Scalable Control of Monotone Systems

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Classical Control Theory is Not Scalable

- Riccati solvers require $O(n^3)$ flops and $O(n^2)$ memory
- MPC requires even more
Towards a Scalable Control Theory

- Gradient descent, LQ optimization, fixed control structure
  - Karl Mårtensson PhD April 2012, >30000 states
  - Onofrio Semeraro PhD February 2013, >100000 states

- Distributed MPC using price mechanisms
  - Pontus Giselsson PhD thesis Nov 2012

- Today: Exploiting monotone/positive systems
Outline

- **Examples**
  - Positive/Monotone Systems
  - Scalable Verification and Synthesis
  - Positively Dominated Systems
  - Optimal Power Flow
Example 1: Transportation Networks

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{bmatrix} =
\begin{bmatrix}
-1 - \ell_{31} & \ell_{12} & 0 & 0 \\
0 & -\ell_{12} - \ell_{32} & \ell_{23} & 0 \\
\ell_{31} & \ell_{32} & -\ell_{23} - \ell_{43} & \ell_{34} \\
0 & 0 & \ell_{43} & -4 - \ell_{34} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} +
\begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
\end{bmatrix}
\]

How do we select \( \ell_{ij} \) to minimize the gain from \( w \) to \( x \)?
Application projects in Lund:

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics
Example 2: A vehicle formation
Example 2: Vehicle Formations

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \ell_{13} (x_3 - x_1) + w_1 \\
\dot{x}_2 &= \ell_{21} (x_1 - x_2) + \ell_{23} (x_3 - x_2) + w_2 \\
\dot{x}_3 &= \ell_{32} (x_2 - x_3) + \ell_{34} (x_4 - x_3) + w_3 \\
\dot{x}_4 &= -4x_4 + \ell_{43} (x_3 - x_4) + w_4
\end{align*}
\]

How do we select $\ell_{i,j}$ to minimize the gain from $w$ to $x$?
Example 3: Mass-spring system

\[
\ddot{x}_i + d_i \dot{x} + k_i x_i = \sum_j \ell_{ij} (x_j - x_i) + w_i \quad i = 1, \ldots, N
\]

Given masses \( m_i \) and local spring constants \( k_i \), select the \( \ell_{ij} \) to minimize the gain from \( w \) to \( x \)?
Outline

- Examples
- Positive/Monotone Systems
  - Scalable Verification and Synthesis
  - Positively Dominated Systems
  - Optimal Power Flow
A linear system is called *positive* if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

\[
\frac{dx}{dt} = Ax + Bu \quad \quad y = Cx
\]

Equivalently, \(A\), \(B\) and \(C\) have nonnegative coefficients except for the diagonal of \(A\).

**Examples:**

- Probabilistic model with \(x_k\) the probability of state \(k\).
- Economic system with \(x_k\) the quantity of commodity \(k\).
- Chemical reaction with \(x_k\) the concentration of reactant \(k\).
- Ecological system with \(x_k\) the population of species \(k\).
Positive Systems and Nonnegative Matrices

Classics:
- Perron (1907) and Frobenius (1912)
- Leontief (1936)
- Hirsch (1985)

Books:
- Gantmacher (1959)
- Berman and Plemmons (1979)
- Luenberger (1979)

Recent control related work:
- Tanaka and Langbort (2010)
Suppose the matrix $A$ has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exists a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

(iii) There exists a vector $z > 0$ such that $A^Tz < 0$.

(iv) There is a diagonal matrix $P > 0$ such that $A^TP + PA < 0$. 
Solving the three alternative inequalities gives three different Lyapunov functions:

\[ A\xi < 0 \quad A^T P + PA \prec 0 \quad A^T z < 0 \]

\[ V(x) = \max_k \left( \frac{x_k}{\xi_k} \right) \quad V(x) = x^T P x \quad V(x) = z^T x \]
For the nonlinear system $\dot{x} = f(x)$, let $x(t) = \phi(x_0, t)$ be the solution starting from $x_0$. The system is called monotone if $x \leq y$ implies $\phi(x, t) \leq \phi(y, t)$ for all $t \geq 0$.

