A SPECTRAL CONDITION FOR ASYMPTOTIC CONTROLLABILITY AND STABILIZATION AT SINGULAR POINTS

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Abstract

In this paper we present a spectral condition for the exponential stabilization of nonlinear control systems with constrained control range at singular points. The spectral approach in particular allows to formulate an equivalence result between exponential null controllability and exponential stabilization by means of a discrete feedback law. The key tool used is a discounted optimal control problem for the corresponding projected semilinear system, which also admits a numerical solution.

Keywords: nonlinear systems, singular points, stabilization, Lyapunov spectrum

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1 Introduction

In this paper we will present a spectral condition for the exponential stabilization of nonlinear control systems with constrained control range at singular points, i.e. systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), y(t), u(t)) \\ \dot{y}(t) &= g(y(t), u(t)) \end{aligned} \tag{1.1}$$

on $\mathbb{R}^d \times M$ where $x \in \mathbb{R}^d$ and $y \in M$, M some Riemannian Manifold and f and g are vectorfields which are C^2 in x, Lipschitz in y and continuous in u. The control function $u(\cdot)$ may be chosen from the set $\mathcal{U} := \{u : \mathbb{R} \to U \mid u(\cdot) \text{ measurable}\}$ where $U \subset \mathbb{R}^m$ is compact, i.e. we have a constrained set of control values.

For each pair (x_0, y_0) of initial values the trajectories of (1.1) will be denoted by the pair $(x(t, x_0, y_0, u(\cdot)),\ y(t, y_0, u(\cdot)))$ and we assume them to exist uniquely for all times.

Our interest lies on the stabilization of the x-component at a *singular point* x^*, i.e. a point where $f(x^*, y, u) = 0$ for all $(y, u) \in M \times U$. For simplicity we will frequently assume $x^* = 0$. Such singular situations do typically occur if the control enters in the parameters of an uncontrolled systems at a fixed point, for instance when the restoring force of a nonlinear oscillator is controlled.

Note that our general setup covers several models: The additional equation for y allows us to model systems where time varying parametric excitations governed by an additional (nonlinear) control or dynamical system enter the system to be stabilized. The case in which the control u does not enter explicitly in the function f and the case in which f does not depend on y occur as special situations in this setup, hence they are also covered.

The main tool used for the stabilization is the linearization of (1.1) at the singular point which is given by

$$\begin{aligned} \dot{z}(t) &= A(y(t), u(t))z(t) \\ \dot{y}(t) &= g(y(t), u(t)) \end{aligned} \tag{1.2}$$

Here $A(y, u) := \frac{d}{dx} f(x^*, y, u) \in \mathbb{R}^{d \times d}$ and $f(x, y, u) = A(y, u)x + \tilde{f}(x, y, u)$ where the estimate $\|\tilde{f}(x, y, u)\| \le C_f \|x\|^2$ for some constant C_f holds in a neighborhood of x^*.

As above we denote the trajectories of (1.2) by $(z(t, z_0, y_0, u(\cdot)),\ y(t, y_0, u(\cdot)))$ for the pair of initial values (z_0, y_0).

The approach we follow is based on optimal control techniques. More precisely, we consider the Lyapunov exponents of the linearization and formulate a discounted optimal control problem in order to minimize these exponents, an idea that has first been presented in [6]. Lyapunov exponents have recently turned out to be a suitable tool for the stability analysis of semilinear systems, see e.g. [3] and [4], and also for their stabilization [5]. However, due to the fact that for discounted optimal control problems optimal feedback laws are in general not available, we modify the feedback concept and introduce *discrete feedback laws* that are based on a discrete time sampled approximation of the given continuous time system. Using this approach it could be shown in [5] that for semilinear systems satisfying an accessibility condition exponential null controllability is equivalent to exponential stabilizability by discrete feedbacks. Here we will present an extension of this result to nonlinear systems of the type (1.1) without assuming any accessibility.

We start by characterizing the null controllability and stability of (1.2).

2 The semilinear system

In our definitions of null controllability we need the notion of an invariant set for the subsystem on M.

