $u^\star(0), u^\star(1), \ldots, u^\star(N-1)$ which depend on the state x_0. Implementing the first element of this sequence, i.e., $u^\star(0)$, yields a new state $x(1)$. Iterative application of this construction provides a control sequence on the infinite time interval.

We obtain a closed loop representation by applying the map $\mu_N : X \to U$ which is given in Definition 1, below, as a static state feedback law $\mu : X \to U$ according to the rule $x_\mu(0) = x_0$,

$$x_\mu(n+1) = f(x_\mu(n), \mu(x_\mu(n))). \tag{7}$$

Definition 1. For $N \in \mathbb{N}_{\geq 2}$ we define the MPC feedback law $\mu_N(x_0) := u^\star(0)$, where $u^\star$ is a minimizing control for (6) with initial value x_0.

Remark 2. For simplicity of exposition we assume that the infimum in (6) is a minimum, i.e., that a minimizing control sequence $u^\star$ exists.

In this paper we consider the conceptually simplest MPC approach imposing neither terminal costs nor terminal constraints. In order to measure the suboptimality degree of the MPC feedback for the infinite horizon problem we define

$$V_\infty^\mu(x_0) := \sum_{n=0}^{\infty} l(x_\mu(n), \mu(x_\mu(n))).$$

3. CONTROLLABILITY AND PERFORMANCE BOUNDS

Before we go into the details we state the following proposition which is a cornerstone of our analysis. To this end, we define $l^\star(x) := \min_{u \in U} l(x, u)$.

Proposition 3. Assume there exists $\alpha \in (0, 1]$ such that for all $x \in X$ the inequality

$$V_N(x) \geq V_N(f(x, \mu_N(x)) + \alpha l(x, \mu_N(x)) \tag{8}$$

holds. Then for all $x \in X$ the estimate

$$\alpha V_\infty(x) \leq \alpha V_\infty^{\mu_N}(x) \leq V_N(x) \leq V_\infty(x)$$

holds. If, in addition, there exist $\mathcal{K}_\infty$-functions α_1, α_2 ($\alpha_i(0) = 0$, α_i is continuous, strictly monotonically increasing, and unbounded for $i = 1, 2$) such that the inequalities

$$l^\star(x) \geq \alpha_1(\|x\|), \qquad V_N(x) \leq \alpha_2(\|x\|)$$

hold for all $x \in X$, then the origin is a globally asymptotically stable equilibrium for (7) with $\mu = \mu_N$ from Definition 1 and Lyapunov function V_N.

Proof. For a proof we refer to [Altmüller, Grüne, and Worthmann, 2009, Proposition 1].

Proposition 3 is based on ideas from relaxed dynamic programming, cf. Grüne, Rantzer [2008]. The relaxed Lyapunov inequality (8) is the decisive condition in Proposition 3. Thus, the question arises how to verify the existence of a positive value for the corresponding suboptimality index α and – in case of existence – how to compute or estimate it.

In this context, we introduce assumptions characterizing the dynamics of the control system in consideration. Based on these assumptions we deduce several consequences for our optimal control problem in order to answer the stated questions. In order to facilitate this we will formulate these assumptions not in terms of the trajectory but in terms of the running cost l along a trajectory. The following assumption guarantees that the system is exponentially stabilizable.

Assumption 4. (Exponential Controllability). For each $x_0 \in X$ there exists a control function $u_{x_0} \in \mathcal{U}$ satisfying the estimate

$$l(x_{u_{x_0}}(n), u_{x_0}(n)) \leq C\sigma^n l^\star(x_0) \tag{9}$$

for all $n \in \mathbb{N}_0$. Here $C \geq 1$ denotes the overshoot and $\sigma \in (0, 1)$ the decay rate.

In order to describe the system's behavior more accurately, we introduce Assumption 5.

Assumption 5. (Growth Condition). For each $x_0 \in X$ there exists a control function $u_{x_0} \in \mathcal{U}$ satisfying the inequality

$$l(x_{u_{x_0}}(n), u_{x_0}(n)) \leq L^n l^\star(x_0) \tag{10}$$

with $L \geq 1$ for all $n \in \mathbb{N}_0$.

