Some remarks on the combinatorial properties of the explicit MPC

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Preliminaries

Let us consider the MPC formulation:

$$\min_{u_0,\dots,u_{N-1}} \sum_{k=0}^{N-1}\left(u_k^T R u_k + x_k^T Q x_k\right) + x_N^T Q_f x_N$$
$$\text{s.t.}\quad G_k x_k + H_k u_k \le b_k,$$
$$x_{k+1} = A x_k + B u_k,\ k = 0\dots N-1$$

where N is the prediction horizon length, Q, R, P are the cost matrices and there are q constraints.

Written in compact form, it is

$$\begin{array}{ll}\min\limits_{u} & \frac{1}{2}u^T H u + x_0^T F u \\ \text{s.t.} & Gu \le W + Ex_0\end{array} \xrightarrow{\text{KKT form}} \begin{array}{l} Hu + F^T x_0 + G^T\lambda = 0 \\ \lambda \ge 0 \\ Gu - W - Ex_0 \le 0 \\ \lambda \times (Gu - W - Ex_0) = 0\end{array}$$

Use the structure of matrix G to

- obtain a better upper bound for the number of sets of active constraints
- provide a recursive partial description of the explicit MPC

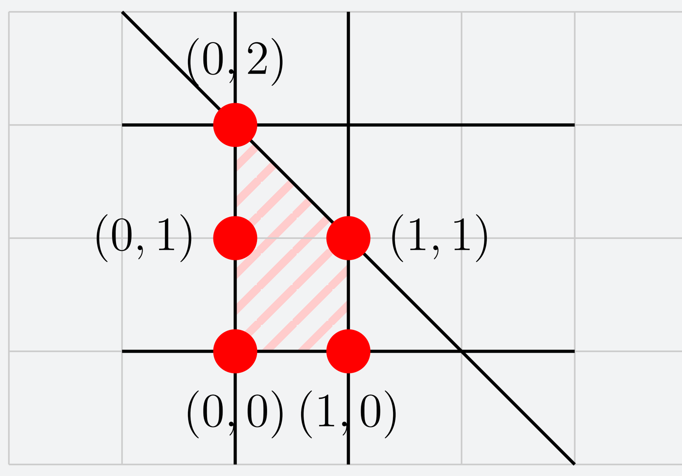
Illustrative example

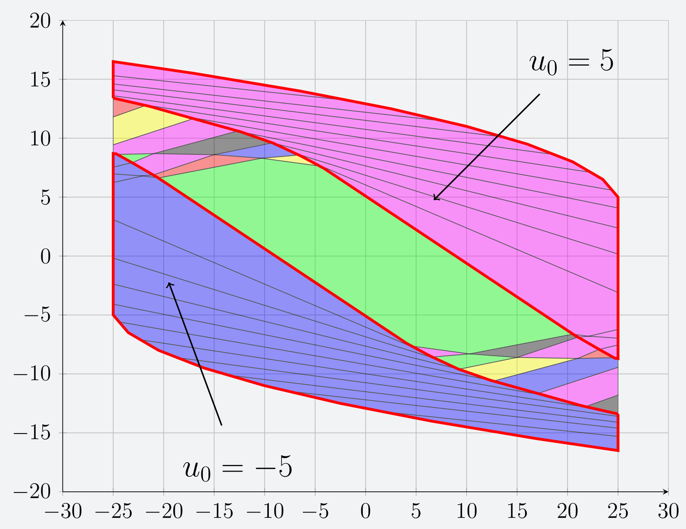
Consider the LTI dynamics

$$x^+ = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}x + \begin{bmatrix}1\\ 0.3\end{bmatrix}u$$

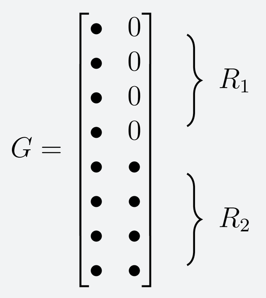
with constraints

$$-5 \le u_k \le 5,\ \begin{bmatrix}-25\\ -25\end{bmatrix} \le x_k \le \begin{bmatrix}25\\ 25\end{bmatrix}$$





For $N = 2$, $m = 1$ and $q_1 = q_2 = 4$ we have the selection conditions:



$$\longrightarrow\quad \begin{array}{l} 0 \le i_1 \le 1 \\ 0 \le i_2 \le 2 \\ 0 \le i_1 + i_2 \le 2\end{array}$$

- the total number of candidate active sets is $\#\mathcal{I}_N^\circ = \binom{4}{0}\cdot\binom{4}{0} + \dots + \binom{4}{0}\cdot\binom{4}{2} = 31 < \sum_{j=0}^{2}\binom{8}{j} = 37 < 2^8 = 64$
- for $N = 12$ we have 51 regions, 32 of them can be obtained from the $N = 11$ order problem.

