ON THE RELATION BETWEEN DISCOUNTED AND AVERAGE OPTIMAL VALUE FUNCTIONS

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Abstract: We investigate the relation between discounted and average deterministic optimal control problems for nonlinear control systems. In particular we are interested in the corresponding optimal value functions. Using the concepts of Viability, Chain Controllability and Controllability a global convergence result for vanishing discount rate is obtained. Basic ingredients for the analysis are an Abelian type theorem, controllability properties of the system and the Morse decomposition of the corresponding control flow.

Keywords: Deterministic nonlinear optimal control, discounted optimal control, average optimal control, Morse decomposition, Viable Sets, Chain control sets, Control sets

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1 Introduction

In this paper we investigate the relation between average and discounted deterministic nonlinear optimal control problems for discount rate tending to zero. Whereas the relation between discounted and average *integrals* has already been explored more than a century ago leading to the Abelian and Tauberian theorems (see e.g. [25, Chapter 10]), corresponding results in nonlinear optimal control theory are much more recent. For stochastic optimal control problems in the Markovian setup the corresponding convergence result is almost classical, see e.g. the survey [2], or [28], where also estimates about the rate of convergence are given. The usual assumptions made in the Markovian case, however, exclude the deterministic case. In the deterministic setup, which we will consider in this paper, Colonius [8] in 1989 published a convergence result for vanishing discount rate on invariant control sets, a similar result for arbitrary control sets has been obtained in 1993 by Wirth [27]. These results have in common that assumptions on the optimal trajectories are made which are difficult to check and in general not satisfied even for simple one-dimensional systems. For invariant control sets this restriction could be removed

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in [18, Theorem 2.11]. Arisawa [3], [4] treats a similar problem (under the name *ergodic problem*) but from a somewhat different point of view: Maximal subsets of convergence of the discounted functional are characterized by introducing a controllability concept in connection with attractivity properties, where again invariance plays a crucial role. This problem goes back to Lions [24], who studied the convergence properties of solutions of Hamilton-Jacobi equations. The name *ergodic problem* is motivated by the fact that for an uncontrolled system (i.e. an ordinary differential equation) the convergence property is equivalent to the ergodicity of that system, see [3, Appendix 1].

The main purpose of this paper is to develop results without assuming invariance and without making assumptions on optimal trajectories but by assuming certain qualitative properties of the system. We obtain a global convergence result by merging estimates from three basic concepts — *Viability* (Section 5) allowing us to state results on extremal values of the value functions, *Chain controllability* (Section 6) enabling us to give estimates for all possible trajectories of the system and *Controllability* (Section 7), which is used in order to characterize the behaviour of certain optimal trajectories — into one global picture in Section 8. This kind of approach was inspired by the analysis of the Lyapunov spectrum of bilinear control systems as carried out in [13]. By this procedure we are also able to characterize the subsets of uniform convergence. Furthermore we present a penalizing strategy for the restriction to certain regions of the state space in Section 9. The assumptions we impose can be interpreted as robustness conditions, cp. Remark 8.5, and are generically satisfied for families of systems under an inner pair condition, cp. Remark 8.6.

At the very heart of our analysis two tools are used: In Section 3 we thoroughly investigate the relation between discounted and average functionals using a similar technique as in [18] and [20]. This can be interpreted as a stronger version of the Abelian theorem, allowing also results on uniform convergence. In Section 4 we investigate the control flow associated to our control system (cf. [11]). Here the concept of attractivity (which is also used in [3]) fits into the general framework of dynamical systems from which we adopt the concept of Morse decompositions.

Apart from the main theorem which is presented in Section 8 we have also formulated the partial results in the Sections 5–7 in a self contained way since they provide useful estimates in themselves. Throughout this paper we assume that the state space is a *compact* manifold M; in Section 9, however, we give some hints about how to overcome this restriction.

The applications of our results are immediate, since discounted optimal control problems enjoy a number of features that averaged ones do not have in general: The corresponding optimal value functions are Hoelder continuous and can be characterized as viscosity solution of Hamilton-Jacobi-Bellman equations (cf. [23]), the problems admit a numerical solution (cf. [6], [19]) and the construction of optimal controls in open loop and feedback form (cf. e.g. [5], [7] and [17]).

Nevertheless it is often desirable to solve average optimal control problems, because they can be formulated in order to determine *asymptotic properties* of a given control system. One example is the exponential behaviour of bi- and semilinear systems measured by Lyapunov exponents (cf. [10], [12]). The approximation by a discounted optimal control problem enables us to obtain stabilizing optimal controls of feedback type (see [17] and [20]) and to compute the whole Lyapunov spectrum numerically (cf. [18]). In particular for the analysis of the complete asymptotic behaviour of a system a global convergence result is needed; the result of the present paper in fact closes the gap in the convergence analysis in [18].

2 Problem statement

We consider nonlinear control systems of the type

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) \tag{2.1}$$

on some compact smooth manifold M where the vector fields f_i, $i = 0, \dots m$ are assumed to be Lipschitz and the control function $u(\cdot)$ satisfies

$$u(\cdot) \in \mathcal{U} := \{u : \mathbb{R} \to U \mid u(\cdot) \text{ measurable}\}$$

where $U \subset \mathbb{R}^m$ is compact and convex. For a given initial value $x_0 \in M$ at time $t = 0$ and a given control function $u(\cdot) \in \mathcal{U}$ we denote the trajectories of (2.1) by

$$\varphi(t, x_0, u(\cdot))$$

In order to define the optimal control problems we assume that a cost function

$$g : M \times \mathbb{R}^m \to \mathbb{R}, \;\; g(x, u) := g_0(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{2.2}$$

which is Lipschitz continuous and bounded, i.e. $|g(x, u)| \le M_g$ for some constant M_g, is given.

Using this cost function we define the averaged functionals along a trajectory by

$$\overline{J_0}(x_0, u(\cdot)) := \limsup_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi(s, x_0, u(\cdot)), u(s)) ds \tag{2.3}$$

and

$$\underline{J_0}(x_0, u(\cdot)) := \liminf_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi(s, x_0, u(\cdot)), u(s)) ds \tag{2.4}$$

and for a positive discount rate $\delta > 0$ we define the discounted functional

$$J_\delta(x_0, u(\cdot)) := \delta \int_0^\infty e^{-\delta s} g(\varphi(s, x_0, u(\cdot)), u(s)) ds \tag{2.5}$$

(The scaling of the integral by the discount rate δ is introduced in order to obtain a more consistent notation in what follows.)