In Section 4 we will apply (3) in order to deduce concrete estimates for the growth constant L. The following lemma which is an immediate consequence of Assumptions 4 and 5 uses the definition

$$\gamma_N := \min\left\{\sum_{n=0}^{N-1} C\sigma^n, \sum_{n=0}^{N-1} L^n\right\} \tag{11}$$

which is illustrated in Figure 1.

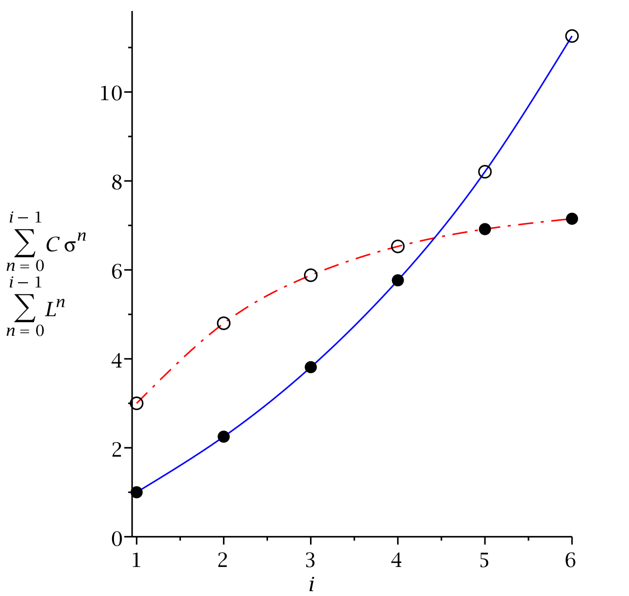

Fig. 1. Visualization of the bounds induced by Assumption 4 (dashed-dotted line) and our growth condition, cf. Assumption 5, (solid line) for $C = 3$, $\sigma = 3/5$, and $L = 5/4$. Each time the minimum which is marked with solid circles coincides with γ_N from (11).

Lemma 6. For each $N \geq 1$ holds the inequality

$$V_N(x_0) \leq l^\star(x_0)\,\gamma_N.$$

Proof. The proof is similar to [Grüne, 2009, Lemma 3.2]. Let γ_N be equal to $\sum_{n=0}^{N-1} L^n$. Then – using u_{x_0} from Assumption 5 – the inequality follows immediately from

$$V_N(x_0) \leq J_N(x_0, u_{x_0}) = \sum_{n=0}^{N-1} l(x_{u_{x_0}}(n), u_{x_0}(n))$$
$$\leq \sum_{n=0}^{N-1} L^n l^\star(x_0) = l^\star(x_0)\,\gamma_N.$$

Otherwise $\gamma_N = \sum_{n=0}^{N-1} C\sigma^n$ holds. Then the assertion follows analogously with u_{x_0} from Assumption 4.

The same slight modifications in the corresponding proofs allow us to transfer [Grüne, 2009, Lemma 3.3, 3.4] to our setting which incorporates the growth condition from Assumption 5. We obtain the estimates

$$J_{N-k}(x_{u^\star}(k), u^\star(k+\,)) \leq \gamma_{N-k}\, l^\star(x_{u^\star}(k)),$$

$k = 0, \ldots, N-1$, for an optimal control $u^\star$ for the finite horizon optimal control problem (5), $N \geq 1$, and γ_N from (11). Similarly, we derive for V_N the inequality

$$V_N(x_{u^\star}(1)) \leq J_j(x_{u^\star}(1), u^\star(1+\cdot)) + \gamma_{N-j}\, l^\star(x_{u^\star}(1+j)),$$

$j = 0, \ldots, N-2$. Based on these inequalities – whose validity relies on Bellman's optimality principle – an optimization problem is deduced in Grüne [2009]. The solution of this optimization problem, which depends on the optimization horizon N, coincides with the parameter $\alpha = \alpha_N$ in the relaxed Lyapunov inequality (8). The following theorem allows for calculating this performance index α_N explicitly.

Theorem 7. Assume Assumptions 4, 5 and let the optimization horizon N be given. Then we obtain for the suboptimality degree α_N from (8) the formula

$$\alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)} \tag{12}$$

with γ_i from (11).

