Embedded optimization using zonotopes: application to time-optimal control

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Introduction

We consider set-theoretic approach for time-optimal control of stable discrete linear systems subject to input constraints:

$$x_{k+1} = Ax_k + Bu_k, \quad ||u_k||_\infty \le 1,$$

where $x \in \mathbb{R}^n$ is the fully observable state vector, $u \in \mathbb{R}^m$ is the control vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We suppose that system is fully controllable and matrix A has no eigenvalues outside unit disk.

Control via reachable sets

The N-step reachable set R_N is the set of states x_N that could be reached from x_0 in N steps:

$$R_N = \{x_N : x_N = A^N x_0 + \sum_{k=0}^{N-1} A^{N-k-1} B u_k, ||u_k||_\infty \le 1\}.$$

In set-theoretic terms, minimum-time control problem can be stated as

$$\min_{u_0,\dots,u_{N-1}} N \text{ so that } x_g \in R_N,$$

where x_g is our *goal state* (usually zero).

Zonotopes and their hyperplane representation

A zonotope is the image of a unit cube under an affine projection, that is, a polytope $Y \subset \mathbb{R}^n$ of the form

$$Y = \{y \in \mathbb{R}^n : y = y_0 + \sum_{i=1}^{q} h_i v_i, \ |v_i| \le 1\}$$

for some matrix $H = [h_1|h_2|\dots|h_q] \in \mathbb{R}^{n\times m}$. Vectors $h_i \in \mathbb{R}^n$ are called "generators" (Ziegler, 1995).

Reachable sets can be represented as zonotopes with matrices of zonotope generators $H = [A^{N-1}B|A^{N-2}B|\dots|B]$, $y = x_g - A^N x_0$, $y_0 = 0$.

Suppose we have computed vectors $d_j, j = \overline{1,q}$, orthogonal to $n-1$ vectors h_i (here M is the total number of all such combinations of h_i having full rank). The *hyperplane representation* of Y in the form of linear inequalities is given by (M. Demenkov, 2007; M. Althoff, 2010)

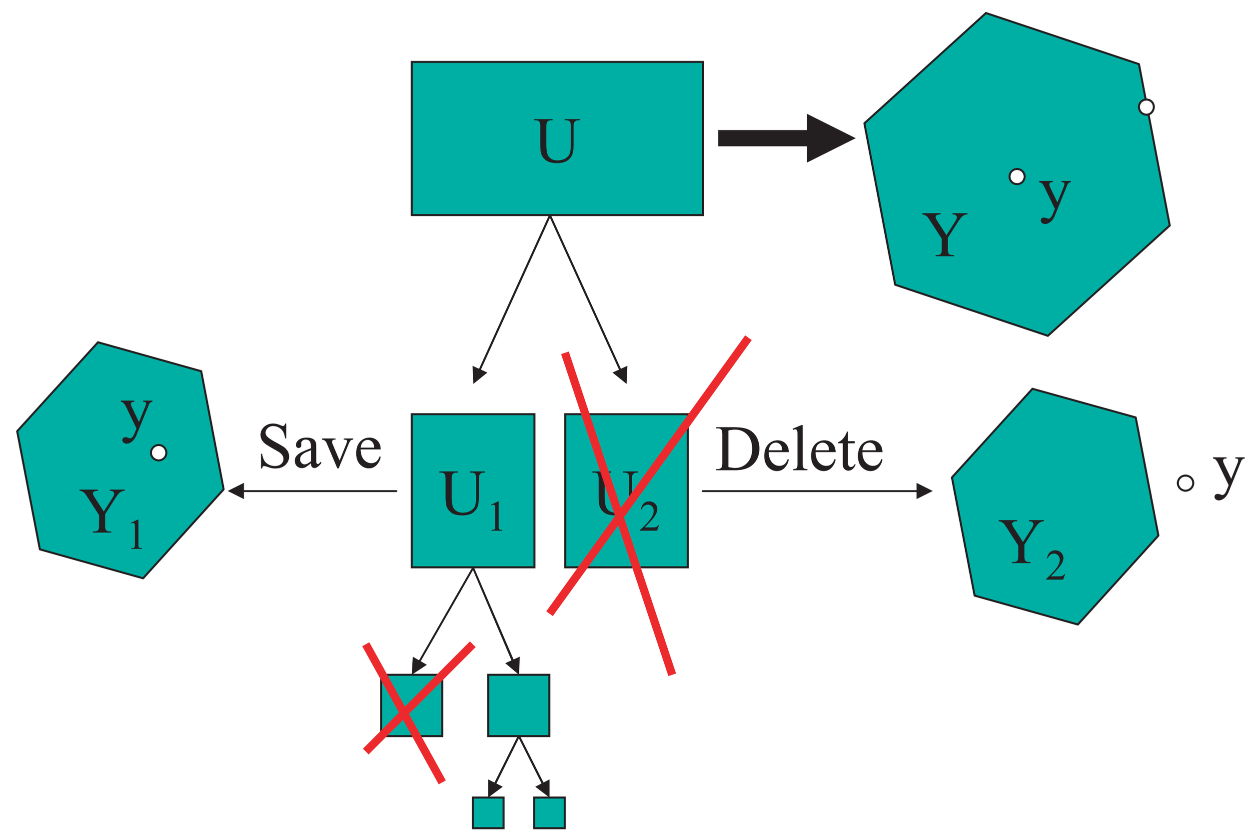
$$\pm d_j^T y \le \sum_{i=1}^{q} d_j^T h_i \text{sign}(d_j^T h_i), j = \overline{1,M}.$$

Computing optimal control sequence via control box bisection

Suppose we want to solve the following equation:

$$y = Hv, v \in \mathbf{U} = \{v : ||v||_\infty \le 1\},$$

to obtain $v \in R^q$ from the given y. Using linear inequalities, we can check if vector y is inside the zonotope Y induced by box $\mathbf{U}$. If the answer is no, we delete the box. If yes, we cut the box into two boxes $\mathbf{U_1}$ and $\mathbf{U_2}$ by half-splitting it along the longest direction and repeat this procedure for both boxes (with modified H and y).