**Theorem.** Let $\dot{x} = f(x)$ be a monotone system such that the origin globally asymptotically stable and the compact set $X \subset \mathbb{R}^n_+$ is invariant. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for $k = 1, \ldots, n$, such that $V(x) = \max\{V_1(x_1), \ldots, V_n(x_n)\}$ satisfies

$$\frac{d}{dt} V(x(t)) = -V(x(t))$$

along all trajectories in $X$.

**Challenge:**
Find nonlinear versions of all other results in this presentation!
Suppose that \( G(s) = C(sI - A)^{-1}B + D \) where \( A \in \mathbb{R}^{n \times n} \) is Metzler, while \( B \in \mathbb{R}^{n \times 1} \), \( C \in \mathbb{R}^{1 \times n} \) and \( D \in \mathbb{R}_+ \). Define \( \|G\|_\infty = \sup \omega |G(i\omega)| \). Then the following are equivalent:

(i) The matrix \( A \) is Hurwitz and \( \|G\|_\infty < \gamma \).

(ii) The matrix \( \begin{bmatrix} A & B \\ C & D - \gamma \end{bmatrix} \) is Hurwitz.

(iii) There is diagonal \( P > 0 \) such that \( \dot{x} = Ax + Bw \) gives

\[
\frac{d}{dt} x(t)^T Px(t) + |Cx(t) + Dw(t)|^2 \leq \gamma^2 |w(t)|^2
\]

(iv) There is \( 0 < p \in \mathbb{R}^n \) such that \( \dot{x} = Ax + Bw \) gives

\[
\frac{d}{dt} (p^T |x(t)|) + |Cx(t) + Dw(t)| \leq \gamma |w(t)|
\]

Moreover, if \( A \) is Hurwitz, then \( \|G\|_\infty = G(0) \).
Suppose that $G(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n \times n}$ is Metzler, while $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and $D \in \mathbb{R}_+$. Define $\|G\|_\infty = \sup_\omega |G(i\omega)|$. Then the following are equivalent:

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(iii) There is diagonal $P > 0$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} x(t)^T Px(t) + |Cx(t) + Dw(t)|^2 \leq \gamma^2 |w(t)|^2$$

(iv) There is $0 < p \in \mathbb{R}^n$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} \left( p^T |x(t)| \right) + |Cx(t) + Dw(t)| \leq \gamma |w(t)|$$

Moreover, if $A$ is Hurwitz, then $\|G\|_\infty = G(0)$. 

Anders Rantzer Scalable Control of Monotone Systems
Suppose that $G(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n \times n}$ is Metzler, while $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and $D \in \mathbb{R}_+$. Define $\|G\|_\infty = \sup_\omega |G(i\omega)|$. Then the following are equivalent:

(i) The matrix $A$ is Hurwitz and $\|G\|_\infty < \gamma$.

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(iii) There is diagonal $P > 0$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} x(t)^T P x(t) + |Cx(t) + Dw(t)|^2 \leq \gamma^2 |w(t)|^2$$

(iv) There is $0 < p \in \mathbb{R}^n$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} \left( p^T |x(t)| \right) + |Cx(t) + Dw(t)| \leq \gamma |w(t)|$$

Moreover, if $A$ is Hurwitz, then $\|G\|_\infty = G(0)$. 
Suppose that $G(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n \times n}$ is Metzler, while $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$ and $D \in \mathbb{R}^+$. Define $\|G\|_\infty = \sup_\omega |G(i\omega)|$. Then the following are equivalent:

(i) The matrix $A$ is Hurwitz and $\|G\|_\infty < \gamma$.

(ii) The matrix $\begin{bmatrix} A & B \\ C & D - \gamma \end{bmatrix}$ is Hurwitz.

(iii) There is diagonal $P > 0$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} x(t)^T Px(t) + |Cx(t) + Dw(t)|^2 \leq \gamma^2 |w(t)|^2$$

(iv) There is $0 < p \in \mathbb{R}^n$ such that $\dot{x} = Ax + Bw$ gives

$$\frac{d}{dt} \left(p^T |x(t)|\right) + |Cx(t) + Dw(t)| \leq \gamma |w(t)|$$

Moreover, if $A$ is Hurwitz, then $\|G\|_\infty = G(0)$. 
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A Scalable Stability Test

Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

\[\begin{bmatrix}
  a_{11} & a_{12} & 0 & a_{14} \\
  a_{21} & a_{22} & a_{23} & 0 \\
  0 & a_{32} & a_{33} & a_{32} \\
  a_{41} & 0 & a_{43} & a_{44}
\end{bmatrix}\begin{bmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \xi_4
\end{bmatrix} < \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix}\]

The first node verifies the inequality of the first row.
The second node verifies the inequality of the second row.