Proof. According to [Grüne et al., 2010, Theorem 5.3] the theorem holds for

$$\gamma_N = \sum_{n=0}^{N-1} C\sigma^n = C\frac{1-\sigma^N}{1-\sigma}.$$

Changing the definition of γ_N to (11) does not affect [Grüne, 2009, Proposition 5.2]. Thus, the crucial step in order to generalize Theorem 5.3 from Grüne et al. [2010] is establishing [Grüne et al., 2010, Lemma 10.1] for γ_i defined via (11) which is possible in consideration of the assumed exponential controllability (9), cf. Grüne, v. Lossow, and Worthmann [2010].

As a consequence of Theorem 7, the performance estimate

$$V_\infty^{\mu_N}(x) \leq V_\infty(x)/\alpha$$

from Proposition 3 holds for our MPC-Feedback.

Remark 8. Theorem 7 remains valid for more general controllability assumptions, for instance, *finite time controllability* with linear overshoot, cf. [Grüne et al., 2010, section 3]. Indeed, the mentioned optimization problem provides suboptimality bounds α_N even for arbitrary $\mathcal{KL}_0$-functions β (see Grüne [2009] for a definition) which exhibit the property $\beta(r, n+m) \leq \beta(\beta(r,n), m)$ for all $r \geq 0, n, m \in \mathbb{N}_0$.

Remark 9. Theorem 7 is also applicable in the context of networked control systems which require the implementation of more than only the first element of the obtained sequence of control values, cf. Grüne, Pannek, Worthmann [2009] for details.

Remark 10. Theorem 7 is also generalizable to functionals including an additional weight on the final term, i.e., we substitute (5) by

$$J_N(x_0, u) = \sum_{n=0}^{N-1} l(x_u(n), u(n)) + \tilde{\omega} l(x_u(N-1), u(N-1)). \tag{13}$$

with $\tilde{\omega} \geq 0$. This may enhance the stability behavior of the underlying system significantly, cf. Grüne et al. [2010], Grüne, v. Lossow, and Worthmann [2010].

Remark 11. The stability assertion following from Theorem 7 is strict for the entire class of systems satisfying the given assumptions, i.e., a negative α_N implies the existence of a control system satisfying inequalities (9), (10) which is not stabilizable, cf. [Grüne, 2009, Theorem 5.3].

Summarizing this section, we presented a condition which enables us to give estimates on the optimal value function on the infinite horizon (4) and – in combination with minor conditions – to conclude asymptotic stability of the closed loop. Moreover, Theorem 7 provides an appropriate tool in order to calculate the corresponding suboptimality degree α_N. However, verifying Assumptions 4, 5 is crucial for the application of these results.

4. DISCUSSION OF THE GROWTH CONDITION

Theorem 7 relies on our controllability assumption as well as the growth condition introduced in the previous section. Hence, in order to apply the corresponding formula for the suboptimality index α_N one has to compute the needed parameters, i.e., overshoot C and decay rate σ for Assumption 4 and the growth bound L for Assumption 5. In order to derive appropriate estimates for C, σ one has to analyze the control system in consideration in detail, cf. Altmüller, Grüne, and Worthmann [2010] and the example in Section 5. Here, we present a guideline for obtaining suitable estimates for our growth condition.

4.1 Sampled–data systems induced by ODEs

In this subsection we focus on the derivation of Assumption 5 for sampled–data system induced by ordinary differential equations, cf. Section 2. Since these are defined in continuous time, we aim at establishing the following criterion which implies Assumption 5.

Criterion 12. Assume the existence of a (piecewise constant) control function $\tilde{u}(\cdot) : \mathbb{R}_0^+ \to U$, i.e. $\tilde{u}(t)$ is constant on $[iT, (i+1)T)$ for $i \in \mathbb{N}_0$, such that the inequality

$$l(\varphi(t; x_0, \tilde{u}), \tilde{u}(t)) \leq e^{L_c t} l^\star(x_0) \tag{14}$$

with $l^\star(x_0) := \min_{u \in U} l(x_0, u)$ holds.

Corollary 13. Let Criterion 12 be satisfied and define the sequence of control values $u(\cdot) : \mathbb{N}_0 \to U$ via $u(n) = \tilde{u}(nT)$. Then the inequality

$$l(x(n), u(n)) \leq L^n l^\star(x_0), \tag{15}$$

holds with $L = e^{L_c T}$ for $x(0) = x_0$, i.e., Assumption 5 is satisfied.