Definition 2.1 A subset $K \subseteq M$ is called invariant for the subsystem of (1.2) on M if for all $y_0 \in K$ and all control functions $u(\cdot) \in \mathcal{U}$ the corresponding trajectory satisfies $y(t, y_0, u(\cdot)) \in K$ for all $t > 0$.

In order to measure the exponential null controllability of (1.2) we define the Lyapunov exponent of a trajectory of (1.2) by

$$\lambda(z_0, y_0, u(\cdot)) := \limsup_{t\to\infty} \frac{1}{t} \ln ||z(t, z_0, y_0, u(\cdot)||$$

Clearly $\lambda(z_0, y_0, u(\cdot)) < 0$ iff the corresponding trajectory converges to the origin exponentially fast. For each pair of initial values we define the infimal Lyapunov exponent by

$$\lambda^*(z_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}} \lambda(z_0, y_0, u(\cdot))$$

and its supremum over $\mathbb{R}^d \setminus \{0\} \times K$ by

$$\tilde{\kappa} := \sup_{(z_0,y_0)\in\mathbb{R}^d\setminus\{0\}\times K} \lambda^*(z_0, y_0)$$

Now we define the concept of a discrete feedback control for (1.2).

Definition 2.2 A *discrete Feedback law* for system (1.2) is given by a mapping $F : \mathbb{R}^d \times K \to U$ and a time step $h > 0$ and is applied to (1.2) by

$$\begin{aligned} \dot{z}(t) &= A(y(t), F(s(\left[\tfrac{t}{h}\right] h), y(\left[\tfrac{t}{h}\right] h)))z(t) \\ \dot{y}(t) &= g(y(t), F(z(\left[\tfrac{t}{h}\right] h), y(\left[\tfrac{t}{h}\right] h))) \end{aligned} \quad (2.1)$$

One of the main advantages of this concept lies the fact that existence and uniqueness of the trajectories of (2.1) are guaranteed even if F is discontinuous. Note that an optimal control approach — which we will apply in the next section — typically results in discontinuous control laws.

Using this definition we can state the main result for semilinear systems.

Theorem 2.3 Let $K \subseteq M$ be a compact invariant set for the subsystem of (1.2) on M. Then for the system (1.2) are equivalent:

(i) $\tilde{\kappa} < 0$

(ii) System (1.2) is asymptotically null controllable over K

(iii) System (1.2) is uniformly exponentially null controllable over K i.e. there exist constants C, $\alpha > 0$, such that for any pair of initial values $(z_0, y_0) \in \mathbb{R}^d \times K$ there exists a control function $u_{(z_0,y_0)}(\cdot) \in \mathcal{U}$ with
$$||z(t, z_0, y_0, u_{(z_0,y_0)}(\cdot))|| \leq Ce^{-\alpha t}||z_0||$$

(iv) There exists a time step $h > 0$ and a discrete Feedback $F : \mathbb{R}^d \times K \to U$ such that (2.1) is uniformly exponentially stable

Hence $\tilde{\kappa}$ is the characteristic value in the Lyapunov spectrum for both null controllability and stabilizability of semilinear systems. A proof of this theorem can be found in [8].

3 Construction of the discrete feedback

The construction of the feedback is obtained by solving an optimal control problem which will minimize the Lyapunov exponent.

From the linearity of (1.2) it follows that $\lambda(z_0, y_0, u(\cdot)) = \lambda(\alpha z_0, y_0, u(\cdot))$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. Hence we can use the projection of the z component to the unit sphere $\mathbb{S}^{d-1}$. Denoting the projected trajectory by $s(t, s_0, y_0, u(\cdot))$ it follows from the chain rule that for $s_0 = \frac{z_0}{||z_0||}$ the Lyapunov exponent can be written as

$$\begin{aligned} &\lambda(s_0, y_0, u(\cdot)) = \\ &\quad \limsup_{t\to\infty} \tfrac{1}{t} \int_0^t q(s(\tau, s_0, y_0, u(\cdot)), \\ &\qquad\qquad y(\tau, y_0, u(\cdot)), u(\tau))d\tau \end{aligned} \quad (3.1)$$

where $q(s, y, u) := s^T A(y, u)s$. This integral is also referred to as an *averaged functional.*

Minimizing (3.1) forms an average time optimal control problem, for which the construction of optimal feedback controls is still an unsolved problem.

