Distributed and boundary model predictive control for the heat equation

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We consider a model predictive control (MPC) scheme without stabilizing terminal constraints applied to the linear heat equation. We apply a method to analyze minimal stabilizing optimization horizons based on an exponential controllability condition. While the method is known to yield conservative quantitative results for concrete examples, our analysis shows that it precisely determines qualitative changes in the horizon for changing parameters and changing problem structures. Particularly, it is able to explain numerically observed differences in the stability behavior of MPC for distributed and for boundary control.

1 Introduction

In this paper we consider the feedback stabilization problem of the linear heat equation with distributed and boundary control. An approach to obtain an optimal stabilizing feedback is to solve an optimal control problem on an infinite time horizon. However, solving this kind of problem is in general computationally infeasible. Model predictive control (MPC), also called receding horizon control (RHC), is a well known method in order to circumvent the solution of optimal control problems on infinite time horizons. The idea of MPC is to replace the problem on an infinite horizon by the iterative solution of finite horizon problems.
The application of MPC to parabolic PDEs for distributed and boundary control can be found, e.g., in [5], [4] and [9]. In [5] and [4], the authors use a modal decomposition technique in order to obtain low dimensional problems and focus on state and control constraints. In [9], a receding horizon method for infinite dimensional systems is presented. All references have in common that the stability of the closed loop solution is guaranteed by adding terminal constraints and control Lyapunov functionals as terminal costs to the finite horizon problem which is solved iteratively. However, as the construction of a suitable terminal regions and costs is a challenging task, here we consider an unconstrained model predictive control scheme, where "unconstrained" means that we do not introduce additional stabilizing terminal constraints or costs. Since unconstrained schemes are much easier to set up and to implement, they are often preferred in practical applications, cf. [15].

For unconstrained MPC schemes, one of the key parameters which needs to be tuned in order to obtain asymptotic stability is the length N of the finite optimization horizon, also called the prediction horizon of the scheme. Under suitable conditions, asymptotic stability can be ensured for sufficiently large optimization horizons, cf. [7, Chapter 6 and the references therein]. Since the computational effort to solve the optimal control problem in each MPC step grows with the length of the optimization horizon, information about the minimal stabilizing horizon is of particular interest.

The contribution of this paper is to explain the qualitative behavior of this horizon in dependence of the used optimization criteria and the system parameters for a linear heat equation. Particularly, we will investigate the differences between distributed and boundary control, the influence of different norms in the stage cost of the finite horizon optimal control problems and the dependence on different parameters of our model problem. In Section 2 we recapitulate the required theory concerning closed loop stability of MPC. The used PDE is introduced in Section 3 in which we also present a motivating numerical example. In the ensuing sections we present the main results: We apply the general theory to the heat equation with distributed control in Section 4 and with boundary control in Section 5 in order to analyze the stability behavior and thus explain our numerical findings.

2 Preliminaries

In this section we summarize the required results concerning MPC and stability. We consider nonlinear discrete time control systems given by

$$z(n+1) = f(z(n), u(n)) \tag{1}$$

with state space Z and set of control values U. Continuous time models can be converted into a discrete form by introducing sampled data systems, where in the case of PDE models Z and U are appropriate function spaces, see the end of Section 3 for the precise formulation for the heat equation considered in this paper. The solution trajectory for given initial state z_0 and control sequence $(u(n))_{n\in\mathbb{N}_0} \subseteq U$ is denoted by $z_u(\cdot) = z_u(\cdot; z_0)$ and the space of control sequences $u : \mathbb{N}_0 \to U$ by $\mathcal{U}$. Our goal is to compute an optimal asymptotically stabilizing feedback law for (1). One approach to obtain such a feedback is to minimize the cost functional

$$J_\infty(z_0, u) := \sum_{n=0}^{\infty} \ell(z_u(n; z_0), u(n)) \tag{2}$$

on the infinite time horizon with continuous stage costs $\ell : Z \times U \to \mathbb{R}_{\geq 0}$ which penalize the distance of the state to the origin, cf. [7]. The corresponding optimal value function is given by

$$V_\infty(z_0) := \inf_{u\in\mathcal{U}} J_\infty(z_0, u). \tag{3}$$

Solving infinite horizon optimal control problems is in general computationally intractable. In order to circumvent this problem we use a receding horizon MPC method. The main idea behind this approach is to truncate the horizon and solve the optimal control problem for the finite horizon cost functional

$$J_N(z_0, u) := \sum_{n=0}^{N-1} \ell(z_u(n; z_0), u(n)) \tag{4}$$

with optimization horizon $N \in \mathbb{N}_{\geq 2}$ and corresponding optimal value function

$$V_N(z_0) := \inf_{u \in \mathcal{U}} J_N(z_0, u). \tag{5}$$

The MPC feedback $\mu_N : Z \to U$ and the corresponding solution z_{μ_N} of the closed loop system

$$z_{\mu_N}(n+1; z_0) = f(z_{\mu_N}(n; z_0), \mu(z_{\mu_N}(n; z_0))) \tag{6}$$

are obtained in the following iterative way:

1. Fix some $N \in \mathbb{N}$, an initial value z_0, set $z_{\mu_N}(0; z_0) := z_0$ and $n := 0$
2. Minimize (4) with initial value $z_{\mu_N}(n; z_0)$ in order to obtain the optimal control sequence $u^\star(0), u^\star(1), \ldots, u^\star(N-1)$ and set $\mu_N(z_{\mu_N}(n; z_0)) := u^\star(0)$
3. Compute $z_{\mu_N}(n+1; z_0)$ according to (6), set $n := n + 1$ and go to 1.