The optimization problem now is to minimize these functionals for any initial value with respect to the control function $u(\cdot) \in \mathcal{U}$. More precisely we consider the optimal value functions

$$\overline{v_0}(x_0) := \inf_{u(\cdot)\in\mathcal{U}} \overline{J_0}(x_0, u(\cdot)) \quad \text{and} \quad \underline{v_0}(x_0) := \inf_{u(\cdot)\in\mathcal{U}} \underline{J_0}(x_0, u(\cdot)) \tag{2.6}$$

and

$$v_\delta(x_0) := \inf_{u(\cdot)\in\mathcal{U}} J_\delta(x_0, u(\cdot)) \tag{2.7}$$

Note that the corresponding maximization problem is obtained by simply replacing g by $-g$.

Both criteria are defined over an infinite time horizon. Here the averaged functionals indeed only measure asymptotic properties, i.e. everything that happens up to some bounded time t_0 does not contribute to the integral. In contrast to this for the discounted functional the boundedness of g implies that essentially only the behaviour on a finite horizon is measured: For any $\delta > 0$ and any $\varepsilon > 0$ there exists $t > 0$ such that

$$|J_\delta(x_0, u(\cdot)) - \delta \int_0^t e^{-\delta s} g(\varphi(s, x_0, u(\cdot)), u(s)) ds| \leq \varepsilon$$

for all $x_0 \in M$ and all $u(\cdot) \in \mathcal{U}$. However, for decreasing $\delta \to 0$ and fixed $\varepsilon > 0$ this time increases. Hence the question, whether v_δ approximates $\overline{v_0}$ and $\underline{v_0}$ for small $\delta > 0$ arises naturally. It is this question that we want to investigate in this article.

3 Discounted and averaged functionals

In this section we will investigate the relation between discounted and averaged integrals, functionals (along trajectories) and value functions. We start with a lemma giving an estimate for these integrals which can be interpreted as a stronger version of the classical Abelian theorem that can be found e.g. in [25, Theorem 10.2].

Lemma 3.1 Let $q : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $|q(s)| < M_q$ for almost all $s \in \mathbb{R}$. Assume there exists a time $T > 0$ such that

$$\frac{1}{t} \int_0^t q(\tau) d\tau < \sigma \text{ for all } t \geq T$$

Then for any $\varepsilon > 0$ and all $0 < \delta < \frac{\varepsilon}{(M_q + |\sigma+\varepsilon|)T}$ the following inequality holds:

$$\delta \int_0^\infty e^{-\delta\tau} q(\tau) d\tau \leq \sigma + \varepsilon$$

A proof of this lemma can be found in [20, Appendix], which uses essentially the same techniques as the proof of [18, Theorems 2.1 and 2.2] combined with a careful evaluation of the constants.

Note that the converse inequalities are easily obtained by replacing g by $-g$.

In order to carry over these results to our functionals and value functions we introduce some further definitions.

Definition 3.2 For the control system (2.1) and the cost function (2.2) we define

$$\begin{aligned}
J_0^t(x,u(\cdot)) &:= \frac{1}{t}\int_0^t g(\varphi(s,x,u(\cdot)),u(s))ds \\
\overline{J_0^t}(x,u(\cdot)) &:= \sup_{\tau\geq t} J_0^\tau(x,u(\cdot)) \\
\underline{J_0^t}(x,u(\cdot)) &:= \inf_{\tau\geq t} J_0^\tau(x,u(\cdot)) \\
\overline{v_0^t}(x) &:= \inf_{u(\cdot)\in\mathcal{U}} \overline{J_0^t}(x,u(\cdot)) \\
\underline{v_0^t}(x) &:= \inf_{u(\cdot)\in\mathcal{U}} \underline{J_0^t}(x,u(\cdot))
\end{aligned}$$

The following lemma shows the relation to the averaged functionals and value functions from Section 2.

Lemma 3.3

$$\lim_{t\to\infty}\overline{J_0^t}(x,u(\cdot)) = \overline{J_0}(x,u(\cdot)), \quad \lim_{t\to\infty}\underline{J_0^t}(x,u(\cdot)) = \underline{J_0}(x,u(\cdot))$$

and

$$\lim_{t\to\infty}\overline{v_0^t}(x) = \overline{v_0}(x), \quad \lim_{t\to\infty}\underline{v_0^t}(x) =: \underline{v_0^\infty}(x) \leq \underline{v_0}(x)$$

Proof: The first two equalities are immediately clear from the definitions. We prove the third assertion, the fourth follows by similar arguments.

Recalling the definition of $\overline{v_0^t}(x)$ and $\overline{v_0}(x)$ using the notation of Definition 3.2 this equality states

$$\lim_{t\to\infty}\inf_{u(\cdot)\in\mathcal{U}}\sup_{\tau\geq t} J_0^\tau(x,u(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}}\limsup_{t\to\infty} J_0^t(x,u(\cdot))$$

We prove the equality by proving both inequalities.

"$\leq$" Fix $\varepsilon>0$ and $u_\varepsilon(\cdot)\in\mathcal{U}$ such that

$$\limsup_{t\to\infty} J_0^t(x,u_\varepsilon(\cdot)) < \inf_{u(\cdot)\in\mathcal{U}}\limsup_{t\to\infty} J_0^t(x,u(\cdot)) + \varepsilon$$

Then there exists $t_\varepsilon\geq 0$ such that

$$J_0^\tau(x,u_\varepsilon(\cdot)) < \inf_{u(\cdot)\in\mathcal{U}}\limsup_{t\to\infty} J_0^t(x,u(\cdot)) + 2\varepsilon$$

for all $\tau\geq t_\varepsilon$. Since $\varepsilon>0$ was arbitrary this implies "$\leq$".

