

Stabilization of discrete-time bilinear systems

The problem of feedback stabilizing a semi-linear discrete-time system is studied. Under an accessibility condition on an associated nonlinear system on the projective space it can be shown that null controllability is equivalent to feedback stabilizability. We present a way in which a stabilizing feedback may be computed using some ideas from discounted optimal control.

1. Introduction

We consider systems on \mathbb{R}^d of the form

$$x(t+1) = A(u(t))x(t) := \left(A_0 + \sum_{i=1}^m u_i(t)A_i \right) x(t), \quad t \in \mathbb{N}, \quad (1)$$

where $A_0, \dots, A_m \in \mathbb{R}^{d \times d}$ do not span a subspace of non-invertible matrices, and the set of admissible control values $U = \text{cl int } U \subset \mathbb{R}^m$ is compact, with connected interior and satisfies $0 \in \text{int } U$. The associated system on the projective space \mathbb{P}^{d-1} is given by

$$\xi(t+1) = \text{IP}A(u(t))\xi(t), \quad t \in \mathbb{N}. \quad (2)$$

Here only those control sequences $u \in U^{\mathbb{N}}$ are admissible for which the corresponding solution $\Phi_u(t, 0)x_0 \neq 0$ for all $t \in \mathbb{N}$, where x_0 is such that it spans ξ_0 . These control values or sequences will be denoted by $U(\xi), U^{\mathbb{N}}(\xi)$. The solution of (2) corresponding to an initial value ξ and a control sequence u is denoted by $\varphi(\cdot, \xi, u)$. Let us note that the setup is a particular case of the systems studied in [3], [4]. Proofs for the statements of the theorems below can be found in [3].

We call system (1) asymptotically null-controllable if for every $x \in \mathbb{R}^d$ there exists a control sequence $u \in U^{\mathbb{N}}$ such that $\lim_{t \rightarrow \infty} \Phi_u(t, 0)x = 0$. System (1) is called (state) feedback stabilizable if there exists a map $F : \mathbb{R}^d \rightarrow U$ such that the system

$$x(t+1) = A(F(x(t)))x(t), \quad t \in \mathbb{N}, \quad (3)$$

is globally asymptotically stable. If F can be chosen such that (3) is exponentially stable, then we call (1) exponentially (state) feedback stabilizable and F is called exponentially stabilizing.

Recall that a nonlinear system is called forward accessible, if for each point the interior of the forward orbit is nonempty. The following theorem states the main result on exponential stabilizability of (1).

Theorem 1. *If (2) is forward accessible, then the following statements are equivalent.*

- (i) *System (1) is asymptotically null controllable.*
- (ii) *System (1) is feedback stabilizable with a piecewise constant feedback F .*
- (iii) *System (1) is exponentially feedback stabilizable with a piecewise constant feedback F .*

2. Construction of the feedback

In this section we will give a constructive approach for the calculation of the exponentially stabilizing feedback for system (1). For this purpose we will base our construction on a dynamic programming technique using the optimal value function of a discounted optimal control problem. Note that this procedure yields an existence result for stabilizing feedbacks and in addition makes the problem numerically feasible, cp. Remark 3.

From the application point of view one reason why feedback stabilization is preferred to open loop asymptotic null controllability lies in the fact that one expects robustness of the closed loop system against small perturbations.

Since in this paper we follow an optimal control approach the resulting feedback will in general be discontinuous, and the desired robustness property cannot be obtained as a simple conclusion from the continuous dependency on the initial value. However, the continuity of the associated value function may be used in order to obtain this property.

The construction of the feedback is related to the following optimal control problem: Define the function $q : \mathbb{P}^{d-1} \times U \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$q(\xi, u) := \begin{cases} \log \frac{\|A(u)x\|}{\|x\|}, & u \in U(\xi) \\ -\infty, & \text{else} \end{cases} \quad (4)$$

and for $\delta > 0$ the δ -discounted cost $J_\delta : \mathbb{P}^{d-1} \times U^{\mathbb{N}} \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by

$$J_\delta(\xi, u) := \begin{cases} \delta \sum_{s=0}^{t-1} e^{-\delta s} q(\varphi(s; \xi, u), u(s)), & u \in U^{\mathbb{N}}(\xi), \\ -\infty & \text{else.} \end{cases} \quad (5)$$

The associated value function on projective space is given by $v_\delta(\xi) := \inf_{u \in U^{\mathbb{N}}} J_\delta(\xi, u)$. For small δ the function v_δ may be interpreted as an approximation of the smallest exponential growth rate that may be attained via an arbitrary control sequence from ξ . Similar to [1] — where continuous time systems are discussed — our construction of the feedback is now based on this approximating property. Define $F_\delta : \mathbb{P}^{d-1} \rightarrow U$ by the following procedure: For any point $\xi \in \mathbb{P}^{d-1}$ such that $v_\delta(\xi) > -\infty$ choose a value $u \in U$ such that

$$q(\xi, u) + e^{-\delta} v_\delta(\varphi(1; \xi, u))$$

becomes minimal. To discuss the case $v_\delta(\xi) = -\infty$ note that this is only possible if $U \neq U_{inv} := \{u \in U \mid \det(A(u)) \neq 0\}$. By assumption we may choose an increasing sequence of compact subsets $U_n \subset U_{inv}$ with nonvoid interior, such that $U = \text{cl} \bigcup_{n=1}^{\infty} U_n$ and such that system (2) with control range restricted to U_n is forward accessible. Denote by $q_n, J_{\delta,n}, v_{\delta,n}$ the function defined by (4), (5) with control range restricted to U_n . If $v_\delta(\xi) = -\infty$ choose a sequence $\{u_n\}$ such that for each $n \in \mathbb{N}$

$$q_n(\xi, u_n) + e^{-\delta} v_{\delta,n}(\varphi(1; \xi, u_n))$$

becomes minimal and choose u to be a limit point of the sequence $\{u_n\}$.

Define a feedback $F_\delta(\xi) := u$ with the choice of u as described above. It may be shown that this feedback law is indeed an optimal control strategy for v_δ . Furthermore it holds that

Theorem 2. *The following statements are equivalent:*

- (i) *System (1) is feedback stabilizable with a piecewise constant feedback F .*
- (ii) *There exists a $\bar{\delta} > 0$ such that for all $0 < \delta < \bar{\delta}$ the feedback F_δ is exponentially stabilizing.*

Remark 3. *By a standard dynamic programming argument the optimal value function v_δ can be characterized as the solution of a discrete Hamilton-Jacobi-Bellman equation. Using the results from [2] the feedback F can be calculated numerically by means of a numerical approximation of v_δ . For this numerical treatment it is advantageous to use one of the approximations U_n that were described above, in order to avoid possible problems caused by singularity.*

3. References

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