Hence we use an approximation of this integral by a discounted optimal control problem with small *discount rate* $\delta > 0$ defined by

$$J_\delta(s_0, y_0, u(\cdot)) := \int_0^\infty e^{-\delta\tau} q(s(\tau, s_0, y_0, u(\cdot)), y(\tau, y_0, u(\cdot)), u(\tau))d\tau \quad (3.2)$$

The function

$$v_\delta(s_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}} J_\delta(s_0, y_0, u(\cdot)) \quad (3.3)$$

is called the *optimal value function* of this discounted optimal control problem.

For $\delta \to 0$ the convergence

$$\sup_{(s_0,y_0)\in\mathbb{S}^{d-1}\times K} \delta v_\delta(s_0, y_0) \to \tilde{\kappa}$$

holds. Furthermore if a control function $u(\cdot) \in \mathcal{U}$ satisfies

$$\delta J_\delta(s(t, s_0, y_0, u(\cdot)), y(t, y_0, u(\cdot)), u(t + \cdot)) \le \sigma$$

for all $t \ge 0$ and some $\sigma \in \mathbb{R}$, $\delta > 0$ the estimate

$$\lambda(s_0, y_0, u(\cdot)) \le \sigma$$

is implied (see [6] or [8] for these results). Hence by constructing an (approximately) optimal control strategy for this discounted problem we do also (approximately) minimize the Lyapunov exponent of (1.2).

In order to do this we approximate $\mathcal{U}$ by

$$\mathcal{U}_h := \{u : \mathbb{R} \to U \mid u|_{[ih,(i+1)h)} \equiv u_i \text{ for all } i \in \mathbb{Z}\}$$

for some time step $h > 0$. This construction is often referred to as the process of sampling, see [10, Section 2.10]. By defining

$$v_\delta^h(s_0, y_0) := \inf_{u(\cdot)\in\mathcal{U}_h} J_\delta(s_0, y_0, u(\cdot))$$

for which the approximation property

$$\|v_\delta - v_\delta^h\|_\infty \le Ch^{\frac{\gamma}{2}}$$

holds for $\gamma = \delta/L$ where L denotes the Lipschitz constant of the projected system, see [1].

In fact, what we obtain by this approximation can be interpreted as a discrete time system

$$s_{i+1} = s(h, s_i, y_i, u_i), \quad y_{i+1} = y(h, y_i, u_i) \quad (3.4)$$

which is also called a sampled system.

By the continuity of all functions involved and the compactness of U we can now define a function $F : \mathbb{S}^{d-1} \times K \to U$ by choosing $F_\mathbb{S}(s_0, y_0) := u \in U$ such that

$$\Big\{\int_0^h e^{-\delta\tau} q(s(\tau, s_0, y_0, u), y(\tau, y_0, u), u)d\tau + e^{-\delta h} v_\delta^h(s(h, s_0, y_0, u), y(h, y_0, u))\Big\}$$

becomes minimal.

The mapping F does now form a feedback law for the discrete time system (3.4) by

$$s_{i+1} = s(h, s_i, y_i, F(s_i, x_i)), \quad y_{i+1} = y(h, y_i, F(s_i, x_i)) \quad (3.5)$$

In fact this yields an optimal feedback law for (3.4) with respect to the control functions from $\mathcal{U}_h$.

By defining $F(z, y) := F_\mathbb{S}(z/\|z\|, y)$ we obtain the desired stabilizing feedback law. This follows from the fact that the application of F to (1.2) according to Definition 2.2 projected to $\mathbb{S}^{d-1}$ yields exactly the same trajectories as (3.5). Hence their Lyapunov exponent is negative, provided $\tilde{\kappa} < 0$ and $\delta > 0$ and $h > 0$ are sufficiently small.

The fact that F is indeed a feedback for the discrete time system (3.4) motivates the name *discrete* feedback. In the literature also the notions *modified* Feedback (see e.g. [9]) or *sampled* Feedback (cp. [2]) are used.