"$\geq$" Fix $\varepsilon>0$ and $t_\varepsilon>0$ such that

$$\lim_{t\to\infty}\inf_{u(\cdot)\in\mathcal{U}}\sup_{\tau\geq t} J_0^\tau(x,u(\cdot)) + \varepsilon > \inf_{u(\cdot)\in\mathcal{U}}\sup_{\tau\geq t_\varepsilon} J_0^\tau(x,u(\cdot))$$

Then there exists a control function $u_\varepsilon(\cdot)\in\mathcal{U}$ such that

$$\lim_{t\to\infty}\inf_{u(\cdot)\in\mathcal{U}}\sup_{\tau\geq t} J_0^\tau(x,u(\cdot)) + 2\varepsilon > \inf_{u(\cdot)\in\mathcal{U}}\sup_{\tau\geq t_\varepsilon} J_0^\tau(x,u(\cdot)) + \varepsilon > \sup_{\tau\geq t_\varepsilon} J_0^\tau(x,u_\varepsilon(\cdot))$$

This implies

$$\lim_{t\to\infty} \inf_{u(\cdot)\in\mathcal{U}} \sup_{\tau\geq t} J_0^\tau(x,u(\cdot)) + 2\varepsilon > \limsup_{t\to\infty} J_0^t(x,u_\varepsilon(\cdot))$$

and since $\varepsilon > 0$ was arbitrary "$\geq$" follows. □

These "finite time" averaged functionals and value functions can now be used to give uniform bounds for J_δ and v_δ for small discount rate $\delta > 0$.

Lemma 3.4 For all $t > 0$, all $\varepsilon > 0$ and $\delta_0 = \frac{\varepsilon}{2M_g t}$ the estimate

$$J_\delta(x,u(\cdot)) \in [\underline{J_0^t}(x,u(\cdot)) - \varepsilon, \overline{J_0^t}(x,u(\cdot)) + \varepsilon]$$

holds for all $\delta \leq \delta_0$. In particular if the limit $\lim_{t\to\infty} J_0^t(x,u(\cdot))$ exists the equality

$$\lim_{\delta\to 0} J_\delta(x,u(\cdot)) = \overline{J_0^t}(x,u(\cdot)) = \underline{J_0^t}(x,u(\cdot))$$

is implied

Proof: Follows immediately from Lemma 3.1 by observing that it is sufficient to consider $|\sigma + \varepsilon| \leq M_g$. □

Corollary 3.5 For all $t > 0$, all $\varepsilon > 0$ and $\delta_0 = \frac{\varepsilon}{2M_g t}$ the estimate

$$v_\delta(x) \in [\underline{v_0^t}(x) - \varepsilon, \overline{v_0^t}(x) + \varepsilon]$$

holds for all $\delta \leq \delta_0$. In particular if $\overline{v_0}(x)$ and $\underline{v_0^\infty}(x)$ agree the equality

$$\lim_{\delta\to 0} v_\delta(x) = \overline{v_0}(x) = \underline{v_0}(x) = \underline{v_0^\infty}(x)$$

is implied.

Proof: Follows immediately from the preceding lemma. □

Hence the goal of this paper will be to give estimates for $\overline{v_0^t}(x)$ and $\underline{v_0^t}(x)$ and characterize the situations in which the limits coincide. In particular we are interested in uniform estimates in t on certain subsets of M which then imply uniform estimates for v_δ for small $\delta > 0$. A special case for these subsets will be those where $\overline{v_0}(x) = \underline{v_0}(x) = \underline{v_0^\infty}(x) \equiv const.$ Keeping Corollary 3.5 in mind we will not — except for the main Theorem 8.4 — explicitly formulate the implications of the estimates in the following sections on v_δ.

Sometimes it will be useful to restrict the state space to some subset $B \subset M$. We will denote the corresponding value functions as follows.

Definition 3.6 For a subset $B \subset M$ we define

$$\overline{v_0}(x,B) := \inf\{\overline{J_0}(x,u(\cdot)) \mid u(\cdot)\in\mathcal{U},\, \varphi(t,x,u(\cdot)) \in B \text{ for all } t \geq 0\}$$

for those points $x \in B$ for which at least one trajectory exists that stays inside B.

In the same way we define $\underline{v_0}(x,B)$, $\underline{v_0^\infty}(x,B)$, $\overline{v_0^t}(x,B)$, $\underline{v_0^t}(x,B)$ and $v_\delta(x,B)$.

We end this section with two lemmas showing some useful properties of averaged functionals which will be used in the next sections.

Lemma 3.7 Let $q : \mathbb{R} \to \mathbb{R}$ be a measurable function, $t > 0$ and $t_1 \in (0, t)$. Let $t_2 = t - t_1$. Then

(i) the following equality holds

$$\frac{1}{t}\int\limits_0^t q(\tau)d\tau = \frac{t_1}{t}\frac{1}{t_1}\int\limits_0^{t_1} q(\tau)d\tau + \frac{t_2}{t}\frac{1}{t_2}\int\limits_0^{t_2} q(\tau + t_1)d\tau$$

(ii) if $|q|$ is bounded by some constant M_q the following estimates hold

$$|\frac{1}{t}\int\limits_0^t q(\tau)d\tau - \frac{1}{t_1}\int\limits_0^{t_1} q(\tau)d\tau| \leq 2M_q\frac{t_2}{t} \text{ and } |\frac{1}{t}\int\limits_0^t q(\tau)d\tau - \frac{1}{t_2}\int\limits_0^{t_2} q(\tau + t_1)d\tau| \leq 2M_q\frac{t_1}{t}$$

Proof: (i) follows by a simple calculation, (ii) from (i) using the property

$$|\frac{1}{s}\int\limits_0^s q(\tau)d\tau| \leq M_q$$

which holds for all $s > 0$. □

Lemma 3.8 Let $q : \mathbb{R} \to \mathbb{R}$ be a measurable function bounded by some constant M_q. Let $t > 0$ be arbitrary and

$$\sigma := \frac{1}{t}\int_0^t q(\tau)d\tau$$

Then for any $\varepsilon > 0$ there exists a time $t^* \leq \frac{(2M_q - \varepsilon)t}{2M_q}$ such that

$$\frac{1}{s}\int_0^s q(t^* + \tau)d\tau \leq \sigma + \varepsilon$$

for all $s \in (0, t - t^*]$. Here $t - t^* \geq \frac{\varepsilon t}{2M_q} \to \infty$ as $t \to \infty$.

Proof: Let

$$\beta := \sup_{s \in (0,t]} \frac{1}{s}\int_0^s q(\tau)d\tau$$

and fix $\varepsilon > 0$. If $\beta \leq \sigma + \varepsilon$ the assertion follows with $t^* = 0$.