... 

Verification is scalable!
A Distributed Search for Stabilizing Gains

Suppose
\[
\begin{bmatrix}
    a_{11} - \ell_1 & a_{12} & 0 & a_{14} \\
    a_{21} + \ell_1 & a_{22} - \ell_2 & a_{23} & 0 \\
    0 & a_{32} + \ell_2 & a_{33} & a_{32} \\
    a_{41} & 0 & a_{43} & a_{44}
\end{bmatrix}
\geq 0 \text{ for } \ell_1, \ell_2 \in [0, 1].
\]

For stabilizing gains $\ell_1, \ell_2$, find $0 < \mu_k < \xi_k$ such that

\[
\begin{bmatrix}
    a_{11} & a_{12} & 0 & a_{14} \\
    a_{21} & a_{22} & a_{23} & 0 \\
    0 & a_{32} & a_{33} & a_{32} \\
    a_{41} & 0 & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
    \xi_1 \\
    \xi_2 \\
    \xi_3 \\
    \xi_4
\end{bmatrix}
+ \begin{bmatrix}
    -1 & 0 \\
    1 & -1 \\
    0 & 1 \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    \mu_1 \\
    \mu_2
\end{bmatrix}
< \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

and set $\ell_1 = \mu_1 / \xi_1$ and $\ell_2 = \mu_2 / \xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).
Let $\mathcal{D}$ be the set of diagonal matrices with entries in $[0, 1]$. Suppose $B, C, D \geq 0$ and $A + ELF$ is Metzler for all $L \in \mathcal{D}$. If $F \geq 0$, then the following are equivalent:

(i) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz and $\|C[sI - (A + ELF)]^{-1}B + D\|_\infty < \gamma$.

(ii) There exist $\xi \in \mathbb{R}^n_+, \mu \in \mathbb{R}^m_+$ with

$$A\xi + E\mu + B < 0 \quad C\xi + D < \gamma \quad \mu \leq F\xi$$

Alternatively, if $E \geq 0$, then (i) is equivalent to

(iii) There exist $p \in \mathbb{R}^n_+, q \in \mathbb{R}^m_+$ with

$$A^Tp + F^Tq + C^T < 0 \quad B^Tp + D < \gamma \quad q \leq E^Tp$$
Example 1: Transportation Networks

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{bmatrix}
= \begin{bmatrix}
-1 - \ell_{31} & \ell_{12} & 0 & 0 \\
0 & -\ell_{12} - \ell_{32} & \ell_{23} & 0 \\
\ell_{31} & \ell_{32} & -\ell_{23} - \ell_{43} & \ell_{34} \\
0 & 0 & \ell_{43} & -4 - \ell_{34} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
+ \begin{bmatrix}
w \\
w \\
w \\
w \\
\end{bmatrix}
\]

How do we select \( \ell_{ij} \in [0, 1] \) to minimize the gain from \( w \) to \( \sum_i x_i \)?
Example 1: Transportation Networks

\[
A = \text{diag}\{-1, 0, 0, -4\} \quad B = (1 \ 1 \ 1 \ 1)^T \\
C = (1 \ 1 \ 1 \ 1) \quad K = 0 \\
L = \text{diag}\{\ell_{31}, \ell_{12}, \ell_{32}, \ell_{23}, \ell_{43}, \ell_{34}\} \\
E = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix} \\
F = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The closed loop matrix is \( A + ELF \).
Minimize $\sum_i \xi_i$ subject to

\begin{align*}
0 & \geq -\xi_1 - \mu_{13} + \mu_{21} + 1 \\
0 & \geq -\mu_{21} - \mu_{23} + \mu_{32} + 1 \\
0 & \geq \mu_{13} + \mu_{23} - \mu_{32} - \mu_{34} + \mu_{43} + 1 \\
0 & \geq -4\xi_4 + \mu_{34} - \mu_{43} + 1
\end{align*}

and $0 \leq \mu_{ij} \leq \xi_j$. Then define $\ell_{ij} = \mu_{ij}/\xi_j$.