The infinite horizon cost of the resulting solution is denoted by

$$J_\infty^{\mu_N}(z_0) = \sum_{n=0}^{\infty} \ell(z_{\mu_N}(n; z_0), \mu_N(z_{\mu_N}(n; z_0))).$$

In order to achieve stability of the closed loop system additional terminal constraints or terminal costs are often introduced in (4), cf. [7, Chapter 5 and the references therein]. In contrast to this approach, here we use an unconstrained MPC algorithm, where the term "unconstrained" indicates that no stabilizing terminal constraints or costs are incorporated in the optimal control problem in Step 2 of the MPC iteration. We remark that state and control constraints coming from modelling considerations can be included by choosing appropriate subsets $\mathbb{Z} \subseteq Z$ and $\mathbb{U} \subseteq U$ but since this topic is not in the focus of this paper we will not go into details concerning feasibility issues.

Now, we summarize the stability condition proposed in [8] which requires the following controllability condition from [6].

Definition 2.1 The system (1) is called exponentially controllable with respect to the stage costs ℓ if there exist an overshoot bound $C \geq 1$ and a decay rate $\sigma \in (0, 1)$ such that for each $z \in Z$ there exists $u_z \in \mathcal{U}$ satisfying

$$\ell(z_{u_z}(n; z), u_z(n)) \leq C\sigma^n \min_{u \in U} \ell(z, u) = C\sigma^n \ell^\star(z) \tag{7}$$

for all $n \in \mathbb{N}_0$.

The exponential controllability in terms of the stage costs can be used to ensure the relaxed Lyapunov inequality

$$V_N(f(z, \mu_N(z))) \leq V_N(z) - \alpha\ell(z, \mu_N(z)), \tag{8}$$

with $\alpha \in (0, 1]$ for all $z \in Z$ which was introduced in [12] in order to conclude stability or a performance bound for the MPC closed-loop. More precisely, the following stability theorem holds, cf. [8] for a proof.

Theorem 2.2 (Stability Theorem) *Let the controllability condition 2.1 hold with overshoot constant $C \geq 1$ and decay rate $\sigma \in (0,1)$. Furthermore, let the prediction horizon N be chosen such that the stability condition*

$$\alpha_N := 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)} > 0 \tag{9}$$

holds with $\gamma_i := C\sum_{n=0}^{i-1}\sigma^n$. Then, the relaxed Lyapunov Inequality (8) *holds with $\alpha = \alpha_N$ for each $z \in Z$ and, consequently, the suboptimality estimate*

$$J_\infty^{\mu_N}(z) \leq \frac{1}{\alpha_N} V_\infty(z), \qquad z \in Z. \tag{10}$$

If, in addition, $\mathcal{K}_\infty$-functions $\underline{\eta}$, $\overline{\eta}$ exist satisfying

$$\underline{\eta}(\|z - z^\star\|) \leq \ell^\star(z) \quad \textit{and} \quad \overline{\eta}(\|z - z^\star\|) \geq V_N(z) \tag{11}$$

for all $z \in Z$, then the trajectory generated by (6) *is asymptotically stable and thus in particular converges to the equilibrium $z^\star$ as $n \to \infty$.*

Our goal is now to use this theorem in order to determine the minimal horizon N which is needed in order to ensure the inequality $\alpha_N > \alpha_0$ for some given α_0. This leads to the following road map for our analysis:

- Find (not necessarily optimal) controls u_z such that the exponential controllability condition is fulfilled.
- Calculate C and σ for this particular control.
- Use this information to explain the qualitative behavior of the minimal horizon N guaranteeing the suboptimality degree $\alpha_N > \alpha_0$ for a given $\alpha_0 \in [0,1)$.

It was already observed in [2] that the minimal horizons N computed from α_N in (9) tend to be conservative for concrete examples. However, the focus in this paper is on the qualitative analysis of N, i.e., on the question how N depends on the problem parameters, on the cost function ℓ and on structure of the control (distributed or boundary control). For this analysis, it will turn out that the road map outlined above yields very precise information and thus valuable structural insight about how the ingredients of the MPC scheme should be tuned.