Otherwise let

$$t^* := \sup\left\{ s \in (0, t] \;\middle|\; \frac{1}{s}\int_0^s q(\tau)d\tau \geq \sigma + \varepsilon \right\}$$

By the continuity in s of this averaged integral the equality

$$\frac{1}{t^*}\int_0^{t^*} q(\tau)d\tau = \sigma + \varepsilon$$

is implied. By Lemma 3.7(ii) it follows from $\frac{1}{t}\int_0^t q(\tau)d\tau = \sigma$ that $t - t^* = t_2 \geq \frac{\varepsilon t}{2M_q}$ and hence $t^* \leq \frac{(2M_q-\varepsilon)t}{2M_q}$. We claim that t^* satisfies the desired property:

Defining $\tilde{q}(s) := q(s) - \sigma - \varepsilon$ it follows from the definition of t^* that

$$\frac{1}{t^*}\int_0^{t^*} \tilde{q}(\tau)d\tau = 0 \quad \text{and} \quad \frac{1}{s}\int_0^s \tilde{q}(\tau)d\tau < 0$$

for all $s \in (t^*, t]$. Hence also

$$\int_0^{t^*} \tilde{q}(\tau)d\tau = 0 \quad \text{and} \quad \int_0^s \tilde{q}(\tau)d\tau < 0$$

holds implying

$$\int_{t^*}^s \tilde{q}(\tau)d\tau < 0$$

for all $s \in (t^*, t]$ which yields the assertion. □

4 The control flow, (ε, T)-chains and their values

As already pointed out in the introduction, the concept of attractivity forms one of the basic tools for the analysis of our problem, since this enables us to formulate results for *all possible trajectories* with initial values in some specified set. Instead of using the control system (2.1) itself we will develop these results in terms of the corresponding *control flow*. Although this requires some definitions it will turn out that this procedure admits an elegant and straightforward approach to the desired results, since techniques from dynamical systems theory can be applied directly. We will start by defining the control flow Φ, see [11] for details.

By endowing the space $\mathcal{U}$ of measurable control functions with the weak*-topology we obtain a compact metric space. On this space for $t \in \mathbb{R}$ we define the right shift by

$$\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}, \quad \Theta(t, u(\cdot)) = u(\cdot + t)$$

This generates a continuous flow on $\mathcal{U}$. Using this shift we define the control flow

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \quad \Phi(t, u(\cdot), x) = (\Theta(t, u(\cdot)), \varphi(t, x, u(\cdot))) \tag{4.1}$$

In fact, this generates a continuous flow on the product space $\mathcal{U} \times M$. For convenience of notation we abbreviate $p = (u(\cdot), x)$ for the elements of this product space. For a set $\mathcal{B} \subset \mathcal{U} \times M$ we denote by

$$\pi_M \mathcal{B} := \{x \in M \,|\, \text{there exists } u(\cdot) \in \mathcal{U} \text{ with } (u(\cdot), x) \in \mathcal{B}\}$$

the natural projection onto M.

Note that the functionals J_δ and J_0^t depend continuously on p due to the fact that f and g are affine in u.

One of the main tools in our analysis is the concept of attractors for flows on metric spaces. In order to define these objects we have to define omega limit sets and invariance. We refer to [1] for more information about flows and dynamical systems.

Definition 4.1 For a subset $\mathcal{B} \subset \mathcal{X}$ the *ω-limit set* is defined by

$$\omega(\mathcal{B}) := \{p \in \mathcal{X} \mid \text{there exist points } p_k \in \mathcal{B} \text{ and times } t_k \to \infty \text{ with } \lim_{k\to\infty} \Phi(t_k, p_k) = p\}$$

The *ω^*-limit set* is defined analogously for the time reversed system.

A subset $\mathcal{B} \subset \mathcal{X}$ is called *forward invariant* if $\Phi(t,p) \in \mathcal{B}$ for all $p \in \mathcal{B}$ and all $t \geq 0$. It is called *backward invariant* if $\Phi(t,p) \in \mathcal{B}$ for all $p \in \mathcal{B}$ and all $t \leq 0$ and *invariant* if it is forward and backward invariant.

Now we can introduce the concept of attractors.

Definition 4.2 Let Φ be a continuous flow on a compact metric space $\mathcal{X}$. A compact invariant set $A \subset \mathcal{X}$ is called an *attractor* if it admits a neighborhood $\mathcal{N}$ such that $\omega(\mathcal{N}) = A$.

For an attractor A the set $A^* := \{p \in \mathcal{X} \mid \omega(p) \not\subset A\}$ is called the *complementary repeller*.

The *domain of attraction* of an attractor A is the set $\mathbf{A}(A) := \mathcal{X} \setminus A^*$.

Note that a repeller is an attractor for the time reversed flow.

The following Lemma on uniform attraction will be used in this section.

Lemma 4.3 Let A be an attractor and $\mathcal{K} \subset \mathcal{X} \setminus A^*$ be a compact set. Let $\mathcal{N}$ be some open neighborhood of A. Then there exists a time T such that

$$\Phi(t,p) \in \mathcal{N} \text{ for all } t \geq T \text{ and all } p \in \mathcal{K}$$

Proof: By the definition of an attractor $\omega(\tilde{\mathcal{N}}) = A$ holds for some open neighborhood $\tilde{\mathcal{N}}$ of A. Hence there exists $T_1 > 0$ such that $\Phi(t,p) \in \mathcal{N}$ for all $p \in \tilde{\mathcal{N}}$ and all $t \geq T_1$.

From the assumptions on $\mathcal{K}$ it follows that for any point $p \in \mathcal{K}$ there exists a time $t_p > 0$ such that $\Phi(t_p, p) \in \tilde{\mathcal{N}}$. The continuity implies that $\Phi(t_p, \tilde{p}) \in \tilde{\mathcal{N}}$ for all $\tilde{p}$ in some neighborhood of p. Since $\mathcal{K}$ is compact we obtain that these times t_p are bounded by some T_2, hence the assertion follows with $T = T_1 + T_2$. □

We will now somewhat generalize the finite time average functionals by introducing (ε, T)-chains and their averaged values. The basic idea is to allow small jumps between finite time trajectory pieces and define infimal chain values by letting these jumps tend to 0 and the time length of the trajectory pieces tend to infinity.

Definition 4.4 For p, $q \in \mathcal{X}$ and $\varepsilon, T > 0$ an *(ε, T)-chain* ζ is given by a number $n \in \mathbb{N}$ together with points in $\mathcal{X}$

$$p_0 = p, p_1, \ldots, p_n = q$$

and times

$$t_0, \ldots, t_{n-1} \geq T$$

such that $d(\Phi(t_i, p_i), p_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. The *total time* of a chain is given by $T(\zeta) := \sum_{i=0}^{n-1} t_i$.

We say that ζ *lies in* $\mathcal{B} \subset \mathcal{X}$ if $\Phi(t, p_i) \in \mathcal{B}$ for all $t \in [0, t_i]$ and all $i = 0, \dots, n-1$.