Optimal solution $\ell_{12} = \ell_{32} = \ell_{43} = 1$ and $\ell_{31} = \ell_{23} = \ell_{34} = 0$. 
Example 2: Vehicle Formations

\[
\begin{align*}
\dot{x}_1 &= -x_1 + \ell_{13}(x_3 - x_1) + w \\
\dot{x}_2 &= \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) + w \\
\dot{x}_3 &= \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + w \\
\dot{x}_4 &= -4x_4 + \ell_{43}(x_3 - x_4) + w
\end{align*}
\]

Select \( \ell_{ij} \in [0, 1] \) to minimize the gain from \( w \) to \( \sum_i x_i \)?
Example 2: Vehicle Formations

Minimize $p_3$ subject to

\[
0 \geq -p_1 - q_{13} + q_{21} + 1 \\
0 \geq -q_{21} - q_{23} + q_{32} + 1 \\
0 \geq q_{13} + q_{23} - q_{32} - q_{34} + q_{43} + 1 \\
0 \geq -4p_4 + q_{34} - q_{43} + 1
\]

and $0 \leq q_{ij} \leq p_j$. Then define $\ell_{ij} = q_{ij}/p_j$.

Optimality: $\ell_{21} = \ell_{23} = \ell_{34} = 1$, and $\ell_{13} = \ell_{32} = \ell_{43} = 0$. 
Example 2: Vehicle Formations

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
-1 - \ell_{13} & 0 & \ell_{13} & 0 \\
\ell_{21} & -\ell_{21} - \ell_{23} & \ell_{23} & 0 \\
0 & \ell_{32} & -\ell_{32} - \ell_{34} & \ell_{34} \\
0 & 0 & \ell_{43} & -4 - \ell_{43}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} + Bw
\]

Select \( \ell_{ij} \in [0, 1] \) to minimize the gain from \( w \) to \( \sum_i x_i \)?
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Example 3: Mass-spring system

\[
\ddot{x}_i + d_i \dot{x} + k_i x_i = \sum_j \ell_{ij} (x_j - x_i) + w_i \quad i = 1, \ldots, N
\]

Given masses $m_i$ and local spring constants $k_i$, select $\ell_{ij} \in [0, \bar{\ell}]$ to minimize the gain from $w_1$ to $x_1$?
Example 3: Mass-spring system

\[ \ddot{x}_i + d_i \dot{x} + k_i x_i = \sum_j \ell_{ij} (x_j - x_i) + w_i \quad d_i \geq k_i \]

In frequency domain:

\[ X_i(s) = \frac{1}{s^2 + d_i s + k_i} \left[ \sum_j \ell_{i,j} (X_j(s) - X_i(s)) + W_i(s) \right] \]
\( G \in RH_{\infty}^{m \times n} \) is called *externally positive* if if the corresponding impulse response \( g(t) \) is nonnegative for all \( t \). The set of all such matrices is denoted \( PH_{\infty}^{m \times n} \).

Suppose \( G, H \in PH_{\infty}^{n \times n} \). Then

- \( GH \in PH_{\infty}^{n \times n} \)
- \( aG + bH \in PH_{\infty}^{n \times n} \) when \( a, b \in \mathbb{R}_+ \).
- \( \|G\|_{\infty} = \|G(0)\| \).
- \( (I - G)^{-1} \in PH_{\infty}^{n \times n} \) if and only if \( G(0) \) is Schur.
$G \in \mathbb{R}^{m \times n}$ is called positively dominated if $|G_{jk}(i\omega)| \leq G_{jk}(0)$ for $\omega \in \mathbb{R}$. The set of all such matrices is denoted $\mathbb{D}H_{\infty}^{m \times n}$.