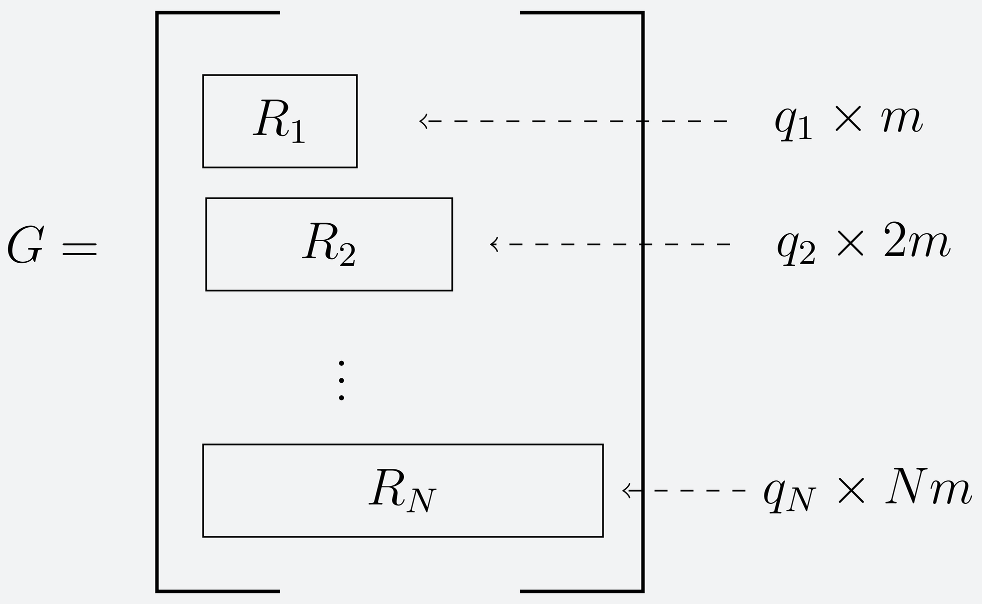
Main idea

Assuming q constraints (the rows in matrix G), we have $\#\mathcal{I}_N^\circ = 2^q$ possible combinations of active constraints.

Definition. *[4]. For an active set of constraints, the* linear independence constraint qualification (LICQ) *holds if the set of active constraint gradients are linearly independent.*

Using the LICQ condition we refine the bound to $\#\mathcal{I}_N^\circ = \sum_{j=0}^{N\cdot m}\binom{q}{j}$.

Up to instant k only the first $\{u_1, \dots, u_k\}$ inputs appear in the constraint description → matrix G is lower-block triangular [5]:



We can use the structure of matrix G to note that there are:

- at most $\min(q_k, k\cdot m)$ independent rows in block R_k
- at most $k \cdot m$ independent rows in the first $R_1, \dots, R_k$ blocks

where each block R_k describes the k^{th} order constraints, has q_k rows ($q = q_1 + \dots q_N$) and $k\cdot m$ columns.

Let i_k denote the number of constraints selected from each block R_k. Then the conditions

$$0 \le i_k \le \min(q_k, k\cdot m),\ \sum_{j=0}^{k} i_j \le k\cdot m,\ \forall k = 1,\dots,N$$

define all the selections of constraints which can be LICQ.

The number of sets of active constraints is bounded by

$$\#\mathcal{I}_N^\circ = \sum_{(i_1,\dots,i_N)\text{ verifies selection}}\left(\prod_{j=1}^{N}\binom{q_j}{i_j}\right)$$

and in the symmetric case $R_k = \begin{bmatrix}\bar{R}_k\\ -\bar{R}_k\end{bmatrix}$, by

$$\#\mathcal{I}_N^\circ = \sum_{(i_1,\dots,i_N)\text{ verifies selection}}\left(\prod_{j=1}^{N}\binom{\frac{q_j}{2}}{i_j}\cdot 2^{i_j}\right).$$

Assume that $\tau\cdot m$ constraints from $R_1, \dots, R_\tau$ blocks are active, then

$$u_{[0,\tau-1]} = \Psi x_0 + \phi$$

are uniquely determined and only the remaining $u_{[\tau,N-1]}$ remain to be determined.

If the $(N-\tau)$-order explicit MPC is available, we determine a subset of the N-order explicit MPC:

$$x_0 \Rightarrow u^*_{[0,N-\tau-1]}(x_0) \longrightarrow x_\tau \Rightarrow u^*_{[\tau,N-1]}(x_\tau)$$

$$x_\tau(x_0, u_{[0,\tau-1]}) = \Psi' x_0 + \Phi'$$

Conclusions

Exploiting the structure of the MPC problem can prove useful:

- the LICQ conditions offer bounds on the number of sets of active constraints
- the search in the state space can be reduced by using information from lower order explicit solutions (useful, e.g., in bi-level optimization [1])

There are interesting issues on the numerical complexity of the problem:

- the selection conditions define an integral polytope whose interior integer points can be counted efficiently [2]
- critical regions with the same initial u_0 can be merged for a compact explicit representation [3]

References

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