The *averaged value* of a chain is given by

$$J_0(\zeta) := \frac{1}{T(\zeta)} \sum_{i=0}^{n-1} t_i J_0^{t_i}(p_i)$$

with $J_0^t(p) := J_0^t(x, u(\cdot))$ from Definition 3.2 for $p = (u(\cdot), x)$.

For a subset $\mathcal{B} \subset \mathcal{X}$ we define the *infimal chain value over* $\mathcal{B}$ for $\varepsilon \to 0$ and $T \to \infty$ by

$$\kappa^*(\mathcal{B}) := \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exist } \varepsilon_k \to 0,\ T_k \to \infty \text{ and } (\varepsilon_k, T_k)\text{-chains} \\ \zeta_k \text{ in } \mathcal{B} \text{ such that } \lim_{k\to\infty} J_0(\zeta_k) \to \mu \end{array} \right\}$$

Note that this setup and hence all results in this section can be generalized to arbitrary flows on compact metric spaces and arbitrary average functionals, provided they can be written in a suitable integral form.

The following equality is an immediate consequence of the previous definitions.

Proposition 4.5 For the infimal chain value over some forward invariant subset $\mathcal{B} \subset \mathcal{X}$ the following equality holds

$$\kappa^*(\mathcal{B}) = \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exist } t_k \to \infty \text{ and points } p_k \in \mathcal{K} \\ \text{such that } \lim_{k\to\infty} J_0^{t_k}(p_k) \to \mu \end{array} \right\}$$

i.e. the jumps in the chains do not change the minimal value over $\mathcal{K}$.

Proof: "$\leq$": This follows from the fact that each trajectory is a (trivial) chain.

"$\geq$": Let ζ be an arbitrary (ε, T)-chain in $\mathcal{B}$. Then by the definition of $J_0(\zeta)$ there exists a time $t_i \geq T$ and a point $p_i \in \mathcal{B}$ in the chain such that $J_0^{t_i}(p_i) \leq J_0(\zeta)$. Hence by the definition of $\kappa^*(\mathcal{B})$ there exist sequences of times $t_k \to \infty$ as $k \to \infty$ and points $p_k \in \mathcal{B}$ such that $\limsup_{k\to\infty} J_0^{t_k}(p_k) \leq \kappa^*(\mathcal{B})$ which implies the assertion. □

For certain points we can even establish a stronger relation between κ^*, J_0^t, $\overline{J_0}$ and $\underline{J_0}$.

Proposition 4.6 For the infimal chain value over some compact forward invariant subset $\mathcal{K} \subset \mathcal{X}$ there exists a point $p \in \mathcal{K}$ such that

$$J_0^t(p) \leq \kappa^*(\mathcal{K}) \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} J_0^t(p) = \kappa^*(\mathcal{K})$$

In particular for this point the limit exists and $\overline{J_0}(p) = \underline{J_0}(p) = \kappa^*(\mathcal{K})$.

Proof: By Proposition 4.5 we find a sequence of points $p_k \in \mathcal{K}$ and times $t_k \to \infty$ as $k \to \infty$ such that $J_0^{t_k}(p_k) < \kappa^*(\mathcal{K}) + \varepsilon_k$ where $\varepsilon_k \to 0$ for $k \to \infty$. Defining $\tilde{\varepsilon}_k := \frac{1}{\sqrt{t_k}} \to 0$ for $k \to \infty$ we apply Lemma 3.8 to $q(s) := g(\Phi(p_k, s))$ for each $k \in \mathbb{N}$ and obtain times t_k^* such that

$$J_0^s(\Phi(t_k^*, p_k)) \leq \kappa^*(\mathcal{K}) + \varepsilon_k + \tilde{\varepsilon}_k$$

for all $s \in (0, t_k - t_k^*]$ where $t_k - t_k^* \geq \frac{\sqrt{t_k}}{2M_g}$. Defining points $\tilde{p}_k := \Phi(t_k^*, p_k)$ and times $\tilde{t}_k := t_k - t_k^* \to \infty$ as $k \to \infty$ we obtain

$$J_0^s(\tilde{p}_k) \leq \kappa^*(\mathcal{K}) + \varepsilon_k + \tilde{\varepsilon}_k$$

for all $s \in (0, \tilde{t}_k]$.

Since $\mathcal{B}$ is compact we may assume that the points $\tilde{p}_k$ converge to some $p \in \mathcal{B}$. Now fix arbitrary $t > 0$ and $\varepsilon > 0$ and consider $J_0^t(p)$. Since J_0^t is continuous we find $k_0 \in \mathbb{N}$ such that $|J_0^t(p) - J_0^t(p_k)| < \varepsilon$ for all $k \geq k_0$. Hence

$$J_0^t(p) < \kappa^*(\mathcal{K}) + \varepsilon_k + \tilde{\varepsilon}_k + \varepsilon$$

follows for all $k \geq k_0$. Since $\varepsilon > 0$ was arbitrary and $\varepsilon_k + \tilde{\varepsilon}_k \to 0$ for $k \to \infty$ we can conclude

$$J_0^t(p) \leq \kappa^*(\mathcal{K})$$

which implies the first assertion since $t > 0$ was arbitrary.

This immediately implies $\limsup_{t\to\infty} J_0^t(p) \leq \kappa^*(\mathcal{K})$. Now assume $\liminf_{t\to\infty} J_0^t(p) < \kappa^*(\mathcal{K})$. This implies the existence of a sequence t_k such that $\lim_{k\to\infty} J_0^{t_k}(p) < \kappa^*(\mathcal{K})$ which contradicts Proposition 4.5. □

The Propositions 4.5 and 4.6 show in particular that when considering infima over compact subsets of $\mathcal{X}$ the chain values, the finite time average values and the averaged values are equivalent. Note, however, that for a single point these equalities will in general not hold. The main advantage of the concept of chains and their values is that we can formulate the following result on continuous dependence for arbitrary times $t > 0$.

Proposition 4.7 Let $\mathcal{K} \subset \mathcal{X}$ be a compact forward invariant set for the control flow Φ. Then for any $\alpha > 0$ there exists a neighborhood $\mathcal{N}(\mathcal{K})$ and a time $T > 0$ such that

$$J_0^t(p) > \kappa^*(\mathcal{K}) - \alpha$$

for all $t \geq T$ and all $p \in \mathcal{N}(\mathcal{K})$ with $\Phi(s,p) \in \mathcal{N}(\mathcal{K})$ for all $s \leq t$.