Suppose $G, H \in \mathbb{D}H_{\infty}^{n \times n}$. Then

- $GH \in \mathbb{D}H_{\infty}^{n \times n}$
- $aG + bH \in \mathbb{D}H_{\infty}^{n \times n}$ when $a, b \in \mathbb{R}_+$.  
- $\|G\|_{\infty} = \|G(0)\|$. 
- $(I - G)^{-1} \in \mathbb{D}H_{\infty}^{n \times n}$ if and only if $G(0)$ is Schur.
Example 3: Mass-spring system

\[ \ddot{x}_i + d_i \dot{x}_i + k_i x_i = \sum_j \ell_{ij} (x_j - x_i) + w_i \]

\[
\left( s^2 + d_i s + k_i + \sum_j \ell_{ij} \right) X_i(s) = \sum_j \left( \ell_{ij} X_j(s) + (\ell_{ij} - \ell_{ij}) X_i(s) \right) + W_i(s)
\]

\[ X = (A + ELF)X + BW \]

The transfer matrices \( B, E \) and \( A + ELF \) are positively dominated for all \( L \in D \) provided that \( d_i \geq k_i + \sum_j \ell_{ij} \).
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- **Optimal Power Flow**
Optimizing Electrical Power Flow

Minimize

$$\frac{1}{T} \sum_k \int_0^T i_k(t)v_k(t) \, dt$$

subject to

$$L \frac{di}{dt} = -Ri + v$$

$$\frac{1}{T} \int_0^T i_k(t)v_k(t) \, dt \leq p_k$$

$$\frac{1}{T} \int_0^T |v_k(t) - v_j(t)|^2 \, dt \leq c_{kj}$$

$$v_k^2 \leq \frac{1}{T} \int_0^T v_k(t)^2 \, dt \leq \bar{v}_k^2$$

for all $k,j$

The optimum is attained for constant $v$ and $i$. Why?
Minimize \[ \sum_k i_k v_k \]
subject to \[ i = Yv \]
and
\[ i_k v_k \leq \bar{p}_k \]
\[ (v_k - v_j)^2 \leq c_{kj} \]
\[ v_k \leq v \leq \bar{v}_k \]
for all \( k, j \)

Notice: \( \bar{p}_k \) negative at loads, positive at generators.
Positive Quadratic Programming

Given $M_0, \ldots, M_K \in \mathbb{R}^{n \times n}$ with nonnegative off-diagonal entries and $b_1, \ldots, b_K \in \mathbb{R}$, the following equality holds:

$$\max_{x} x^T M_0 x = \max_{X \succeq 0} \text{tr}(M_0 X)$$

subject to

- $x \in \mathbb{R}_+^n$
- $x^T M_k x \geq b_k$
- $k = 1, \ldots, K$

Proof

If $X = \begin{bmatrix} x_1^2 & \ldots & x_n^2 \end{bmatrix}$ maximizes the right hand side, then $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ maximizes the left.

Positive Quadratic Programming

Given $M_0, \ldots, M_K \in \mathbb{R}^{n \times n}$ with nonnegative off-diagonal entries and $b_1, \ldots, b_K \in \mathbb{R}$, the following equality holds:

\[
\begin{align*}
\max_{x} & \quad x^T M_0 x \\
\text{subject to} & \quad x \in \mathbb{R}_+^n \\
& \quad x^T M_k x \geq b_k \\
& \quad k = 1, \ldots, K
\end{align*}
\]

\[
\begin{align*}
\max_{tr} & \quad \text{tr}(M_0 X) \\
\text{subject to} & \quad X \succeq 0 \\
& \quad \text{tr}(M_k X) \geq b_k \\
& \quad k = 1, \ldots, K
\end{align*}
\]

Proof

If $X = \begin{bmatrix} |x_1|^2 & * \\ * & \vdots & * \\ \vdots & * & \vdots \\ |x_n|^2 \end{bmatrix}$ maximizes the right hand side, then $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ maximizes the left.

Positive Quadratic Programming

Given \( M_0, \ldots, M_K \in \mathbb{R}^{n \times n} \) with nonnegative off-diagonal entries and \( b_1, \ldots, b_K \in \mathbb{R} \), the following equality holds:

\[
\max_{x} x^T M_0 x = \max_{X} \text{tr}(M_0 X) \\
\text{subject to } x \in \mathbb{R}_+^n \\
x^T M_k x \geq b_k \\
k = 1, \ldots, K
\]

\[
\text{subject to } X \succeq 0 \\
\text{tr}(M_k X) \geq b_k \\
k = 1, \ldots, K
\]

Proof

If \( X = \begin{bmatrix} |x_1|^2 & * \\ * & |x_n|^2 \end{bmatrix} \) maximizes the right hand side, then \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \) maximizes the left.

