Optimal control of passive particles advected by two-dimensional point vortices

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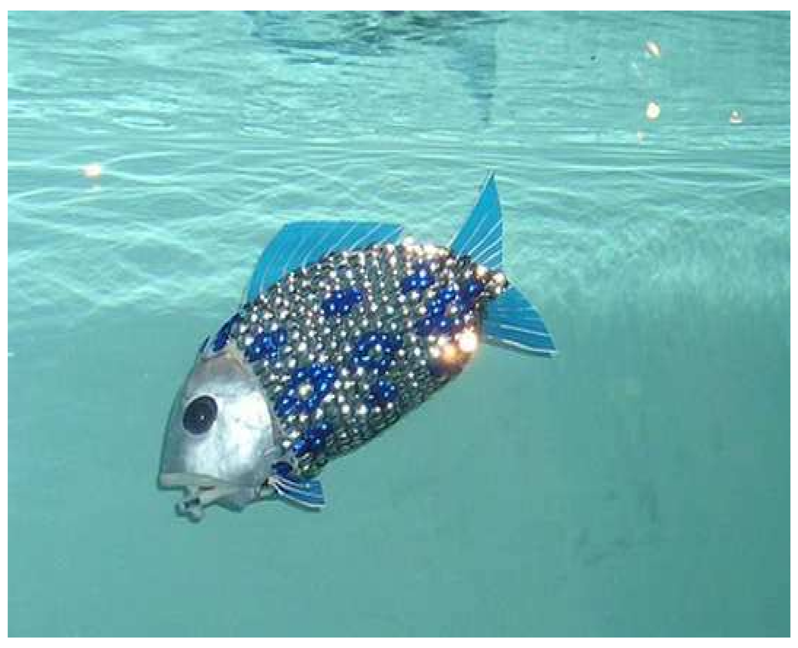
Introduction

As shown below, two types of systems are considered: (i) underwater gliders and (ii) robotic fishes.

Underwater glider (left), robotic fish (right).

- A vortex is a point with circulation that generates a rotational flow field.
- The dynamic equation of any particle advected by the flow generated by N vortices, each one located on z_i, is given by

$$\dot{z}^*(t) = \frac{1}{2\pi i}\sum_{i=1}^{N}\frac{k_i}{z(t) - z_i(t)}$$

Our problem:

- Dynamic system is driven by one vortex.
- Solve an optimal control problem that consists in moving a particle between two given points while minimizing a certain cost functional.
- Apply necessary conditions of optimality in the form of a maximum principle.

Velocity field driven by a vortex

- Vortex at the origin $(0,0)$, with vorticity k.
- $z(t) = x(t) + iy(t)$ is the position of the particle at time t.
- The dynamic equations of the particle in complex and polar forms are

$$\dot{z}^*(t) = \frac{1}{2\pi i}\frac{k}{z(t)} \quad \text{and} \quad \begin{cases} \dot{\rho}(t) = 0 \\ \dot{\theta}(t) = \frac{k}{2\pi}\frac{1}{\rho^2(t)} \end{cases}$$

The position of the particle at time t is given by

$$z(t) = \rho(0)e^{i\left(\frac{kt}{2\pi\rho^2(0)} + \theta(0)\right)} \tag{1}$$

Control problem

- Passive particle placed in the flow with initial position (x_0, y_0).
- $X(t) = (x(t), y(t))$ is the position of the particle at time t.

The control problem can be formulated as follows:

$$\begin{aligned} &\text{Minimize } g(X(T)) \\ &\text{s. t.} \quad \dot{X} = F(X,u),\ X(0) = (x_0,y_0),\ X(T) = (x_f,y_f) \\ &\qquad\ \ \|u(t)\|_\infty \leq 1, \quad \forall t \in [0,T] \end{aligned}$$

- Apply the maximum principle to determine the optimal control u^*.
- Maximization of the Pontryagin's function $H(X,P,u)$.
- Satisfaction of the appropriate boundary conditions.