Proof: Fix $\alpha > 0$. Assume that for arbitrary neighborhoods $\mathcal{N}(\mathcal{K})$ and arbitrary times $T > 0$ there exist points $p_{T,\mathcal{N}} \in \mathcal{N}(\mathcal{K})$ such that $J_0^t(p) < \kappa^*(\mathcal{K}) - \alpha$ for some $t \geq T$ and $\Phi(s,p) \in \mathcal{N}(\mathcal{K})$ for all $s \leq t$

Now choose an arbitrary $\varepsilon > 0$ and a $\delta \in (0, \varepsilon)$ such that for all times $0 \leq s \leq 2T$ and all points p, $q \in \mathcal{X}$ with $d(p,q) < \delta$ the inequalities

$$|J_0^s(p) - J_0^s(q)| < \varepsilon \quad \text{and} \quad d(\Phi(s,p), \Phi(s,q)) < \varepsilon \tag{4.2}$$

hold. The trajectories $\Phi(s, p_{T,\mathcal{N}})$ can now be partitioned into pieces with times $\tau_n \in [T, 2T]$. By choosing $\mathcal{N}(\mathcal{K})$ sufficiently small by the choice of δ for every point $p_m := \Phi(\sum_{n=0}^m \tau_n, p_{T,\mathcal{N}})$ there exists a point q_m in $\mathcal{K}$ such that (4.2) is satisfied. Hence this yields an $(\varepsilon, 2T)$-chain ζ in $\mathcal{K}$ satisfying $J_0(\zeta) \leq \kappa^*(\mathcal{K}) - \alpha + \varepsilon$. Since ε and T were arbitrary a contradiction to the definition of $\kappa^*(\mathcal{K})$ follows. □

The following corollary shows how this result can be extended to attractors and their domain of attraction.

Corollary 4.8 Let $A \subset \mathcal{X}$ be an attractor for Φ. Let $\mathcal{K}$ be a compact subset of the domain of attraction of A. Then for any $\alpha > 0$ there exists a time $T > 0$ such that

$$J_0^t(p) > \kappa^*(A) - \alpha$$

for all $p \in \mathcal{K}$ and all $t \geq T$.

Proof: For any $\alpha > 0$ we find a neighborhood $\mathcal{N}(A)$ of A and a time T_0 such that the assertion of Proposition 4.7 holds with $\alpha/2$. By Lemma 4.3 there exists a time T_1 such that $\Phi(s,p) \in \mathcal{N}(A)$ for all $s \geq T_1$ and all $p \in \mathcal{K}$. Now the assertion follows by Lemma 3.7 by choosing T sufficiently large compared to T_1. □

In the next step we will investigate the finite time average value on nested attractors. We start with two attractors.

Lemma 4.9 Let $A_0 \subset A_1$ be attractors of Φ and $A_0^* \supset A_1^*$ be the complementary repellers. Then for any $\alpha > 0$ there exists a $T > 0$ such that

$$J_0^t(p) \geq \min\{\kappa^*(A_0), \kappa^*(A_1 \cap A_0^*)\} - \alpha$$

for all $p \in A_1$ and all $t \geq T$.

Proof: Fix $\alpha > 0$. Then we find an open neighborhood $\mathcal{N}(A_1 \cap A_0^*)$ and a a time $T_1 > 0$ such that the assertion of Proposition 4.7 holds for $\alpha/2$. For $\mathcal{K} := A_1 \setminus \mathcal{N}(A_1 \cap A_0^*)$ there exists a time $T_2 > 0$ such that also the assertion of Corollary 4.8 also holds for $\alpha/2$.

We claim that

$$T := \max\{T_1 + T_2, \frac{2}{\alpha} M_g \max\{T_1, T_2\}\},$$

where M_g is the bound on g, satisfies the assertion: Pick an arbitrary point $p \in A_1$. For $p \in \mathcal{K}$ the assertion follows from Corollary 4.8. For $p \in A_1 \cap A_0^*$ the assertion follows from Proposition 4.7. For all other p define $t_0 := \min\{t \geq 0 \,|\, \Phi(t,p) \in \mathcal{K}\}$. In order to estimate $J_0^t(p)$ for $t \geq T$ we distinguish three cases:

(i) $t_0 < T_1$: Lemma 3.7(ii) implies

$$J_0^t(p) \geq J_0^{t-t_0}(\underbrace{\Phi(t_0,p)}_{\in\mathcal{K}}) - 2M_q \frac{t_0}{t} \geq \kappa^*(A_0) - \alpha$$

(ii) $t_0 \geq T_1, t \leq t_0 + T_2$: Here Lemma 3.7(ii) implies

$$J_0^t(p) \geq J_0^{t_0}(p) - 2M_q \frac{t - t_0}{t} \geq \kappa^*(A_1 \cap A_0^*) - \alpha$$

(iii) $t_0 \geq T_1, t \geq t_0 + T_2$: In this case Lemma 3.7(i) implies

$$J_0^t(p) \geq \frac{t_0}{t} J_0^{t_0}(p) + \frac{t-t_0}{t} J_0^{t-t_0}(\underbrace{\Phi(t_0,p)}_{\in\mathcal{K}}) \geq \frac{t_0}{t}\kappa^*(A_1 \cap A_0^*) + \frac{t-t_0}{t}\kappa^*(A_0) - \frac{\alpha}{2}$$

Hence in all three cases the assertion follows. □

The main goal of this section is to give uniform estimates for $\overline{J_0^t}$ and $\underline{J_0^t}$ on a Morse decomposition of $\mathcal{X}$ corresponding to the flow Φ. In order to obtain such a result we need the following definitions.