Note that this optimal discrete feedback can be approximated numerically as described in [5] and [7].

4 The nonlinear system

We will now return to our original system (1.1). We apply the discrete feedback as constructed in the last section to (1.1) via

$$\begin{aligned} \dot{x}(t) &= f(x(t), y(t), F(x([\tfrac{t}{h}]\, h), y([\tfrac{t}{h}]\, h))) \\ \dot{y}(t) &= g(y(t), F(x([\tfrac{t}{h}]\, h), y([\tfrac{t}{h}]\, h))) \end{aligned} \quad (4.1)$$

The robustness result [8, Lemma 6.3] states that in a sufficiently small neighborhood of x^* this optimal discrete feedback still yields approximately optimal trajectories on finite time intervals. More precisely if we denote the trajectories of (4.1) by the pair $(x_F(t, x_0, y_0), y_F(t, y_0))$ then for any $\varepsilon > 0$ there exists a neighborhood $U(x^*)$ and a discrete feedback F such that the estimate

$$\frac{1}{T} \ln \|x_F(t, x_0, y_0)\| \le \kappa + \varepsilon$$

holds for all $x_0 \in U(x^*)$ and some bounded $T \in [c(\varepsilon), C(\varepsilon)]$. What is remarkable about this result is the fact that F may be discontinuous, hence one may not argue using continuous dependency on the initial value. Instead, the main ingredients for proving this inequality are the Hoelder continuity of v_δ^h and the relation between the discounted and average time functionals.

Note that this inequality — obtained by estimating the effect of the nonlinear perturbation on the discounted functional — does only hold on finite time intervals. Hence we have to use induction in order to conclude that these trajectories do indeed converge to x^* exponentially fast, even uniformly, if $\kappa + \varepsilon < 0$. This yields the following main theorem for the nonlinear case for which the detailled proof is given in [8].

Note that for the nonlinear systems the different asymptotic controllability concepts from Theorem 2.3 are not equivalent. In particular it is not sufficient that (1.1) is (locally) exponentially controllable to x^*; in order to formulate the desired result we *need* the notion of *uniform* exponential controllability.

Theorem 4.1 Consider system (1.1). Let $K \subseteq M$ be a compact invariant set for the subsystem of $\overline{(1.1)}$ on M. Then the following properties are equivalent:

(i) $\tilde{\kappa} < 0$

(ii) (1.1) is (locally) uniformly exponentially controllable to x^* over K

(iii) (1.2) is asymptotically null controllable over K

(iv) There is $h > 0$ and a discrete feedback F such that (4.1) is (locally) uniformly exponentially stable at x^* over K

This theorem shows in particular that any attempt to stabilize (1.1) at a singular point by using its linearization must fail if uniform exponential controllability is not satisfied, because the linearized system will not even be asymptotically null controllable. Conversely, exponential discrete feedback stabilization is always possible under this condition. We have therefore obtained the strongest result possible within the linearization approach.

Again $\tilde{\kappa}$ forms the characteristic value for this property, hence uniform exponential stabilizability is determined by this value of the Lyapunov spectrum.

It may be worth noting that a related result has been developed in [2] using Lyapunov functions: There it is shown that for nonlinear systems asymptotic controllability to a (not necessarily singular) point x implies stabilizability by means of a discrete feedback, where in order to reach x the step size h must tend to 0. The result can therefore be interpreted as a kind of practical stabilization. In contrast to this practical stability here we obtain *exponential* stability using a discrete feedback with a *fixed* step size.

5 Conclusions

In this paper we presented results on the relation between null controllability and exponential stabilization by using a discrete feedback law for nonlinear systems at singular points using the characteristic value $\tilde{\kappa}$ of the Lyapunov spectrum.

The construction of the feedback is obtained by minimizing the Lyapunov exponent of the linearized system, which forms a semilinear system. For semilinear systems both asymptotic null controllability and exponential stabilizability by a discrete feedback turned out to be equivalent to the negativity of $\tilde{\kappa}$.

For general nonlinear systems the equivalence between uniform exponential controllability and uniform exponential stabilizability has been shown. Again the spectral condition $\tilde{\kappa} < 0$ is necessary and sufficient for these properties.

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