Maximal solutions for a class of singular Hamilton-Jacobi equations

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Abstract

In this paper we present results about the solutions of a class of singular Hamilton-Jacobi equations. Since these equations in general do not have a unique solution we define the notion of maximal solutions for which a stability result can be proved. Furthermore we present a discretization scheme for their numerical computation and give an estimate for the discretization error.

1 Introduction

We present some results recently obtained in [1], [2] concerning the the study of Hamilton-Jacobi equations of the following type

$$H(x, Du) = f(x) \qquad x \in \Omega, \tag{1}$$

coupled with a pointwise Dirichlet boundary condition

$$u(x) = g(x) \qquad x \in \partial\Omega, \tag{2}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is continuous in both variables and verifies

$$H(x,0) = 0, \qquad H(x,p) > 0 \quad \text{if } |p| \neq 0, \tag{3}$$

$$\lim_{|p|\to+\infty} H(x,p) = +\infty \quad \text{uniformly for } x \in \overline{\Omega} \tag{4}$$

$$\begin{array}{c} t \to H(x,tp) \text{ is strictly increasing in } [0,1] \\ \text{for any } (x,p) \in \overline{\Omega} \times \mathbb{R}^N. \end{array} \tag{5}$$

and the function f is nonnegative and continuous in $\overline{\Omega}$.

A significant example of equations of the previous type is given by the Eikonal equation

$$|Du| = f(x) \qquad x \in \Omega, \tag{6}$$

(this equation arises in the study of the Shape-from-Shading problem in image analysis, in control theory, in geometric optics, etc.).

It is well known (see [4], [5]) that if $K = \{x \in \overline{\Omega} : f(x) = 0\} \subset \Omega$ is not empty then the uniqueness of the viscosity solution for problem (1)–(2) fails to hold. Roughly speaking, this is due to the fact that the controlled dynamical system of which equation (1) represents the dynamic programming equation in a neighborhood of K can move with almost infinite speed paying a neglectible cost.

Among the infinitely many viscosity solutions of the problem, the maximal one turns out to be the value function of the exit-time control problem associated to (1)–(2) and therefore it is interesting to find a way to characterize this solution. Without any assumption on K, it was shown in [2] that the maximal solution of (1)–(2) can be characterized as the unique viscosity subsolution of problem (1)-(2) which is also a supersolution, in an appropriate sense, of an associated singular equation. Moreover, the notion of singular solution is stable for perturbation of the equation. Finally, in [1] it is proved that an approximation scheme defined via discretization of a "regular" equation obtained by adding a positive constant to f converges to the maximal solution of (1)-(2).

2 Maximal solutions and exit time control problems

In general, in order to associate a control problem to equation (1), the hamiltonian H is assumed to be convex in p. As pointed out in [2], the previous assumption can be replaced in the present case by the weaker assumption

$$\begin{array}{l} \mathcal{Z}(x) = \{p \in \mathbb{R}^N : H(x,p) \le f(x)\} \\ \text{is convex for any } x \in \overline{\Omega}. \end{array} \tag{7}$$

We introduce the gauge function ρ and the support function δ of the set $\mathcal{Z}(x)$, namely

$$\begin{aligned} \rho(x,p) &= \inf\{\lambda > 0 : \frac{p}{\lambda} \in \mathcal{Z}(x)\} \\ &= \{\lambda > 0 : H\left(x, \frac{p}{\lambda}\right) = f(x)\} \\ \delta(x,p) &= \max\{pq : \ p \in \mathcal{Z}(x)\}, \end{aligned}$$

for $(x,p) \in \Omega \times \mathbb{R}^N$ (both these functions are convex and homogeneous in the variable p) and we consider the