Definition 4.10 Let Φ be a continuous flow on a compact metric space $\mathcal{X}$. Let $\emptyset = A_0 \subset A_1 \subset \ldots \subset A_d = \mathcal{X}$ be a sequence of attractors and let $\mathcal{X} = A_0^* \supset A_1^* \supset \ldots \supset A_d^* = \emptyset$ be the complementary repellers. Then for any $i = 1, \ldots, d$ the set

$$\mathcal{M}_i = A_{d-i+1} \cap A_{d-i}^*$$

is called a *Morse set* and the collection $\mathcal{M}_i$, $i = 1, \ldots, d$ is called a *Morse decomposition.* For any Morse set $\mathcal{M}_i$ we define the corresponding *domain of attraction* by

$$\mathbf{A}(\mathcal{M}_\mathbf{i}) := \{p \in \mathcal{X} \,|\, \omega(p) \subset \mathcal{M}_i\}$$

Lemma 4.11 For any sequence of Morse sets $\mathcal{M}_{j_1}, \ldots, \mathcal{M}_{j_2}$, $1 \le j_1 \le j_2 \le d$ we have

$$\bigcup_{i=j_1,\ldots,j_2} \mathbf{A}(\mathcal{M}_i) = A_{d-j_2}^* \setminus A_{d-j_1+1}^*$$

Proof: "$\subseteq$" Let $p \in \mathbf{A}(\mathcal{M}_i)$ for some $i \in \{j_1, \ldots, j_2\}$. This implies $\omega(p) \subset A_{d-i+1}$, hence $p \notin A_{d-i+1}^* \supset A_{d-j_1+1}^*$.

On the other hand $\omega(p) \in \mathcal{M}_i$ implies $\omega(p) \subset A_{d-i}^*$. Since $p \notin A_{d-i}^*$ implies $\omega(p) \subset A_{d-i}$ and $A_{d-i} \cap A_{d-i}^* = \emptyset$ it follows that $p \in A_{d-i}^* \subset A_{d-j_2}^*$.

"$\supseteq$" Let $p \in A_{d-j_2}^* \setminus A_{d-j_1+1}^*$. Then by the inclusion of the repellers there exists $i \in \{j_1, \ldots, j_2\}$ such that $p \in A_{d-i}^* \setminus A_{d-i+1}^*$ By the invariance $p \in A_{d-i}^*$ implies $\omega(p) \subset A_{d-i}^*$. By the definition of the repeller $p \notin A_{d-i+1}^*$ implies $\omega(p) \subset A_{d-i+1}$, hence $\omega(p) \subset \mathcal{M}_i$. □

We will now discuss the order of the Morse sets. Note that the attractor sequence induces a total order of the Morse sets. However, it is possible that different attractor sequences generate the same Morse sets but with a different order. Hence we define a stronger order relation for the Morse sets.

Definition 4.12 Consider a Morse decomposition $\mathcal{M}_1, \ldots, \mathcal{M}_d$ for the flow Φ. Then for two Morse sets $\mathcal{M}_i \neq \mathcal{M}_j$ we define $\mathcal{M}_i < \mathcal{M}_j$ if there exist points $p_1, \ldots, p_k$ and Morse sets $\mathcal{M}_{i_0} = \mathcal{M}_i, \mathcal{M}_{i_1}, \ldots, \mathcal{M}_{i_d} = \mathcal{M}_j$ such that $\omega^*(p_l) \in \mathcal{M}_{l_{i-1}}$ and $\omega(p_l) \in \mathcal{M}_{l_i}$ for all $l = 1, \ldots, d$.

Remark 4.13 Note that $\mathcal{M}_i < \mathcal{M}_j$ implies $i < j$.

However, even a stronger relation to the attractor sequence can be established.

Proposition 4.14 Let $\mathcal{M}_1, \ldots \mathcal{M}_d$ be a Morse decomposition of the flow Φ generated by an attractor sequence $A_0 \subset A_1 \subset \ldots \subset A_d$. Then for any Morse set $\mathcal{M}_i$ there exists an attractor sequence $\tilde{A}_0 \subset \tilde{A}_1 \subset \ldots \subset \tilde{A}_d$ generating a Morse decomposition $\tilde{\mathcal{M}}_1, \ldots \tilde{\mathcal{M}}_d$ satisfying

$$\tilde{\mathcal{M}}_{l(j)} = \mathcal{M}_j$$

for all $j \in \{1, \ldots, d\}$ and a bijective function $l : \{1, \ldots, d\} \to \{1, \ldots, d\}$ and

$$\tilde{\mathcal{M}}_{l(i)} < \tilde{\mathcal{M}}_j$$

for all $j > l(i)$.

Proof: Step 1: We first show the following property: For any Morse set $\mathcal{M}_i$ with $\mathcal{M}_i \not< \mathcal{M}_{i+1}$ there exists an attractor sequence $\tilde{A}_0 \subset \tilde{A}_1 \subset \ldots \subset \tilde{A}_d$ generating a Morse decomposition $\tilde{\mathcal{M}}_1, \ldots \tilde{\mathcal{M}}_d$ with $\tilde{\mathcal{M}}_j = \mathcal{M}_j$ for all $j \in \{1, \ldots, d\}$ with $j \neq i$, $j \neq i+1$, and $\tilde{\mathcal{M}}_i = \mathcal{M}_{i+1}$, $\tilde{\mathcal{M}}_{i+1} = \mathcal{M}_i$.

W.l.o.g. we may assume $i + 1 = d$, otherwise we may restrict Φ to A^*_{d-i-1}. Hence we consider $\mathcal{M}_d = A_1$ and $\mathcal{M}_{d-1} = A_2 \cap A_1^*$. From the assumption $\mathcal{M}_{d-1} \not< \mathcal{M}_d$ we can conclude that a $p \in \mathcal{X}$ with $\omega(p) \subset \mathcal{M}_d$ and $\omega^*(p) \subset \mathcal{M}_{d-1}$ can not exist. Hence for any $p \in A_2 \setminus A_1$ (which implies $\omega^*(p) \subset \mathcal{M}_{d-1}$) it follows that $p \in A_1^*$.

This implies that $A_1^* \supset A_2 \setminus A_1$, hence $A_2 = A_1 \cup (A_2 \cap A_1^*) = \mathcal{M}_d \cup \mathcal{M}_{d-1}$ which means that A_2 consists of two non connected compact sets. Hence each of these sets itself is an attractor.

Define $\tilde{A}_1 = \mathcal{M}_{d-1}$ and $\tilde{A}_j = A_j$ for all $j \in \{2, \ldots, d\}$. Obviously $\tilde{A}_1^* := \mathbf{A}(A_1) \cup A_2^*$ is the repeller corresponding to $\tilde{A}_1$. Hence

$$\tilde{A}_2 \cap \tilde{A}_1^* = A_2 \cap (\mathbf{A}(A_1) \cup A_2^*) = A_2 \cap \mathbf{A}(A_1)$$

Now let $p \in A_2$. Then either $p \in A_1$ or $p \in \mathcal{M}_{d-1}$ implying $p \notin \mathbf{A}(A_1)$. Hence $A_2 \cap \mathbf{A}(A_1) = A_1 = \mathcal{M}_d$. Since $\tilde{A}_1 \subset \tilde{A}_2 \subset \ldots \subset \tilde{A}_d$ is obvious the assertion follows.