Minimum energy problem

- Cost function is the total control power consumption $\int_0^T u^2\,dt$.
- Controlled dynamics for the Mayer form: $F(\rho,\theta,w,u) = (u, \frac{k}{2\pi\rho^2}, u^2)$.
- The Pontryagin's function is

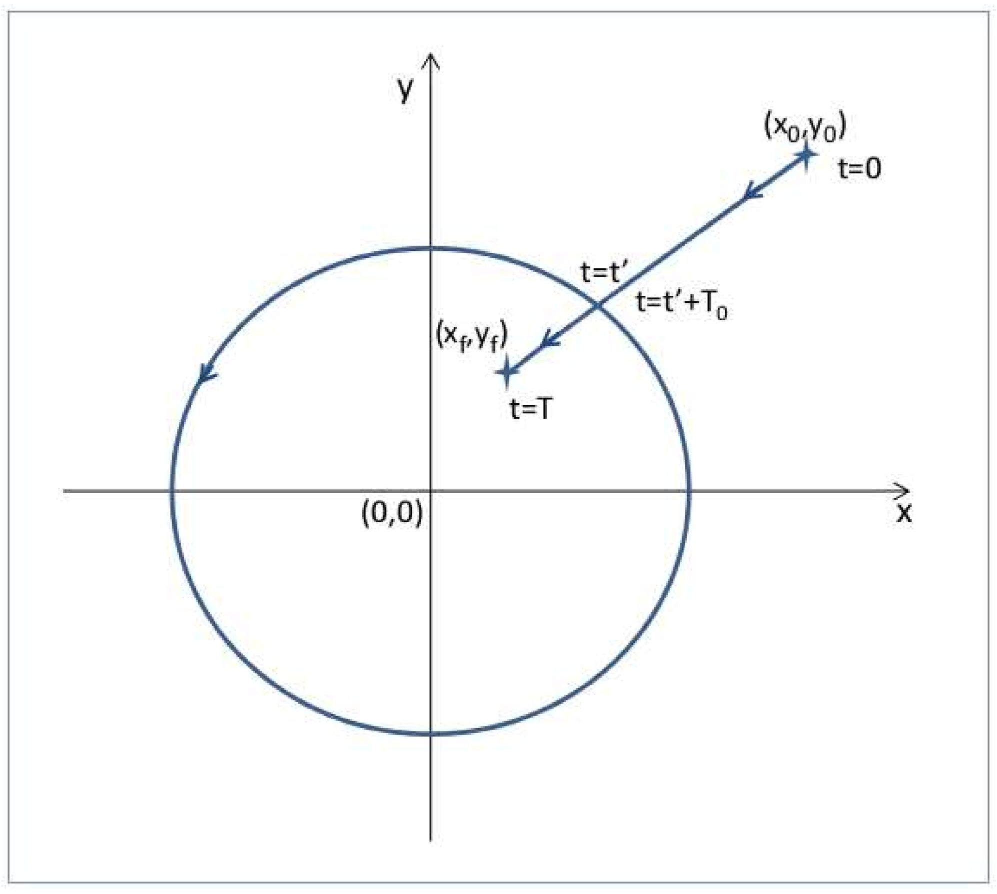
$$H(t,\rho,\theta,w,p_\rho,p_\theta,p_w) = p_\rho u + p_\theta\frac{k}{2\pi\rho^2} + p_w u^2 \tag{2}$$

- Time interval and position of the initial and end points, $[0,T]$, (x_0,y_0) and (x_f,y_f) are given.
- T: $\|(x_f,y_f)\|_2^2 \leq \frac{k}{4\pi^2} \leq \|(x_0,y_0)\|_2^2$ with $T_0 = \|(x_0,y_0)\|_2 - \|(x_f,y_f)\|_2$.
- Minimum cost $w(T) = T_0$ provided by the control

$$u^*(t) = \begin{cases} -1 & t \in [0,t') \\ 0 & t \in [t', t'+T-T_0) \\ -1 & t \in [t'+T-T_0, T] \end{cases},\quad t' = \rho(0) - \frac{\sqrt{k(T-T_0)}}{2\pi}.$$

- The equations of motion

$$\rho(t) = \begin{cases} \rho(0) - t \\ \rho(t') \\ \rho(0) + T - T_0 - t \end{cases},\ \theta(t) = \begin{cases} \theta(0) + \frac{kt}{2\pi\rho(0)(\rho(0)-t)} \\ \theta(0) + \frac{k(\rho(0)t - t'^2)}{2\pi\rho(0)(\rho(0)-t')^2} \\ \theta(0) + \theta_1 + \frac{kt}{2\pi\rho(0)(\rho(0)+T-T_0-t)} \end{cases}$$



Trajectory of the particle.

- The problem is well posed because ρ is always positive, for all $t \in [0,T]$.

Conclusions and future work

- Generalize this control problem to any point on the 2D plane.
- Extend the problem to N vortices.
- Derive optimality conditions in the form of a maximum principle.

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