The decisive question arising in this context is whether Criterion 12 is restrictive or not. In order to address this issue we consider the cost function $l : X \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_0^+$ defined by $l(x, u) := \frac{1}{2}\|x\|^2 + \lambda\|u\|^2$ with $\lambda \geq 0$ which is often used in practical applications. Since $l(x_0, 0) = l^\star(x_0)$, i.e., the minimum is attained in 0, we define $\tilde{u} \equiv 0$. Using (3) we obtain

$$\begin{aligned} l(\varphi(t; x_0, \tilde{u}), \tilde{u}(t)) &= \frac{1}{2}\|\varphi(t; x_0, \tilde{u})\|^2 + \lambda\|\tilde{u}(t)\|^2 \\ &= \frac{1}{2}\|\varphi(t; x_0, 0)\|^2 + \lambda\|0\|^2 \\ &\leq \frac{1}{2} e^{2L_g t}\|x_0\|^2 = e^{L_c t} l^\star(x_0) \end{aligned}$$

with $L_c := 2L_g$, i.e. Inequality (14) holds true.

Before we proceed with the analysis of the growth condition from Assumption 5, we take a closer look at the interconnection of the proposed controllability condition 4 and the growth condition from Assumption 5. To this end, we assume the existence of a control function $\hat{u}(\cdot)$ satisfying the inequality

$$l(\varphi(t; x_0, \hat{u}), \hat{u}(t)) \leq C e^{-\mu t} l^\star(x_0), \quad \mu > 0 \tag{16}$$

for each $x \in X$. This implies our controllability assumption at each sampling instant nT_0 with $\sigma_0 = e^{-\mu T_0}$, $n \in \mathbb{N}_0$ (see the connection between Criterion 12 and Corollary 13) and fix the continuous time optimization horizon $[0, N_0 T_0)$. Sampled–data systems typically require sufficiently fast sampling, cf. Nešić, Teel [2004]. In order to investigate this issue systematically we consider the sequence of sampling periods $T_0, T_0/2, T_0/4, \ldots$, i.e., $T_k = 2^{-k} T_0$. This determines the discrete optimization horizons $N_0, 2N_0, 4N_0, \ldots$, i.e. $N_k = 2^k N_0$. The corresponding decay rate from (16) is $\sigma_k = e^{-\mu T_k}$. Hence, we consider the sequence

$$(T_k,\, N_k,\, \sigma_k)_{k \in \mathbb{N}_0} = (2^{-k} T_0,\, 2^k N_0,\, e^{-\mu T_k})_{k \in \mathbb{N}_0} \tag{17}$$

of parameter combinations consisting of sampling period, optimization horizon, and decay rate. Note that the interval $[0, T_k)$ on which the first element of the calculated control value sequence is applied scales down as well.

Neglecting our growth condition, i.e., applying Theorem 7 with $L := C + C\sigma - 1$, the corresponding suboptimality estimates become arbitrary bad, cf. Grüne, v. Lossow, and Worthmann [2009] for a proof. Thus, we can not conclude asymptotic stability. However, our growth condition counteracts this drawback, cf. Figure 2. Note that the constant L in Assumption 5 converges to one as the sampling period tends to zero. Hence, γ_i coincides with the sum corresponding to Assumption 5 for small i, cf. Figure 1, i.e., our growth condition provides more precise estimates for the system's behavior and compensates conservatism caused by the overshoot constant C. This explains why our growth condition resolves problem occurring for fast sampling – independently from the exact constant L in (10).

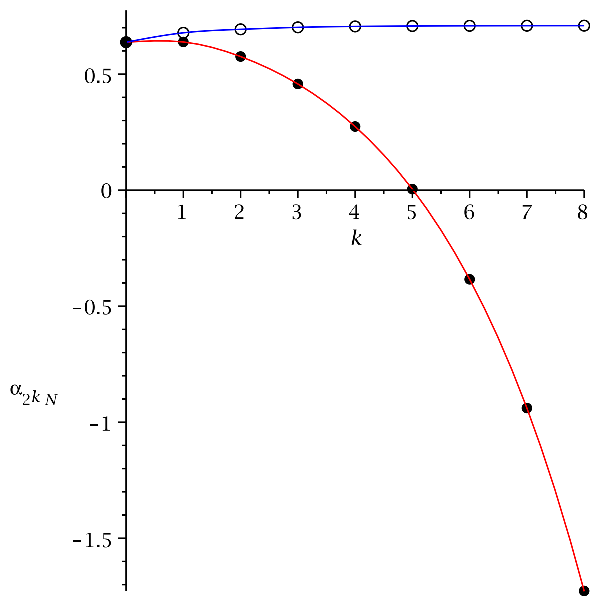

Fig. 2. We depict the suboptimality estimates obtained from Theorem 7 for the sequence (17). The solid circles mark the suboptimality estimates obtained without our growth condition (10), whereas the other curve illustrates the results which take (10) into account.