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equation

$$\rho(x, Du) = 1 \qquad x \in \Omega. \tag{8}$$

Observe that this equation is singular on the set K, since, if $x \in K$, then $\mathcal{Z}(x) = \{0\}$ and therefore $\rho(x, 0) = 0$ and $\rho(x, p) = +\infty$ if $|p| \neq 0$. We define the following nonsymmetric semidistance on $\overline{\Omega} \times \overline{\Omega}$

$$L(x,y) = \inf \left\{ \int_0^T \delta(\xi(t), -\dot{\xi}(t))dt \;\middle|\; \begin{array}{l} T > 0,\ \xi(t) \in \\ W^{1,\infty}([0,T], \overline{\Omega}) \\ \text{s.t. } \xi(0) = x, \\ \xi(T) = y \end{array} \right\}$$

It can be shown that L induces a topology τ_L on $\overline{\Omega}$ This topology, in some sense, retains the information on the structure of the set K.

The notion of singular supersolutions which allows us to characterize the maximal subsolution of (1)–(2) is based on two points. The first point is to adapt the test functions used in the definition of viscosity solution to the topology τ_L. The second point is to use the singular equation (8) to give a notion of strict subsolution of (1) (note that because of the presence of the zero set K there is no canonical notion of strict subsolution of (1)).

Definition 1 *Given a l.s.c. function $v : \Omega \to \mathbb{R}$, a Lipschitz continuous function ϕ is called L-subtangent to v at $x_0 \in \Omega$ if x_0 is a minimizer of $v - \phi$ in a neighborhood A of $B_L(x_0)$ (or, equivalently, in a τ_L-neighborhood of x_0).*

We remark that the convexity assumption (7) allows us to use Lipschitz continuous test functions instead of C^1 test functions as in the standard definition of viscosity solution.

Definition 2 *Given an open subset A of Ω, a function v is said to be a strict subsolution of (1) in A if v satisfies*

$$\rho(x, Dv) \leq \theta \qquad \textit{for } x \in A,$$

in the viscosity sense, for some $\theta \in (0, 1)$.

We can now introduce the definition of singular viscosity supersolution.

Definition 3 *A l.s.c. function $v : \Omega \to \mathbb{R}$ is said to be a singular supersolution of equation (1) in Ω if, for any $x_0 \in \Omega$, it does not admit an L-subtangent at x_0 which is a strict subsolution of (1) in a neighborhood of $B_L(x_0)$ (i.e. in a τ_L-neighborhood of x_0).*

In the following theorem, we compare viscosity subsolutions (in the standard sense of Crandall-Lions) and singular supersolutions of equation (1).

Theorem 1 *Let $u \in USC(\overline{\Omega})$, $v \in LSC(\overline{\Omega})$ be a viscosity subsolution and a singular supersolution of equation (1), respectively. Then*

$$\min_{\overline{\Omega}} \{v - u\} = \min_{\partial\Omega} \{v - u\}.$$

We now turn to the characterization of the optimal control problem associated to the singular viscosity supersolution of (1).

Let V be the value function of the control problem with the dynamics

$$\begin{cases} \dot{\xi}(t) = q(t) & t \in [0, \infty) \\ \xi(0) = x, \end{cases}$$

where $x \in \Omega$ and q is any bounded measurable function on $[0, +\infty)$ such that $T := \inf\{t > 0 : \xi(t) \notin \Omega\} < +\infty$, and with the cost functional

$$J(x, q) = \int_0^T \delta(\xi(t), -\dot{\xi}(t))dt + g(\xi(T)).$$

Then the following relation holds true.

Theorem 2 *The function V is a viscosity subsolution and a singular supersolution of equation (1) and satisfies $\lim_{y \to x, y \in \Omega} V(y) \leq g(x)$ for any $x \in \partial\Omega$. Hence V is the maximal subsolution of problem (1)–(2).*

We conclude this section stating a particular case, used in the next section, of a general stability theorem proved in [2].