Since the qualitative behavior does not change significantly for different choices of α_0, in the remainder of the paper we focus on the minimal stabilizing horizon, i.e., we set $\alpha_0 = 0$. For this setting we visualize in Figure 1 the influence of the constants C and σ on the minimal stabilizing horizon N: the areas in Figure 1 indicate those values of the pairs (C, σ) corresponding to a certain minimal stabilizing horizon. The key information from this visualization which will be crucial in our subsequent analysis is the different impact of the constants: For fixed σ it is always possible to obtain stability with the shortest possible horizon $N = 2$ by reducing C. In contrast to this, for fixed C it is in general impossible to arbitrarily reduce N by reducing σ. Therefore, the overshoot constant C plays an important role in the stability analysis.

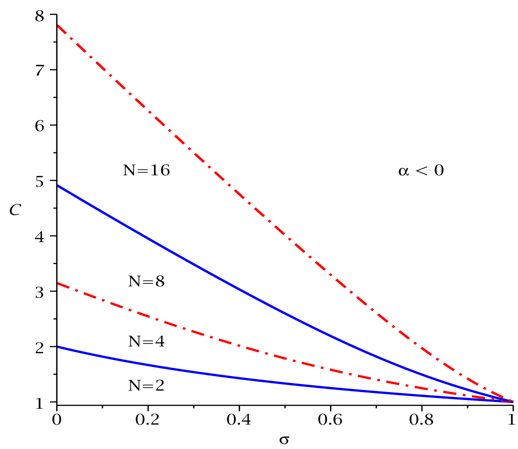

Fig. 1 Stability areas for different optimization horizons N in dependence of C and σ

3 Heat Equation

In this section we introduce the equation which we want to stabilize using MPC. We consider the one dimensional linear heat equation with either distributed or boundary control. In the first case the control system is given by

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) + \mu y(x,t) + v(x,t) && \text{on} \quad \Omega \times (0,\infty) \\ y(0,t) &= y(L,t) = 0 && \text{on} \quad (0,\infty) \\ y(x,0) &= y_0(x) && \text{on} \quad \Omega \end{aligned} \tag{12}$$

with homogeneous Dirichlet boundary conditions and distributed control $v(x,t)$, i.e., we are able to control the whole domain $\Omega = (0,L)$. $y_0 \in H_0^1(\Omega)$ denotes the initial condition. In addition to the diffusion term (y_{xx}) there is a destabilizing reaction term (μy) with reaction parameter $\mu \in \mathbb{R}^+$. It is well known, cf. [3], that the uncontrolled equation ($v(x,t) \equiv 0$) is unstable for $\mu \geq \lambda_1$ where $\lambda_1 = \lambda_1(\Omega)$ denotes the smallest eigenvalue of $(-\partial_{xx})$ in $H_0^1(\Omega)$. Figure 2 displays the unstable solution of (12) with $\mu = 15$ and initial condition $y_0(x) = \frac{1}{5}\sin(\pi x)$ on $\Omega = (0,1)$. The theory concerning existence and regularity of the optimal control in the case of quadratic cost functionals is well known and can be found e.g. [13].

In the case of boundary control we assume that we can only act on the right boundary via a Dirichlet condition, i.e., we consider

$$\begin{aligned} y_t(x,t) &= y_{xx}(x,t) + \mu y(x,t) && \text{on} \quad \Omega \times (0,\infty) \\ y(0,t) &= 0 && \text{on} \quad (0,\infty) \\ y(1,t) &= v(t) && \text{on} \quad (0,\infty) \\ y(x,0) &= y_0(x). && \text{on} \quad \Omega \end{aligned} \tag{13}$$

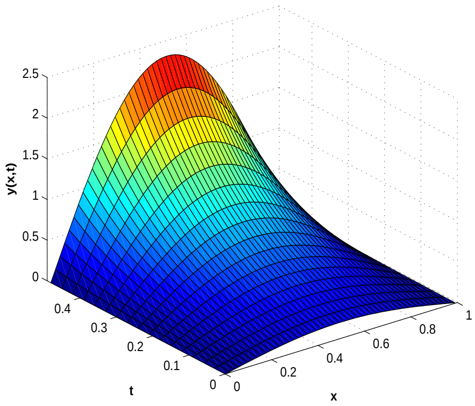

Fig. 2 Solution of the heat equation (12) with $\mu = 15$ and initial condition $y(x, 0) = \frac{1}{5}\sin(\pi x)$.

Existence and regularity results of the optimal control for quadratic cost — which are more complicated to derive than for distributed control — can be found in [11].