The key observation in order to deduce inequality (14) is that the cost function does – in many cases – not depend explicitly on the state, but rather on its norm. Thus, there exists a function $\bar{l} : \mathbb{R}_0^+ \times U \to \mathbb{R}_0^+$ with

$$\bar{l}(\|x\|, u) = l(x, u). \tag{18}$$

For the previously considered example this yields $\bar{l}(r, u) = \frac{1}{2}r^2 + \lambda\|u\|^2$. Here, assuming that the cost function is monotonically increasing in its first argument does not seem to be a restriction at all. As a consequence, using Gronwall's inequality yields

$$\bar{l}(\|\varphi(t; x_0, \tilde{u})\|, \tilde{u}(t)) \leq \bar{l}(e^{L_g t}\|x_0\|, \tilde{u}(t))$$

with $\tilde{u}(t) \equiv 0$. Then the growth condition from Assumption 5 follows for cost functions $\bar{l}(x, u) := \sum_{n=0}^{d} c_i x^{e_i} + f(u)$ with coefficients c_i and exponents $e_i \in \mathbb{R}^+$, $i = 0, \ldots, d$ as well as arbitrary functions $f : U \to \mathbb{R}_0^+$ with $f(u) = 0$ if and only if $u = 0$. Moreover, we emphasize that this includes cost functions which are not differentiable at 0, e.g. choose $e_0 = 1/2$.

Remark 14. Since we have not specified the norm $\|\cdot\|$ the deduced results include cost functions of type

$$l(x, u) = x^T Q x + u^T R u$$

with positive definite matrices Q, R.

4.2 Cost functions defined via integrals

In many practical applications cost functions $l : \mathbb{R}^n \times U$,

$$l(x, u) := \int_0^T \|\varphi(t; x, \tilde{u})\|^2 dt + \lambda \int_0^T \|\tilde{u}(t)\|^2 dt,$$

with regularization parameter $\lambda > 0$ and control function $\tilde{u}(t) = u$ for $t \in [0, t)$ are used. In contrast to the last subsection this cost function $l(x, u)$ evaluates x and u not only at the sampling instances, but on the entire sampling interval $[0, T)$. This allows – as a consequence – for the existence of $\bar{u}$ such that $l^\star(x) = l(x, \bar{u}) < l(x, 0)$ holds which makes the validation of Assumption 5 more difficult.

We deduce our growth condition from Assumption 5 for this setting, i.e., we show the existence of a sequence of control values $u(\cdot)$ such that the inequality

$$\int_{nT}^{(n+1)T} \|\varphi(t; x, \tilde{u})\|^2 + \lambda \|\tilde{u}(t)\|^2 \, dt \leq L^n l^\star(x_0) \quad (19)$$

holds with $\tilde{u}(nT + t) = u(n)$ for $t \in [0, T)$. For the (technical) proof which relies on a more elaborated version of Gronwall's inequality, cf. Hille [1969], in order to take the control value $\bar{u} \neq 0$ into account, we refer to the appendix. Indeed, we preserve the property $L \to 1$ for $T \to 0$ which guarantees that the growth bound L is close to one for sufficiently fast sampling.

4.3 From ordinary to partial differential equations

In this subsection our goal consists of transferring the derived results to infinite dimensional systems. This is motivated by sampled–data systems induced by partial differential equations with linear operators which allows for a wide range of applications. Typically, these operators are – in contrast to the finite dimensional case – unbounded, cf. Guo, Luo, and Morgul [2009].

