

# Existence of Minimizers for Fractional Variational Problems Containing Caputo Derivatives

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## Summary

We study dynamic minimization problems of the calculus of variations with Lagrangian functionals containing Riemann–Liouville fractional integrals, classical and Caputo fractional derivatives. Under assumptions of regularity, coercivity and convexity, we prove existence of solutions.

## Introduction

Fractional calculus of variations

- was first introduced in 1996, by Fred Riewe, because classical Lagrangian and Hamiltonian mechanics is unable to deal with nonconservative forces like friction;
- is a generalization of classical calculus of variations, considering derivatives (and integrals) of real or complex order.

Many results regarding to traditional variational problems were extended to the non-integer case e.g.,

- necessary optimality conditions;
- Noether's theorem.

However, if we ask about results addressed to the existence of solutions for problems of fractional calculus of variations, we will see that they are rare and are discussed only in

- M. Klimek, 'Existence–uniqueness result for a certain equation of motion in fractional mechanics', *Bulletin of the Polish Academy of Science*, vol. 58, 2010, no.4, 73–78.
- L. Bourdin, 'Existence of weak solutions for fractional Eler–Lagrange equations', *Journal of Mathematical Analysis and Applications*, vol. 399, 2013, no. 1, 239–251.

It is important to remark that existence theorems are essential ingredients of the deductive method for solving variational problems, which

- starts with the proof of existence;
- next one shows necessary conditions;
- finally candidates are examined to arrive at a solution.

These arguments make the question of existence an emergent topic, which requires serious attention and more interest.

## Basic Notions

Let  $a, b \in \mathbb{R}$  ( $a < b$ ),  $d \in \mathbb{N}^*$ ,  $1 \leq r \leq \infty$ . We will denote by:

- $\|\cdot\|$  the standard Euclidean norm of  $\mathbb{R}^d$ ;
- $L^r := L^r(a, b; \mathbb{R}^d)$  the usual space of  $r$ -Lebesgue integrable functions endowed with its usual norm  $\|\cdot\|_{L^r}$ .
- $W^{1,r} := W^{1,r}(a, b; \mathbb{R}^d)$  the usual  $r$ -Sobolev space endowed with its usual norm  $\|\cdot\|_{W^{1,r}}$ ;
- $\mathcal{C} := \mathcal{C}([a, b]; \mathbb{R}^d)$  the standard space of continuous functions;
- $\mathcal{C}_c^\infty := \mathcal{C}_c^\infty([a, b]; \mathbb{R}^d)$  the standard space of infinitely differentiable functions compactly supported in  $]a, b[$ .

Moreover, let us remind that the compact embedding  $W^{1,r} \hookrightarrow \mathcal{C}$  holds.

We define the left and the right Riemann–Liouville fractional integrals  $I_-^\alpha$  and  $I_+^\alpha$  of order  $\alpha \in \mathbb{R}$  ( $\alpha > 0$ ) by

$$I_-^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(y)d\tau}{(t-y)^{1-\alpha}}, \quad t > a$$

and

$$I_+^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(y)d\tau}{(y-t)^{1-\alpha}}, \quad t < b,$$

respectively. Here  $\Gamma(\alpha)$  denotes the Euler's Gamma function. Note that, operators  $I_-^\alpha$  and  $I_+^\alpha$  are well defined a.e. on  $(a, b)$  for  $f \in L^1$ .

Let  $0 < \alpha < 1$ . Then the left and the right Caputo fractional derivatives of order  $\alpha$  are given by

$$\forall t \in ]a, b[, \quad {}_cD_-^\alpha[f](t) := I_-^{1-\alpha}[\dot{f}](t) \quad \text{and} \quad \forall t \in ]a, b[, \quad {}_cD_+^\alpha[f](t) := -I_+^{1-\alpha}[\dot{f}](t),$$

respectively. Later in the text, we will make use of the following property yielding boundedness of Riemann–Liouville fractional integrals in the space  $L^p$ .

**Property 1.** *The left Riemann–Liouville fractional inegral  $I_-^\alpha$  with  $\alpha > 0$  is linear and bounded operator in  $L^r$ :*

$$\forall f \in L^r, \quad \|I_-^\alpha[f]\|_{L^r} \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \|f\|_{L^r}.$$

## Main Results

Let  $1 < p < \infty$ ,  $p'$  be the adjoint of  $p$  and let  $\alpha \in \mathbb{R}$  ( $0 < \alpha < 1$ ). We deal with variational functional

$$\mathcal{L} : E \longrightarrow \mathbb{R} \\ u \longmapsto \int_a^b L(u, I_-^\alpha[u], \dot{u}, {}_cD_+^\alpha[u], t) dt,$$

and ask about existence of minimizers for  $\mathcal{L}$ . We assume that

- $E$  is weakly closed subset of  $W^{1,p}$ ;
- $\dot{u}$  is the derivative of  $u$ ;
- $L$  is a Lagrangian of class  $\mathcal{C}^1$ :

$$L : (\mathbb{R}^d)^4 \times [a, b] \longrightarrow \mathbb{R} \\ (x_1, x_2, x_3, x_4, t) \longmapsto L(x_1, x_2, x_3, x_4, t).$$

Moreover, we will denote by  $\partial_i L$  partial derivatives of  $L$  with respect to its  $i$ -th argument,  $i = 1, 2, 3, 4$ .

Now, using general assumptions of regularity, coercivity and convexity, we prove fractional analog of Tonelli theorem, ensuring the existence of minimizer for  $\mathcal{L}$ .