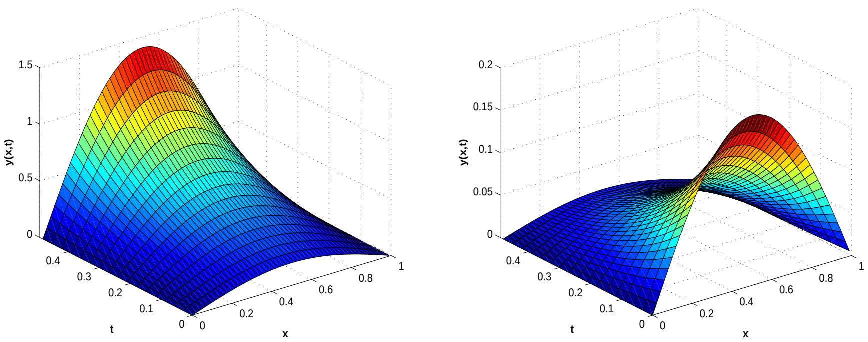

Fig. 3 Closed loop solution of the heat equation (12) with stage costs (15) (left) and stage costs (16) (right) with optimization horizon $N = 2$ and $\Omega = (0, 1)$.

In [6] it was shown that the choice of the stage cost has a significant influence on the optimization horizon which is required to guarantee a desired suboptimality degree. In order to motivate the study in this paper we present numerical simulations of our MPC algorithm which show that the minimal stabilizing horizon depends on the used stage cost as well as on the system parameters. We consider (12) with distributed control and initial function $y_0(x) = \frac{1}{5}\sin(\frac{\pi}{L}x)$. In Figure 3 the trajectory of the closed loop solution is displayed on $\Omega = (0, 1)$ for two different stage costs (15) and (16) (for details on the costs see Section 4). From the convergence behavior for growing t it is clearly visible that the horizon $N = 2$ is sufficient to guarantee stability for stage cost (16) but not for (15). This behavior changes to the opposite

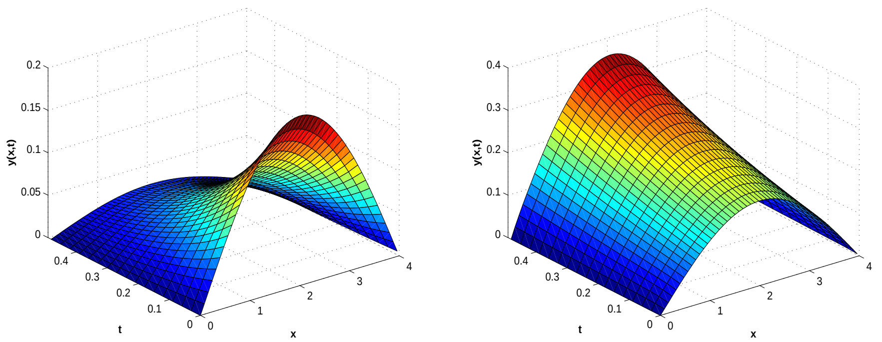

Fig. 4 Closed loop solution of the heat equation (12) with stage costs (15) (left) and stage costs (16) (right) with optimization horizon $N = 9$ and $\Omega = (0, 4)$.

if we enlarge the domain to $\Omega = (0, 4)$, cf. Figure 4 for $N = 9$: now the closed loop system is stable for stage cost (15) but not for (16). Hence, our goal in the remainder of the paper is to explain this at the first glance surprising behavior.

In order to use the theory presented in Section 2 we have to rewrite the continuous time PDE as a discrete time system (1). Let $\varphi(x, t; v, y_0)$ be the solution of (12) with control v and initial function y_0. Then we define the sampled data system with sampling period $T > 0$ as

$$z(n + 1) := \varphi(x, T; u(n), z(n)) \tag{14}$$

with $z(0) = y_0$. In the distributed case the discrete control $u(n) \in L^\infty([0, T), L^2(\Omega))$ is defined as $u(n) := v(\cdot, t + nT)|_{[nT,(n+1)T)}$. In the boundary case we define $u(n) := v(t + nT)|_{[nT,(n+1)T)}$ with $u(n) \in L^\infty([0, T), \mathbb{R})$. In the sequel, this relation between v and u will be implicitly used when verifying (7). The discrete time n corresponds to the continuous time nT which implies $z(n) = y(\cdot, nT) \in H^1(\Omega) =: Z$.

4 Distributed Control

4.1 Numerical observations

The analysis in [1] and the example in Section 3 show that the minimal stabilizing horizon strongly depends on the choice of the stage cost. A common choice (cf. [13]) for the stage costs ℓ is given by

$$\ell(z(n), u(n)) = \frac{1}{2}\|y(x, nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|v(x, nT)\|^2_{L^2(\Omega)}, \tag{15}$$

i.e., we penalize the state as well as the control in the $L^2(\Omega)$-norm. The value $\lambda > 0$ denotes a so called regularization or Tikhonov parameter.

An alternative choice of ℓ is to penalize the gradient of the state y instead of the state itself. Since we consider homogeneous Dirichlet boundary conditions this stage cost also identifies the equilibrium $y^\star \equiv 0$, i.e., $\ell(y,u) > 0$ for $y \neq y^\star$ and $\ell(y^\star, 0)) = 0$. This leads to

$$\ell(z(n), u(n)) = \frac{1}{2}\|y_x(x, nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|v(x, nT)\|^2_{L^2(\Omega)}. \tag{16}$$

Note that because of the boundary conditions the norm $\|y_x\|_{L^2(\Omega)}$ is equivalent to $\|y\|_{H^1_0(\Omega)}$.