Note: The problem is convex in \(|x_1|^2, \ldots, |x_n|^2|\).
Optimizing Electrical Power Flow

Minimize

subject to \( i = Yv \) and

\[
\begin{align*}
\sum_k i_k v_k & \\
i_k v_k & \leq \bar{p}_k \\
(v_k - v_j)^2 & \leq c_{kj} \\
v_k & \leq u_k & \leq \bar{u}_k
\end{align*}
\]

for all \( k, j \)

Notice: All mixed terms have the right sign!
The KYP Lemma for Positive Systems

Let $A \in \mathbb{R}^{n \times n}$ be Metzler and Hurwitz, while $B \in \mathbb{R}^{n \times m}$ and the pair $(A, B)$ is controllable. Suppose that all entries of $Q \in \mathbb{R}^{(n+m) \times (n+m)}$ are non-negative, except for the last $m$ diagonal elements. Then the following are equivalent:

1. For $\omega \in [0, \infty]$ is true that
   
   \[
   \begin{bmatrix}
   (i\omega I - A)^{-1}B \\
   I
   \end{bmatrix}^* Q
   \begin{bmatrix}
   (i\omega I - A)^{-1}B \\
   I
   \end{bmatrix} \preceq 0
   \]

2. \[
   \begin{bmatrix}
   -A^{-1}B \\
   I
   \end{bmatrix}^* Q
   \begin{bmatrix}
   -A^{-1}B \\
   I
   \end{bmatrix} \preceq 0.
   \]

3. There exists a diagonal $P \succeq 0$ such that
   \[
   Q + \begin{bmatrix}
   A^T P + PA & PB \\
   B^T P & 0
   \end{bmatrix} \preceq 0
   \]

4. There exist $x, p \in \mathbb{R}^n_+$ and $u \in \mathbb{R}^m_+$ such that
   \[
   Ax + Bu \geq 0 \text{ and } Q \begin{bmatrix}
   x \\
   u
   \end{bmatrix} + \begin{bmatrix}
   A^T \\
   B^T
   \end{bmatrix} p \leq 0
   \]
Main Step of Proof

(2) implies that \[ x \quad w \parallel T \quad Q \quad x \quad w \leq 0 \] for \( x \in \mathbb{R}_+^n \), \( w \in \mathbb{R}_+^m \) satisfying

\[
0 \leq x_i(A_ix + B_iw) \quad i = 1, \ldots, n
\]

Positive quadratic programming gives existence of \( \tau_1, \ldots, \tau_n \geq 0 \) such that the quadratic form

\[
\sigma(x, w) = \begin{bmatrix} x \\ w \end{bmatrix}^T Q \begin{bmatrix} x \\ w \end{bmatrix} + \sum_i \tau_i x_i(A_ix + B_iw)
\]

is negative semi-definite.

(3) follows with \( P = \frac{1}{2} \text{diag}(\tau_1, \ldots, \tau_n) \succeq 0 \).
Optimizing Electrical Power Flow

Minimize

\[ \frac{1}{T} \sum_{k} \int_{0}^{T} i_k(t)v_k(t)dt \]

subject to

\[ L \frac{di}{dt} = -Ri + v \]

\[ \frac{1}{T} \int_{0}^{T} i_k(t)v_k(t)dt \leq \bar{p}_k \]

\[ \frac{1}{T} \int_{0}^{T} |v_k(t) - v_j(t)|^2 dt \leq c_{kj} \]

\[ v_k^2 \leq \frac{1}{T} \int_{0}^{T} v_k(t)^2 dt \leq \bar{v}_k^2 \]

for all \( k, j \)

The optimum is attained for constant \( v \) and \( i \). Why?

Constant Lagrange relaxation optimum satisfies all constraints!

\( i_k \in \mathbb{R} \)

\( v_k \in \mathbb{R} \)
Optimizing Electrical Power Flow

Minimize

$$\frac{1}{T} \sum_k \int_0^T i_k(t)v_k(t)dt$$

subject to

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Classical hard problems solvable for monotone systems:

- Static output feedback
- $H_\infty/L_1$ optimal decentralized controllers
- No need for global information
- Verification and synthesis scale linearly!

Further reading:
CDC 2011/2012, full paper on our web page
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