Corollary 1 *Let v_ϵ be a locally uniformly bounded sequence of viscosity solutions of*

$$\begin{array}{ll} H(x, Du_\epsilon) = f_\epsilon(x) & x \in \Omega \\ u_\epsilon = 0 & x \in \partial\Omega \end{array} \tag{9}$$

where $f_\epsilon(x) = \max\{f(x), \epsilon\}$. Then

$$\lim_{\epsilon \to 0} v_\epsilon(x) = V(x)$$

locally uniformly in $\overline{\Omega}$, where V is the maximal solutions of (1)-(2) with $g = 0$.

3 Approximation of the maximal solution

In this section we construct a numerical scheme for the approximation of the maximal solution which is based on a regularization technique. For the proofs of the results stated in this section we refer to [1].

Throughout this section we assume for simplicity that $g(x) = 0$, i.e. we impose homogeneous Dirichlet boundary conditions.

For any $\epsilon > 0$ we define

$$f_\varepsilon(x) := \max\{f(x), \epsilon\}$$

and define the function $\delta_\epsilon(x, p)$ analogous to δ from Section 2 now with f_ϵ instead of f. Note that also δ_ϵ is convex and homogeneous in p.

Then by standard techniques we obtain that the (unique) solution u_ϵ of (9) is the optimal value function of the optimal control problem with dynamics

$$\begin{cases} \dot{\xi}(t) = q(t) & t \in [0,1] \\ \xi(0) = x, \end{cases}$$

where $x \in \Omega$ and q is any measurable function on $[0, +\infty)$ with $\|q(t)\| = 1$ and with cost functional

$$J(x,q) = \int_0^T \delta_\epsilon(\xi(t), q(t))dt$$

where $\xi(T) \in \partial\Omega$. Note that for $\|p\| = 1$ the function δ_ε is bounded from below by some positive constant, hence the boundedness of the solution immediately implies that the optimal trajctories hit $\partial\Omega$ in finite time.

We are going to apply a two stage discretization of this optimal control problem first in time and then in space; a similar scheme has been used in [3] for discounted optimal control problems.

For the time discretization we fix a time step $h > 0$ and look for the solution of the equation

$$u_{\epsilon,h}(x) = \inf_{\|q\|=1} \{h\delta_\epsilon(x,q) + u_{\epsilon,h}(x + hq)\} \tag{10}$$

for $x \in \Omega$ with the boundary condition $u_{\epsilon,h}(x) = 0$ for $x \in \partial\Omega$.

This equation indeed has a unique bounded solution $u_{\epsilon,h}$ which satisfies the following convergence property. Here

$$\omega_{\delta_\epsilon}(\alpha) := \sup\{|\delta_\epsilon(x,p) - \delta_\epsilon(y,p)| \,:\, \|p\| = 1,\, \|x-y\| < \alpha\}$$

Theorem 3 *Assume that $\omega_{\delta_\epsilon}(h)/\epsilon \to 0$ as $\epsilon \to 0$. Then*

$$u_{\epsilon,h} \to V$$

locally uniformly in $\bar{\Omega}$ where V is the maximal solution of (1)-(2) with $g = 0$.

In fact, what makes this convergence work is just the stability of the maximal solution of (1)-(2) combined with the consistency of the scheme.

We will now introduce the space discretization in order to transform (10) into a finite dimensional problem. For this purpose we chose a grid Γ covering $\overline{\Omega}$ consisting of simplices S_j with nodes x_i and look for the solution of (10) in the space

$$W := \{w \in C(\overline{\Omega}, \mathbb{R}) \,|\, \nabla w \equiv \text{const on } S_j\}$$

of piecewise linear functions on Γ. By the parameter k we denote the maximal diameter of the simplices S_j.

Thus we end up with the fully discrete scheme

$$u^k_{\epsilon,h}(x_i) = \inf_{\|q\|=1} \{h\delta_\epsilon(x_i,q) + u_{\epsilon,h}(x_i + hq)\} \tag{11}$$

for all nodes $x_i \in \Omega$ with the boundary condition $u^k_{\epsilon,h}(x) = 0$ for the nodes $x_i \in \partial\Omega$.