**Definition 1.** *We will say that  $\mathcal{L}$  is **regular**, if for any  $u \in W^{1,p}$  it satisfies*

- $L(u, K[u], \dot{u}, K[\dot{u}], t) \in L^1$ ;
- $\partial_1 L(u, I_-^\alpha[u], \dot{u}, {}_cD_+^\alpha[u], t) \in L^1$ ;
- $\partial_2 L(u, I_-^\alpha[u], \dot{u}, {}_cD_+^\alpha[u], t) \in L^{p'}$ ;
- $\partial_3 L(u, I_-^\alpha[u], \dot{u}, {}_cD_+^\alpha[u], t) \in L^{p'}$ ;
- $\partial_4 L(u, I_-^\alpha[u], \dot{u}, {}_cD_+^\alpha[u], t) \in L^{p'}$ .

**Definition 2.** *We will say that  $\mathcal{L}$  is **coercive**, if it satisfies*

$$\lim_{\substack{u_{W^{1,p}} \rightarrow \infty \\ u \in E}} \mathcal{L}(u) = +\infty.$$

Next result gives Tonelli-type theorem for Lagrangian functionals containing Caputo fractional derivatives.

**Theorem 1 (TONELLI-TYPE THEOREM).** *Let us assume that:*

- $L$  is regular;
- $\mathcal{L}$  is coercive on  $E$ ;
- $L(\cdot, t)$  is convex on  $(\mathbb{R}^d)^4$  for any  $t \in [a, b]$ .

*Then, there exists a minimizer for  $\mathcal{L}$ .*

In order to make the hypotheses of Theorem 1 more concrete, we give more precise sufficient conditions on the Lagrangian  $L$ , that imply regularity and coercivity of functional  $\mathcal{L}$ . In this fashion, we define the family of sets denoted by  $\mathcal{P}_M$ , for any  $M \geq 1$ .

For  $M \geq 1$ , we define  $\mathcal{P}_M$  to be the set of maps  $P : (\mathbb{R}^d)^4 \times [a, b] \rightarrow \mathbb{R}^+$  such that for any  $(x_1, x_2, x_3, x_4, t) \in (\mathbb{R}^d)^4 \times [a, b]$ :

$$P(x_1, x_2, x_3, x_4, t) = \sum_{k=0}^N c_k(x_1, t) x_2^{d_{2,k}} x_3^{d_{3,k}} x_4^{d_{4,k}},$$

with  $N \in \mathbb{N}$  and where, for any  $k = 0, \dots, N$ ,  $c_k : \mathbb{R}^d \times [a, b] \rightarrow \mathbb{R}^+$  is continuous and satisfies  $d_{2,k} + d_{3,k} + d_{4,k} \leq (p/M)$ .

**Proposition 1.** *Suppose that there exist  $P_0 \in \mathcal{P}_1$ ,  $P_1 \in \mathcal{P}_1$ ,  $P_2 \in \mathcal{P}_{p'}$ ,  $P_3 \in \mathcal{P}_{p'}$  and  $P_4 \in \mathcal{P}_{p'}$  such that for any  $(x_1, x_2, x_3, x_4, t) \in (\mathbb{R}^d)^4 \times [a, b]$ :*

- $|L(x_1, x_2, x_3, x_4, t)| \leq P_0(x_1, x_2, x_3, x_4, t)$ ;
- $\partial_1 L(x_1, x_2, x_3, x_4, t) \leq P_1(x_1, x_2, x_3, x_4, t)$ ;
- $\partial_2 L(x_1, x_2, x_3, x_4, t) \leq P_2(x_1, x_2, x_3, x_4, t)$ ;
- $\partial_3 L(x_1, x_2, x_3, x_4, t) \leq P_3(x_1, x_2, x_3, x_4, t)$ ;
- $\partial_4 L(x_1, x_2, x_3, x_4, t) \leq P_4(x_1, x_2, x_3, x_4, t)$ .

*Then,  $L$  is regular.*

Namely, let us consider  $u_0 \in \mathbb{R}^d$  and  $E = W_a^{1,p}$ , where  $W_a^{1,p} := \{u \in W^{1,p}, u(a) = u_0\}$ . Note that,  $W_a^{1,p}$  is a weakly closed subset of  $W^{1,p}$ , because the compact embedding  $W^{1,p} \hookrightarrow \mathcal{C}$  holds.

Next proposition gives sufficient condition for coercive Lagrangian  $L$  implying its coercivity.

**Proposition 2.** *Assume that, for any  $(x_1, x_2, x_3, x_4, t) \in (\mathbb{R}^d)^4 \times [a, b]$ :*

$$L(x_1, x_2, x_3, x_4, t) \geq c_0 x_3^p + \sum_{k=1}^N c_k x_1^{d_{1,k}} x_2^{d_{2,k}} x_3^{d_{3,k}} x_4^{d_{4,k}},$$

*with  $c_0 > 0$  and  $N \in \mathbb{N}^*$  and where, for any  $k = 1, \dots, N$ ,  $c_k \in \mathbb{R}$  and*

$$0 \leq d_{1,k} + d_{2,k} + d_{3,k} + d_{4,k} < p.$$

*Then,  $\mathcal{L}$  is coercive on  $W_a^{1,p}$ .*

**Example 1.** *As an example, let us consider the following Lagrangian*

$$L(x_1, x_2, x_3, x_4, t) = \frac{1}{2} \sum_{i=1}^4 x_i^2.$$

*It is not difficult to verify, that  $L$  is convex and satisfies hypotheses of Propositions 1 and 2 with  $p = 2$ . Therefore, there exists a minimizer of  $\mathcal{L}$  defined on  $W^{1,2}$ .*