L	$\lambda_1(L)$	$N_{\|y\|}$	$N_{\|y_x\|}$
1	π^2	6	2
2	$\pi^2/4$	8	5
3	$\pi^2/9$	8	8
π	1	8	8
4	$\pi^2/16$	8	10
5	$\pi^2/25$	8	12

Table 1 Minimal stabilizing horizon in dependence of the stage cost and the domain determined from numerical simulations of the MPC closed loop

Now, we want to investigate the example from Section 3 in a systematic way. For this purpose we first look at minimal stabilizing horizons determined from numerical simulation of the MPC closed loop for the different stage cost (15) and (16) with different domains $\Omega = (0, L)$. In this example we choose the initial function $y_0(x) = \sin(\frac{\pi}{L}x)$ and the destabilizing reaction parameter $\mu = 15$. Since $\lambda_1 = (\pi/L)^2 < 15 = \mu$ the origin is an unstable equilibrium for $L \geq 1$. The sampling time is given by $T = 0.01$ and the regularization parameter by $\lambda = 0.01$. The results are presented in Table 4.1. The first two rows display the varying interval length L and the corresponding eigenvalue λ_1. The minimal stabilizing horizon observed in the numerical example is denoted by $N_{\|y\|}$ for stage costs (15) and by $N_{\|y_x\|}$ for stage costs (16). We observe that up to $L = \pi$ the values $N_{\|y_x\|}$ are smaller than $N_{\|y\|}$. This behavior changes to the opposite for $L > \pi$. Furthermore, it is observable that $N_{\|y\|}$ grows for small L, but then it remains constant. In contrast to this $N_{\|y_x\|}$ increases monotonically.
Our goal is to give a qualitative explanation of this behavior with the methods presented in Section 2.

4.2 Derivation of C and σ

In order to perform the first step of our road map we have to find a control u_z for any initial value z such that the exponential controllability condition (7) is satisfied. A key issue in this method is that we do not need the optimality of this control. It is well known, cf. [3], that the linear feedback

$$v(x,t) := -Ky(x,t) \tag{17}$$

stabilizes (12) for a sufficient large constant $K \in \mathbb{R}$. This property is used in the following theorem in order to establish (7).

Theorem 4.1 *The heat equation* (12) *with control (17), $K > \mu - \lambda_1$, and stage costs (15) fulfills the exponential controllability condition* (7)*. The corresponding constants are given by $\sigma = e^{-2T(\lambda_1-\mu+K)} \in (0,1)$ and $C = (1 + \lambda K^2) \in \mathbb{R}$.*

P r o o f. The particular control (17) reduces the equation (12) to

$$y_t(x,t) = y_{xx}(x,t) + (\mu - K)y(x,t). \tag{18}$$

Note that we get a more regular solution than in the general case, i.e., $y(x,t) \in C^\infty(\bar{\Omega} \times (0,\infty))$, because there is no longer a control function in the equation. In order to obtain an estimation for σ we use classical Lyapunov methods presented in e.g. [10]. For this purpose we define

$$V(t) := \frac{1}{2}\|y(x,t)\|^2_{L^2(\Omega)} = \frac{1}{2}\int_0^L y(x,t)^2\ dx \tag{19}$$

and differentiate $V(t)$ with respect to time

$$\begin{aligned}\dot{V}(t) &= \int_0^L y(x,t)y_t(x,t)\ dx = \int_0^L y(x,t)(y_{xx}(x,t) + (\mu - K)y(x,t))\ dx \\ &= y(x,t)y_x(x,t)\Big|_0^L - \int_0^L y_x(x,t)^2\ dx + (\mu - K)\int_0^L y(x,t)^2\ dx \\ &\le (-\lambda_1 + \mu - K)\int_0^L y(x,t)^2\ dx = 2(-\lambda_1 + \mu - K)V(t)\end{aligned}$$

where we used the Wirtinger inequality in the last estimate. This yields

$$V(t) \le e^{-2t(\lambda_1 - \mu + K)}V(0) \tag{20}$$

and thus

$$\ell^\star(z(n)) = \frac{1}{2}\|y(x,nT)\|^2_{L^2(\Omega)} \le \frac{1}{2}e^{-2nT(\lambda_1-\mu+K)}\|y(x,0)\|^2_{L^2(\Omega)} = \sigma^n \ell^\star(z_0)$$

with $\sigma := e^{-2T(\lambda_1-\mu+K)}$.
The overshoot constant C is determined by including the control effort

$$\begin{aligned}\ell(z(n),u(n)) &= \frac{1}{2}\|y(x,nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|v(x,nT)\|^2_{L^2(\Omega)} \\ &= \frac{1}{2}\|y(x,nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}K^2\|y(x,nT)\|^2_{L^2(\Omega)} \\ &= C\ell^\star(z(n)) \quad \text{with} \quad C := (1 + \lambda K^2).\end{aligned}$$

By combining the previous inequalities we obtain the desired inequality (7). □

The next theorem is the counterpart of Theorem 4.1 for stage cost (16).