We can give the following estimate for the discretization error.

Theorem 4 *Let $u^k_{\epsilon,h} \in W$ be the unique solution of (11). Then the estimate*

$$|u_\epsilon(x) - u^k_{\epsilon,h}(x)| \le$$
$$C\left[\frac{k\omega_{\delta_\epsilon}(k)}{h\alpha(\epsilon)} + \frac{k}{\beta(\epsilon)h} + \frac{h}{\alpha(\epsilon)}\right] \max\{u_\epsilon(x), u^k_{\epsilon,h}(x)\} + Ch$$

holds for all sufficiently small $k > 0$ with

$$\alpha(\epsilon) = \inf_{x\in\Omega, \|q\|=1} \delta_\epsilon(x,q)$$

and

$$\beta(\epsilon) = \inf_{x\in\Omega, \|q\|=\|p\|=1} \frac{\delta_\epsilon(x,p)}{\delta_\epsilon(x,q)}$$

and with some constant C independent from ϵ, h and k.

Note that this estimate is stonger than the usual L_∞ estimate since essentially the error scales linearly with the value of the value functions which is 0 at $\partial\Omega$. The reason for this behaviour origins in the fact that the error is estimated along the optimal trajectories whose length depends on the optimal value.

A particular nice formulation of this estimate can be obtained if we consider the Eikonal equation (6) and assume that f is uniformly Lipschitz. In this case the estimate becomes

$$|u_\epsilon(x) - u^k_{\epsilon,h}(x)| \le$$
$$C\left[\frac{k^2}{h\epsilon} + \frac{k}{h} + \frac{h}{\epsilon}\right] \max\{u_\epsilon(x), u^k_{\epsilon,h}(x)\} + Ch$$

for some constant C independent from ϵ, h and k. In particular this implies convergence of the scheme if $\varepsilon \to 0$, $h/\varepsilon \to 0$ and $k/h \to 0$.

In the rest of this paper we discuss the error that we obtain when equation (1) is replaced by equation (9), i.e. the error introduced by the regularization of the problem.

Corollary 1 implies that u_ε converges to V, where V is the maximal subsolution of (1). Unfortunately, in general this convergence can be arbitrary slow. We can, however, give a general criterion allowing us to derive an estimate for this regularization error.

Definition 4 *Let $K \subset \mathbb{R}^d$ be a compact set. For each connected component K_i of K we define the* inner diameter *$d(K_i)$ by*

$$d(K_i) := \sup_{x,y\in K_i} d_{K_i}(x,y)$$

where

$$d_{K_i}(x,y) := \inf\left\{ T \ge 0 \;\middle|\; \begin{array}{l} \exists \xi(t) \in W^{1,\infty}([0,T], \overline{\Omega}) \\ s.t.\ \xi(0) = x,\ \xi(T) = y, \\ \|\dot{\xi}(t)\| = 1\ a.e. \end{array} \right\}$$

and for K we define the inner diameter by

$$d(K) := \sum_{K_i} d(K_i)$$

where the sum is taken over all connected components of K.

Using this definition we can state the following estimate for the regularization error.

Theorem 5 *Let V be the maximal subsolution of (1) and let u_ϵ be the unique viscosity solution of (9).*
Then the estimate

$$\|V - u_\epsilon\|_\infty \leq \epsilon d(K_\epsilon)$$

holds where $K_\epsilon := \{x \in \Omega \,|\, f(x) \leq \epsilon\}$.

Observe that if f is piecewise polynomial then K_ε is bounded for all $\epsilon > 0$ and hence linear convergence follows. Piecewise polynomial maps are in particular interesting since they include the case where f is obtained from experimental data by some polynomial interpolation (e.g. using piecewise linear interpolations, multidimensional splines...).

We have therefore obtained estimates for the discretization error and the regularization error allowing us to give a priori estimates for the whole numerical approximation.

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