Ignorant policies for centralized distributed controllers

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Motivation

Ultimate goal: The construction of a centralized controller for weakly coupled systems
Approach: Optimality based design $\rightsquigarrow$ well-posedness & performance

Dynamic Programming

Control system and value function

Dynamics $\boldsymbol{f} : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, target set $\mathcal{T} \subset \mathcal{X}$
Cost function $c : \mathcal{X} \times \mathcal{U} \to [0, \infty]$ with $c(\mathcal{T}, \boldsymbol{u}) = \{0\}$ for all $\boldsymbol{u} \in \mathcal{U}$
Value function

$$J\left(\boldsymbol{x}, (\boldsymbol{u}(k))_{k\in\mathbb{N}}\right) = \sum_{k\in\mathbb{N}} c\big(\boldsymbol{x}(k), \boldsymbol{u}(k)\big),$$

$$\text{s.t. } \boldsymbol{x}(k+1) = \boldsymbol{f}\big(\boldsymbol{x}(k), \boldsymbol{u}(k)\big),\ \boldsymbol{x}(0) = \boldsymbol{x}$$

Optimal value function (OVF)

$$V(\boldsymbol{x}) = \inf_{\boldsymbol{u}(1),\boldsymbol{u}(2),\ldots} J\left(\boldsymbol{x}, (\boldsymbol{u}(k))_{k\in\mathbb{N}}\right)$$

Controller design: Bellman equation

$$V(\boldsymbol{x}) = \inf_{\boldsymbol{u}\in\mathcal{U}} \left\{c(\boldsymbol{x}, \boldsymbol{u}) + V\big(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\big)\right\}$$
$$\boldsymbol{F}(\boldsymbol{x}) = \arg\min_{\boldsymbol{u}\in\mathcal{U}} \left\{c(\boldsymbol{x}, \boldsymbol{u}) + V\big(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\big)\right\}$$

Discretization = dynamic game

Partition $\mathcal{P} = \{P_1, \ldots, P_p\}$, canonical projection $[\,\cdot\,]_{\mathcal{P}} : \mathcal{X} \to \mathcal{P}$

$$V_{\mathcal{P}}(P) = \inf_{\boldsymbol{u}\in\mathcal{U}} \sup_{\boldsymbol{y}\in P} \left\{c(\boldsymbol{y}, \boldsymbol{u}) + V_{\mathcal{P}}\left([\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u})]_{\mathcal{P}}\right)\right\}$$

- Used for robust controller design [1]
- Computable for medium space and control dimensions.

Weakly Coupled Systems

Non-coupled system

State space $\mathcal{X} = \bigotimes_{i=1}^n \mathcal{X}_i$, control set $\mathcal{U} = \bigotimes_{i=1}^n \mathcal{U}_i$

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{u}_1) \\ \vdots \\ \boldsymbol{f}_n(\boldsymbol{x}_n, \boldsymbol{u}_n) \end{bmatrix}, \qquad c(\boldsymbol{x}, \boldsymbol{u}) = \sum_{i=1}^n c_i(\boldsymbol{x}_i, \boldsymbol{u}_i)$$

OVF of the form

$$V(\boldsymbol{x}) = \sum_{i=1}^n V_i(\boldsymbol{x}_i)$$

Weakly coupled system

Costs still non-coupled, but

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{u}_1) \\ \vdots \\ \boldsymbol{f}_n(\boldsymbol{x}, \boldsymbol{u}_n) \end{bmatrix}, \quad \text{such that } \frac{\partial \boldsymbol{f}_i(\boldsymbol{x}, \boldsymbol{u}_i)}{\partial \boldsymbol{x}_j} \text{ small if } i \neq j$$

Is this regularity inherited by the OVF?

$$V(\boldsymbol{x}) \overset{?}{\approx} \sum_{i=1}^n V_i(\boldsymbol{x}_i)$$

Negative answer provided in [2].

Numerical Approach

Enforce regularity by taking suboptimality into account.

Split dynamic programming

$$V(\boldsymbol{x}) = \inf_{\boldsymbol{u}\in\mathcal{U}} \left\{c(\boldsymbol{x}, \boldsymbol{u}) + V(\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}))\right\}$$
$$= \inf_{\boldsymbol{u}_i\in\mathcal{U}_i} \left\{\hat{c}_i(\boldsymbol{x}, \boldsymbol{u}_i) + V(\hat{\boldsymbol{f}}_i(\boldsymbol{x}, \boldsymbol{u}_i))\right\}$$

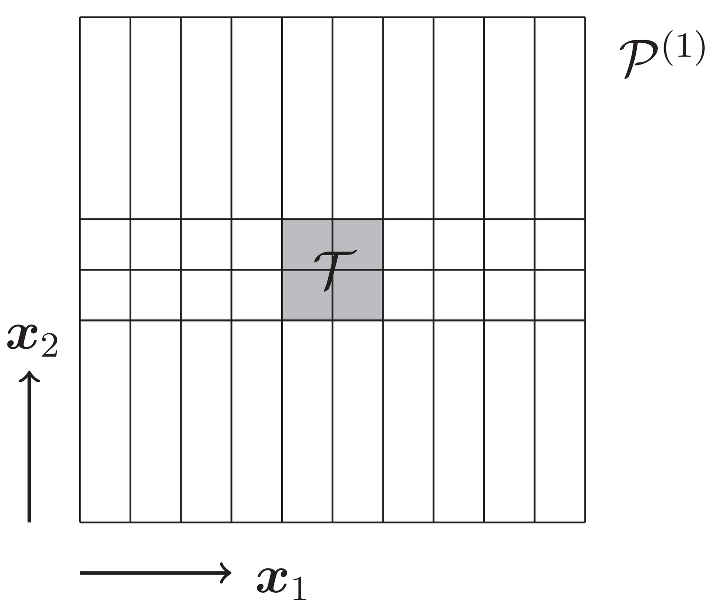
with
$\hat{c}_i(\boldsymbol{x}, \boldsymbol{u}_i) = c(\boldsymbol{x}, \tilde{\boldsymbol{u}})$ und $\hat{\boldsymbol{f}}_i(\boldsymbol{x}, \boldsymbol{u}_i) = \boldsymbol{f}(\boldsymbol{x}, \tilde{\boldsymbol{u}})$
where $\tilde{\boldsymbol{u}} = \big(\boldsymbol{F}_1(\boldsymbol{x}), \ldots, \boldsymbol{F}_{i-1}(\boldsymbol{x}), \boldsymbol{u}_i, \boldsymbol{F}_{i+1}(\boldsymbol{x}), \ldots, \boldsymbol{F}_n(\boldsymbol{x})\big)$
Optimal pair $(V, \boldsymbol{F})$ solves the coupled Bellman system