As an example, consider time-optimal input sequence computation for

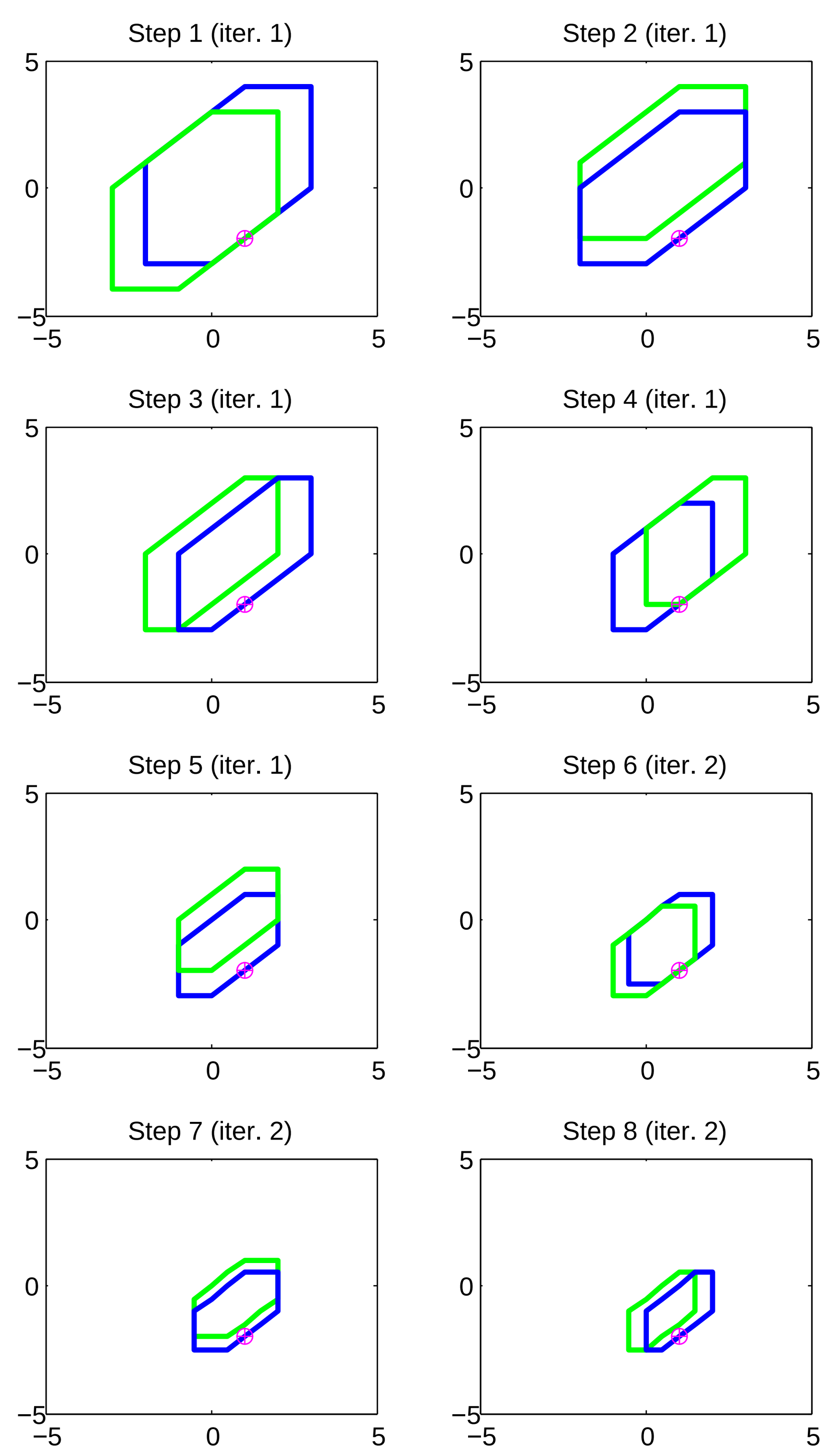
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, x_g = 0.$$

In this case, $\min N = 5$ and the matrix of generators is:

$$H = [A^4 b \;\; A^3 b \;\; A^2 b \;\; Ab \;\; b] = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 \end{bmatrix},$$

$$y = -A^5 x_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

In the following figure one can see the evolution of two reachable sets (induced by $\mathbf{U_1}$ and $\mathbf{U_2}$) during bisection procedure. The small crossed circle represents y.



- G.M. Ziegler, *Lectures on polytopes*, Vol. 152 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.
- M. Althoff, O. Stursberg and M. Buss. Computing reachable sets of hybrid systems using a combination of zonotopes and polytopes. *Nonlinear Analysis: Hybrid Systems*. vol. 4, 2010, pp. 233—249.
- M. Demenkov. "Interval bisection method for control allocation", *in Proc. 17th IFAC Symposium on Automatic Control in Aerospace*, Toulouse, 2007.

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