Theorem 4.2 *The heat equation* (12) *with control (17), $K > \mu - \lambda_1$, and stage costs (16) satisfies the exponential controllability condition* (7). *The corresponding constants are given by $\sigma = e^{-2T(\lambda_1-\mu+K)} \in (0,1)$ and $C = (1 + \frac{\lambda K^2}{\lambda_1}) \in \mathbb{R}$.*

Proof. Since we do not change the control we obtain the same reduced PDE (18) as in the proof of Theorem 4.1. In particular, we have the same regularity results. Similar to (19) we define

$$V(t) := \frac{1}{2}\|y_x(x,t)\|^2_{L^2(\Omega)} = \frac{1}{2}\int_0^L y_x(x,t)^2\,dx. \tag{21}$$

Differentiating with respect to time yields

$$\begin{aligned}\dot V(t) &= \int_0^L y_x(x,t)y_{xt}(x,t)\,dx = y_t(x,t)y_x(x,t)\Big|_0^L - \int_0^L y_{xx}(x,t)y_t(x,t)\,dx\\ &= -\int_0^L (y_{xx}(x,t))^2\,dx - (\mu-K)\int_0^L y_{xx}(x,t)y(x,t)\,dx\\ &\le (-\lambda_1+\mu-K)\int_0^L y_x(x,t)^2\,dx = 2(-\lambda_1+\mu-K)V(t)\end{aligned}$$

and thus

$$\ell^\star(y(n)) = \frac{1}{2}\|y_x(x,nT)\|^2_{L^2(\Omega)} \le \frac{1}{2}e^{-2nT(\lambda_1-\mu+K)}\|y_x(x,0)\|^2_{L^2(\Omega)} = \sigma^n\ell^\star(y_0)$$

with $\sigma = e^{-2T(\lambda_1-\mu+K)}$. In order to estimate C we consider

$$\begin{aligned}\ell(z(n),u(n)) &= \frac{1}{2}\|y_x(x,nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|v(x,nT)\|^2_{L^2(\Omega)}\\ &= \frac{1}{2}\|y_x(x,nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}K^2\|y(x,nT)\|^2_{L^2(\Omega)}\\ &\le C\ell^\star(z(n)) \quad \text{with} \quad C := (1+\frac{\lambda K^2}{\lambda_1})\end{aligned} \tag{22}$$

and get the desired inequality as in the proof of Theorem 4.1. □

4.3 Explanation of numerical observations

The results in Theorem 4.1 and 4.2 can now be used to explain the behavior observed by numerical simulations and summarized in Table 4.1. Obviously, we obtain for stage costs (15) and (16) the same decay rate $\sigma = e^{-2T(\lambda_1-\mu+K)}$. The main difference in the estimates of the controllability constants is the occurrence of the eigenvalue λ_1 in the overshoot constant C in (22). For $L < \pi$ there is $\lambda_1 > 1$ and the value C for the stage cost (16) is smaller than the corresponding C for (15). In view of Figure 1 we expect a shorter horizon for this smaller C value. For $L > \pi$ the order reverses. This corresponds exactly to the numerical results presented in Table 4.1. Note that we choose the initial function y_0 in such way that it is the corresponding eigenfunction to the eigenvalue λ_1 and therefore, the Wirtinger inequality is tight.

Furthermore, we can use the theorems to explain the growth behavior as well. For stage cost (15) the eigenvalue λ_1 (and therefore L) influences only the decay rate $\sigma = e^{-2T(\lambda_1-\mu+K)}$ but not the overshoot bound $C = (1+\lambda K^2)$. In order to ensure that $\sigma \in (0,1)$ the parameter

K has to compensate $\mu - \lambda_1$. Thus, for small λ_1 we need a large parameter K and this leads to large C and a long horizon N. In the second row we see that there is a big decay (compared with μ) from $\lambda_1 = \pi^2$ to $\lambda_1 = \pi^2/4$ for the interval length $L = 1$ and $L = 2$. For larger L the decay in λ_1 is comparatively small and the K and therefore the horizon N does not change considerably. The same argument holds true for the influence of λ_1 on σ in the case of stage cost (16). However, in contrast to (15) the overshoot bound $C = (1 + \frac{\lambda K^2}{\lambda_1})$ also depends on the eigenvalue λ_1. Since λ_1 converges to zero for growing interval length L we obtain a large constant C and we expect a large horizon N, which is exactly what we observe in the simulation.