In order to establish the growth condition from Assumption 5 for this setting we choose $\tilde{u} \equiv 0$. Then [Pazy, 1983, Theorem 1.2.2] provides the estimate

$$T(t) \leq M e^{\omega t}, \qquad 0 \leq t < \infty \quad (20)$$

with $\omega \geq 0$, $M \geq 1$ for the $\mathcal{C}_0$-semigroup $T(\cdot)$ whose infinitesimal generator is the above mentioned linear operator. For the corresponding sampled–data system with sampling period $T > 0$ and cost function $\frac{1}{2}\|x\|^2 + \lambda\|u\|^2$ this yields Assumption 5 with $M^2 e^{2\omega T}$. Note that this constant does not necessarily converge to one for a sampling period tending to zero. Nevertheless, the proposed growth condition may tighten the estimate from Theorem 7.

5. NUMERICAL EXAMPLE

In order to verify our analytical results we analyze the example of a synchronous generator model

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -b_1 x_3(t) \sin x_1(t) - b_2 x_2(t) + P \\
\dot{x}_3(t) &= b_3 \cos x_1(t) - b_4 x_3(t) + E + u(t)
\end{aligned}$$

with parameters $b_1 = 34.29$, $b_2 = 0.0$, $b_3 = 0.149$, $b_4 = 0.3341$, $P = 28.22$, and $E = 0.2405$, cf. Bazanella et al. [2003]. Our goal consists of stabilizing the corresponding sampled–data system at the unique equilibrium $x_\star \approx (1.12, 0.0, 0.914)$ by nonlinear model predictive control. In particular, we choose sampling intervals of length $T = 0.0025$ and impose the cost functional $J_N(x, u)$ with stage cost

$$l(x, u) = \int_0^T \|\varphi(t; x, \tilde{u}) - x_\star\|^2 + \lambda \tilde{u}(t)^2 \, dt,$$

$\tilde{u}(\cdot) = u$ on $[0, T)$ and regularization parameter $\lambda = 1/200$.

In order to apply our theoretically deduced results we require the overshoot C and the decay rate σ from our controllability Assumption 4. Here, we estimate these parameters numerically. To this end, we rewrite (9) as

$$l(x(n), u(n)) \,/\, l^\star(x_0) \leq C\sigma^n. \quad (21)$$

Moreover, we restrict the set of initial values to $X_0 = [0.42, 1.82] \times [-0.7, 0.7] \times [0.214, 1.614]$ which we discretize using a discretization grid $G \subset X_0$ with diameter $d < 0.05$ in each direction. Since even the computation of the reachable set is challenging for a continuous time controller, cf. Grüne [2001], this restriction seems to be necessary for a numerical examination.

For the described setting we obtain evolutions for the quotient on the left hand side of (21) as displayed in Figure 3. As a consequence, the minimal achievable overshoot C

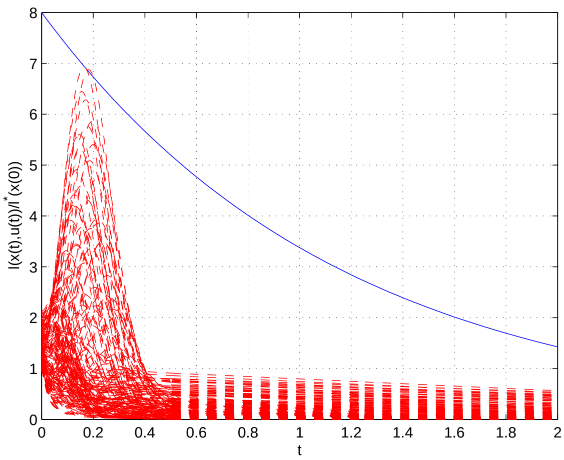

Fig. 3. Evolution of relative stage costs for a representative choice initial values $x_0 \in G$

is given by 6.9246 (with decay rate $\sigma \approx 0$). If, however, we allow for the larger overshoot $C_a = 8$ we obtain the decay rate $\sigma_a = 0.4221$. Note that the running costs corresponding to the quotients shown in Figure 3 are actually at the optimization tolerance 10^{-10} at time instant $t = 2$, hence no further improvement can be expected from this point onwards.

Furthermore, we compute the growth bound $L = 1.64829$ of the system according to Subsection 4.2 in consideration of the Lipschitz constants $L_g = 68.32333$ and $L_u = 1$. Note that our growth bound L is – despite its complicated derivation, cf. Appendix A – only slightly larger than $e^{2L_gT} \approx 1.4072$, i.e. the growth constant corresponding to the cost functional $l(x,u) = \|x\|^2 + \lambda\|u\|^2$.