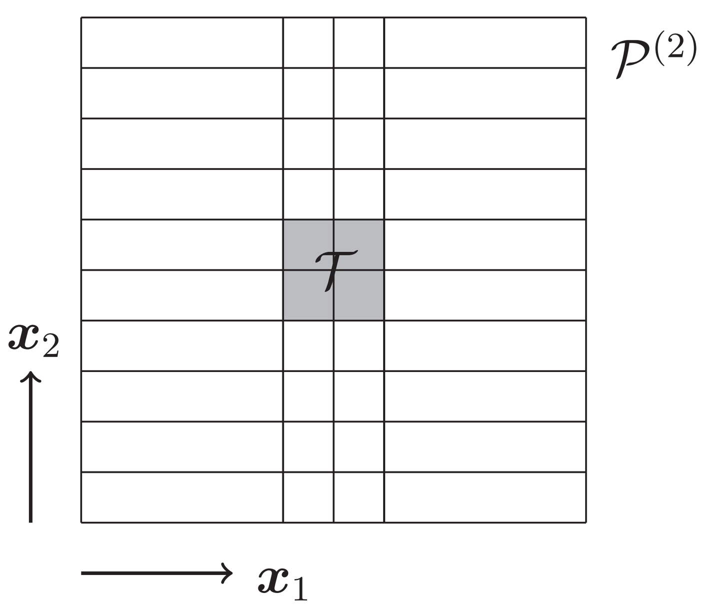
$$V^{(i)}(\boldsymbol{x}) = \inf_{\boldsymbol{u}_i\in\mathcal{U}_i} \left\{\hat{c}_i(\boldsymbol{x}, \boldsymbol{u}_i) + V^{(i)}\big(\hat{\boldsymbol{f}}_i(\boldsymbol{x}, \boldsymbol{u}_i)\big)\right\}$$
$$\boldsymbol{F}_i(\boldsymbol{x}) = \arg\inf_{\boldsymbol{u}_i\in\mathcal{U}_i} \left\{\hat{c}_i(\boldsymbol{x}, \boldsymbol{u}_i) + V^{(i)}\big(\hat{\boldsymbol{f}}_i(\boldsymbol{x}, \boldsymbol{u}_i)\big)\right\} \qquad i = 1, \ldots, n$$

Cyclical iterative solution.

Different partitions

Solve subproblems on different partitions





$$V_{\mathcal{P}^{(i)}}(P) = \inf_{\boldsymbol{u}_i\in\mathcal{U}_i} \sup_{\boldsymbol{y}\in P} \left\{\hat{c}_i(\boldsymbol{y}, \boldsymbol{u}_i) + V_{\mathcal{P}^{(i)}}\big([\hat{\boldsymbol{f}}_i(\boldsymbol{y}, \boldsymbol{u}_i)]_{\mathcal{P}^{(i)}}\big)\right\}$$
$$\boldsymbol{F}_{\mathcal{P}^{(i)}}(P) = \arg\inf_{\boldsymbol{u}_i\in\mathcal{U}_i} \sup_{\boldsymbol{y}\in P} \left\{\hat{c}_i(\boldsymbol{y}, \boldsymbol{u}_i) + V_{\mathcal{P}^{(i)}}\big([\hat{\boldsymbol{f}}_i(\boldsymbol{y}, \boldsymbol{u}_i)]_{\mathcal{P}^{(i)}}\big)\right\} \qquad (1)$$

Idea: Due to weak coupling, $\boldsymbol{F}_i$ does not need to vary strongly in $\boldsymbol{x}_j$ $(j \neq i)$ to achieve stability.

Main Results & Outlook

Theorem

a) Every solution of (1) satisfies $J(\boldsymbol{x}, \boldsymbol{F}(\cdot)) \leq \min_i V_{\mathcal{P}^{(i)}}(\boldsymbol{x})$
b) Global convergence for identical partitions, i.e. $\mathcal{P}^{(i)} = \mathcal{P}^{(j)}$ for all i, j
c) Quadratic convergence for LQR problems to the optimal solution (compare with [3])

Current and future work

- Communicating partitions
- Event-based design
- Extension to sparsely coupled systems

References

[1] L. Grüne and O. Junge. Global optimal control of perturbed systems. *J. Optim. Theory Appl.*, 136(3):411–429, 2008.
[2] P. Koltai and O. Junge. Optimal value functions for weakly coupled systems: a posteriori estimates. ZAMM 2013, DOI: 10.1002/zamm.201100138.
[3] M. L. Puterman and S. L. Brumelle. On the convergence of policy iteration in stationary dynamic programming. *Mathematics of Operations Research*, 4(1):60–69, 1979.

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