5 Boundary Control

5.1 Numerical observations

In this section we consider the boundary control problem (13) for the heat equation. Here we exclusively consider the stage cost

$$\ell(z(n), u(n)) = \frac{1}{2}\|y(x, nT)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}|v(nT)|^2 \tag{23}$$

which is analogous to (15). Again, we want to analyze the dependence on parameter for distributed and boundary control and first provide numerical simulation results which are given in Table 5.1. The minimal stabilizing horizon determined from the simulations is denoted by (N_B) for boundary and (N_D) for distributed control. The varying parameters are the reaction coefficient μ and the regularization parameter λ. In the second row the minimal stabilizing horizon for $\lambda = 0.001$ is displayed. It is observable that in the distributed case the smallest possible horizon $N = 2$ is sufficient in order to achieve stability for a reaction term $\mu \leq 20$. In contrast to that we obtain larger horizons in the case of boundary control. Even for this small value of λ it is in general not possible to get stability for $N = 2$. Furthermore, we see that the horizon in the distributed case increases dramatically for growing λ. For the boundary case the horizon only changes in a moderate way depending on λ. In order to explain these observations we proceed as in Section 4.

5.2 Derivation of C and σ

In order to obtain a control that fulfills the controllability condition (7) we use the technique of backstepping, cf. [16] for details. The idea behind this method is to transform (13) into a so called ”target system” which is exponentially stable. The coordinate transformation is done with a Volterra kernel

$$w(x, t) = y(x, t) - \int_0^x k(x, z) y(z, t)\, dz. \tag{24}$$

A suitable target system is given by

$$\begin{aligned} w_t(x, t) &= w_{xx}(x, t) + (\mu - K) w(x, t) \\ w(0, t) &= 0 \\ w(1, t) &= 0 \\ w(x, 0) &= w_0 \end{aligned} \tag{25}$$

	$\lambda = 0.001$		$\lambda = 0.01$		$\lambda = 0.05$	
μ	N_D	N_B	N_D	N_B	N_D	N_B
11	2	2	3	2	7	2
12	2	3	4	3	11	3
13	2	3	4	3	14	4
14	2	4	5	4	15	5
15	2	5	6	5	16	6
16	2	6	6	6	16	7
17	2	7	7	7	16	8
18	2	7	7	7	16	8
19	2	8	8	8	16	9

Table 2 Minimal stabilizing horizon for distributed N_D and boundary N_B control with varying λ and μ determined from numerical simulations of the MPC closed loop

Note that this is exactly the reduced system we used in the proof of Theorem 4.1, where K is again the gain parameter. One advantage of this backstepping approach is that the resulting Volterra kernel is given as an explicit formula. For our system (13) and the target system (25) the transformation and the inverse transformation are given by

$$\begin{aligned} w(x,t) &= y(x,t) + \int_0^x k_1(x,z)y(z,t)\;dz \\ &= y(x,t) + \int_0^x Kz\frac{I_1\left(\sqrt{K(x^2-z^2)}\right)}{\sqrt{K(x^2-z^2)}}y(z,t)\;dz \end{aligned} \tag{26}$$

$$\begin{aligned} y(x,t) &= w(x,t) + \int_0^x k_2(x,z)w(z,t)\;dz \\ &= w(x,t) - \int_0^x Kz\frac{J_1\left(\sqrt{K(x^2-z^2)}\right)}{\sqrt{K(x^2-z^2)}}w(z,t)\;dz, \end{aligned} \tag{27}$$

where $I_1(x)$ denotes the first modified Bessel function and $J_1(x)$ the first classical Bessel function. The feedback control law coming from the Volterra transformation is given by

$$u(t) = -\int_0^1 Kz\frac{I_1\left(\sqrt{K(1-z^2)}\right)}{\sqrt{K(1-z^2)}}y(z,t)\;dz. \tag{28}$$

A deriviation that this transformation converts the system (13) into (25) can be found in [16]. For a proof concerning the kernel properties see [14].
In the subsequent Theorem 5.2 we use this control in order to establish (7). For its proof we need the following lemma whose proof is straightforward and thus omitted.

Lemma 5.1 *Let* $u(x) \in L^2(0,1)$ *and* $u(x) = v(x) + \int_0^x k_i(x,z)v(z)\;dz, (i = 1,2)$. *Then the estimate*

$$\|u(x)\|^2_{L^2(\Omega)} \leq (1+L_i)^2\|v(x)\|^2_{L^2(\Omega)} \tag{29}$$

holds with the constant $L_i := \left(\int_0^1 \int_0^x k_i(x,z)^2 \, dz \, dx\right)^{1/2}$.