Thus, we have determined the needed constants from Assumptions 4 and 5 in order to apply Theorem 7. Neglecting our growth condition, i.e., Assumption 5, we obtain stability for a NMPC horizon length $N = 36$. However, taking our growth condition into account Theorem 7 ensures stability for a significantly smaller optimization horizon of $N = 29$. This improvement is remarkable, in particular if one considers that our numerical simulations show that $N = 20$ is actually the minimal optimization horizon which stabilizes all initial values $x_0 \in G$. Moreover, leaving the optimization horizon unchanged, i.e., $N = 36$, increases the respective suboptimality index α_N from 0.258 to 0.448. Thus, we characterize the stability behavior of the MPC closed loop much more accurate by incorporating our growth condition.

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Appendix A. PROOF OF INEQUALITY (19)

The goal of this section consists of proving our growth condition for cost functions defined via integrals for sampled–data systems, cf. 4.1, i.e. estimating the growth bound such that Inequality (19) holds. For that purpose, we define $u(\cdot) = \bar{u}$, i.e. $\tilde{u}(\cdot) \equiv \bar{u}$ and carry out the following calculation in order to apply a version of Gronwall's inequality which takes the impact of the constant control function into account

$$\|\varphi(t;x,\tilde{u})\| \leq \|x\| + \int_0^t \|g(\varphi(s;x,\tilde{u}),\bar{u})\|\,ds$$

$$\leq \|x\| + \int_0^t \|g(\varphi(s;x,\tilde{u}),\bar{u}) - g(0,\bar{u})\| + \|g(0,\bar{u})\|\,ds$$

$$\leq \|x\| + \int_0^t L_g\|\varphi(s;x,\tilde{u})\| + \|g(0,\bar{u})\|\,ds$$

$$= \|x\| + t\|g(0,\bar{u})\| + L_g\int_0^t \|\varphi(s;x,\tilde{u})\|\,ds.$$

Applying [Hille, 1969, Theorem 1.5.7], i.e., Gronwall's inequality, and assuming Lipschitz-continuity of $g(0,\cdot)$ (this is, e.g., for control affine systems, typically satisfied) yields

$$\|\varphi(t;x,\tilde{u})\| \leq e^{L_g t}\|x\| + \|\bar{u}\|(e^{L_g t} - 1)\frac{L_u}{L_g}.$$

Hence, it follows with the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_T^{2T} \|\varphi(t;x_0,\tilde{u})\|^2\,dt \\
&= \int_T^{2T} \|\varphi(T;\varphi(t-T;x_0,\tilde{u}),\tilde{u}(T+\cdot))\|^2\,dt \\
&\leq \int_0^T \left(e^{L_gT}\|\varphi(t;x_0,\tilde{u})\| + \|\bar{u}\|(e^{L_gT}-1)L_u/L_g\right)^2\,dt \\
&\leq \int_0^T e^{2L_gT}\|\varphi(t;x_0,\tilde{u})\|^2 + \left((e^{L_gT}-1)L_g/L_u\right)^2\|\bar{u}\|^2 \\
&\quad + (e^{L_gT}(e^{L_gT}-1)(\|\varphi(t;x_0,\tilde{u})\|^2 + \|\bar{u}\|^2)L_u/L_g\,dt \\
&\leq \tilde{L}\left(\int_0^T \|\varphi(t;x_0,\tilde{u})\|^2\,dt + \lambda\int_0^T \|\bar{u}\|^2\,dt\right),
\end{aligned}$$

$\tilde{L} := \max\{c_s,\, c_u\}$ with $c_s = e^{2L_gT} + e^{L_gT}(e^{L_gT}-1)L_u/L_g$ and $c_u = [(e^{2L_gT} - e^{L_gT})L_u/L_g + ((e^{L_gT}-1)L_u/L_g)^2]/\lambda$. Using this result we obtain the estimate

$$l(x(n),u(n)) \leq Ll(x(n-1),u(n-1)) \leq \ldots \leq L^n l^\star(x_0)$$

with $L := \max\{c_s,\, c_u+1\}$ which implies the desired Inequality (19).