Theorem 5.2 *The boundary controlled heat equation (13) with control (28),* $K > \mu - \lambda_1$*, and stage costs (23) satisfies the exponential controllability condition* (7)*. The corresponding constants are given by* $\sigma = e^{-2T(\lambda_1 - \mu + K)} \in (0,1)$ *and* $C = (1 + \lambda K^2 \eta(K))\xi(K) \in \mathbb{R}$ *with*

$$\eta(K) := \int_0^1 \left(x \frac{I_1(\sqrt{K(1-x^2)})}{\sqrt{K(1-x^2)}} \right)^2 dx \quad \textit{and} \quad \xi(K) := (1 + L_1(K))^2 (1 + L_2(K))^2.$$

P r o o f. Since the target system (25) is equivalent to the reduced PDE (18) we can use the information about the decay rate. By combining Theorem 4.1 and Lemma 5.1 we obtain

$$\begin{aligned} \|y(x,t)\|_{L^2}^2 &\leq (1+L_2)^2 \|w(x,t)\|_{L^2}^2 \leq (1+L_2)^2 e^{-2t(\lambda_1 - \mu + K)} \|w(x,0)\|_{L^2}^2 \\ &\leq (1+L_1)^2 (1+L_2)^2 e^{-2t(\lambda_1 - \mu + K)} \|y(x,0)\|_{L^2}^2 \end{aligned}$$

and thus

$$\ell^\star(z(n)) = \frac{1}{2} \|y(x,nT)\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \xi(K) \sigma^n \|y(x,0)\|_{L^2(\Omega)}^2 = \xi(K) \sigma^n \ell^\star(z_0) \quad (30)$$

with $\xi(K) := (1 + L_1(K))^2 (1 + L_2(K))^2$ and $\sigma := e^{-2T(\lambda_1 - \mu + K)}$. Furthermore, we get

$$\begin{aligned} \ell(z(n), u(n)) &= \frac{1}{2} \|y(x,nT)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \left| v(nT) \right|^2 \\ &= \frac{1}{2} \|y(x,nT)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \left| \int_0^1 K y(x,nT) x \frac{I_1(\sqrt{K(1-x^2)})}{\sqrt{K(1-x^2)}} \, dx \right|^2 \\ &\leq \frac{1}{2} \|y(x,nT)\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} K^2 \left(\int_0^1 y(x,nT)^2 \, dx \right) \left(\int_0^1 \left(x \frac{I_1(\sqrt{K(1-x^2)})}{\sqrt{K(1-x^2)}} \right)^2 dx \right) \\ &= \tilde{C} \ell^\star(z(n)) \end{aligned} \quad (31)$$

with $\tilde{C} := (1 + \lambda K^2 \eta(K))$ and $\eta(K) := \int_0^1 \left(x \frac{I_1(\sqrt{K(1-x^2)})}{\sqrt{K(1-x^2)}} \right)^2 dx$.

By combining (30) and (31) we obtain

$$\ell(z(n), u(n)) \leq \tilde{C} \ell^\star(z(n)) \leq \tilde{C} \xi(K) \sigma^n \ell^\star(z_0) \quad (32)$$

with $C := \tilde{C}\xi(K)$ the desired exponentially controllability condition. □

5.3 Explanation of numerical observations

We can use Theorem 4.1 and 5.2 to explain the observations from Table 5.1. Since the choosen target system (25) is equivalent to the reduced system (18), we obtain the same decay rate σ. Therefore, the different behavior of distributed and boundary control must be due to the overshoot bound C. In the distributed case this constant was given by $C = (1 + \lambda K^2)$. Obviously, it is possible to bring C arbitrary close to one by reducing the regularization parameter λ. Therefore, we can always achieve stability with the smallest possible horizon $N = 2$ (see

Figure 1). This explains why stability is obtained for $N = 2$ in the case of distributed control with $\lambda = 0.001$.
In order to explain why this is not possible in the case of boundary control we have to investigate the overshoot bound $C = (1 + \lambda K^2 \eta(K))\xi(K)$ from Theorem 5.2 in detail. For small λ we obtain $C \approx \xi(K)$. Since $\xi(K)$ is monotonically increasing with $\xi(K) > 1$ for $K \in \mathbb{R}^+$ we cannot obtain $C \to 1$ as $\lambda \to 0$ and therefore we cannot expect stability for $N = 2$ for small values of λ.

Furthermore, we see that the horizon in the distributed PDE increases dramatically for growing λ. This can be directly explained by considering C in Theorem 4.1: The increasing λ leads to a larger C and therefore to a larger horizon. In order to explain why this occurs only in a moderate way in the boundary case, we consider the influence of $\eta(K)$ on the constant C. $\eta(K)$ is a positive and monotonically increasing function with $\eta(22.81) \approx 1$. Since the reaction parameter μ is small and therefore also K we obtain $\xi(K) < 1$. This reduces the influence of the regularization parameter λ and explains the observation.

6 Conclusions

In this paper we have analyzed the minimal stabilizing optimization horizon N for MPC without terminal constraints for the heat equation. Particularly, we have analyzed the dependence of N on various problem parameters, on the stage cost and the differences between distributed and boundary control. While the method we apply is known to yield conservative quantitative values for concrete examples, our analysis shows that it very precisely explains numerical observations about the qualitative dependence of N with respect to changes in the model parameters and the problem structure. It can thus yield valuable insight into how to choose and tune the ingredients of an MPC scheme in order to obtain stability with small optimization horizons and thus with low numerical effort.

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