Step 2: Now consider the Morse set $\mathcal{M}_i$ from the assumption. Let k be the number of Morse sets with $i < j$ and $\mathcal{M}_i \not< \mathcal{M}_j$. If $k = 0$ we are done. Otherwise let $j > i$ be minimal with $\mathcal{M}_i \not< \mathcal{M}_j$. Then the transitivity of the order relation implies $\mathcal{M}_l \not< \mathcal{M}_j$ for all $l \in \{i, \ldots, j-1\}$. Applying Step 1 iteratively to $j-1, \ldots, i$ we obtain an attractor sequence generating a Morse decomposition with $l(i) > l(j)$, and the number k as defined above has decreased by 1.

Applying Step 2 iteratively now yields the assertion. □

Our main theorem now gives the complete picture on a Morse decomposition of $\mathcal{X}$.

Theorem 4.15 Let Φ be a continuous flow on a compact metric space $\mathcal{X}$. Let $\emptyset = A_0 \subset A_1 \subset \ldots \subset A_d = \mathcal{X}$ be a sequence of attractors and consider the corresponding Morse sets $\mathcal{M}_i = A_{d-i+1} \cap A^*_{d-i}$.

Then for any $i \in \{1, \ldots, d\}$, any compact set $\mathcal{K} \subset \bigcup_{\mathcal{M}_j \geq \mathcal{M}_i} \mathbf{A}(\mathcal{M}_j)$ and any $\alpha > 0$ there exists a $T > 0$ such that

$$J_0^t(p) \geq \min_{\mathcal{M}_j \geq \mathcal{M}_i} \kappa^*(\mathcal{M}_j) - \alpha$$

for all $p \in \mathcal{K}$ and all $t \geq T$.

Proof: Fix $i \in \{1, \ldots, d\}$. By Proposition 4.14 we may assume $\{\mathcal{M}_j \,|\, \mathcal{M}_j \geq \mathcal{M}_i\} = \{\mathcal{M}_j \,|\, j \geq i\}$.

Now from the assumption on $\mathcal{K}$ and Lemma 4.11 it follows that $\mathcal{K}$ lies in the domain of attraction of A_{d-i+1}. Hence Corollary 4.8 implies that

$$J_0^t(p) \geq \kappa^*(A_{d-i+1}) - \alpha$$

for all sufficiently large t.

Therefore we have to give an estimate for $\kappa^*(A_{d-i+1})$. We proceed by induction over $d-i+1 =: k$ and claim that

$$\kappa^*(A_k) = \min_{j=d-k+1,\dots,d} \kappa^*(\mathcal{M}_j)$$

For $k=1$ the assertion follows directly since $A_1 = \mathcal{M}_d$.

Now assume the assertion is true for $k>1$. For the induction step note that the value of a chain cannot be smaller than the minimum over the values of the trajectory pieces in the chain. Hence by Lemma 4.9 we obtain for A_{k+1}

$$\kappa^*(A_{k+1}) = \min\{\kappa^*(A_k), \kappa^*(\underbrace{A_{k+1} \cap A_k^*}_{=\mathcal{M}_{d-k}})\}$$

and the induction assumption yields the assertion. □

We end this section by stating a fact about Morse decompositions, for which we need some additional definitions.

Definition 4.16 A point $p \in \mathcal{U} \times M$ is called *chain recurrent*, if for all $\varepsilon, T > 0$ there exists an (ε, T)-chain from p to p.

The *chain recurrent set* $\mathcal{CR}$ of the flow Φ is the set of all chain recurrent points.

A subset $\mathcal{M} \subset \mathcal{U} \times M$ is called *chain transitive* if for all points p, $q \in \mathcal{M}$ and all $\varepsilon, T > 0$ there exists an (ε, T)-chain from p to q.

Remark 4.17 Note that the maximal invariant chain transitive sets are just the connected components of the chain recurrent set $\mathcal{CR}$. The relation of $\mathcal{CR}$ to the Morse sets follows from a result due to Conley [15] which will be useful for the interpretation of Theorem 4.15 in Section 6: If the chain recurrent set $\mathcal{CR}$ of (4.1) consists of finitely many connected components, then it is always possible to find an attractor sequence such that these connected components are just the corresponding Morse sets. Conversely, if we have a sequence of attractors yielding a finest Morse decomposition (i.e. no further refinement by introducing more attractors is possible), then the corresponding Morse sets are just the connected components of $\mathcal{CR}$. See e.g. [14] for a discussion and proof of this result in the context of control flows.

5 Viable sets

We will now return to the control system (2.1) and "translate" the results from the preceding section. We start by investigating the behaviour of the value functions on viable sets for the control system 2.1, which correspond to the forward invariant sets for the control flow Φ.

Definition 5.1 A subset $B \subset M$ is called *viable* (or *controlled invariant*), if for any $x \in B$ there exist a control function $u_x(\cdot) \in \mathcal{U}$ such that

$$\varphi(t, x, u_x(\cdot)) \in B \text{ for all } t \geq 0$$

We will use this definition in order to characterize properties of the extremal values of the value functions. as carried out in the next two propositions. Here we use the restricted value functions from Definition 3.6. Note that on viable sets B they are well defined for each point in B.

Proposition 5.2 Consider the system (2.1) and a viable set $B \subset M$. Then

$$\sup_{x\in B} \overline{v_0^t}(x,B) \geq \inf_{x\in B} \overline{v_0}(x,B)$$

and

$$\sup_{x\in B} \underline{v_0^t}(x,B) \geq \inf_{x\in B} \underline{v_0}(x,B)$$

for all $t \geq 0$.

Proof: Assume there exists a time $t > 0$ such that $\sup_{x\in B} \overline{v_0^t}(x,B) < \inf_{x\in B} \overline{v_0}(x,B)$. Then there exists $\varepsilon > 0$ such that for any $y \in B$ there exists a control function $u_y(\cdot)$ satisfying $J_0^t(y,u_y(\cdot)) < \inf_{x\in B} \overline{v_0}(x,B) - \varepsilon$ and $\varphi(s,y,u_y(\cdot)) \in B$ for all $s \geq 0$. Starting in some arbitrary point $y_0 \in B$ we may thus iteratively construct a control function $\bar{u}(\cdot)$ by $\bar{u}(\cdot)|_{[0,t]} := u_{y_0}(\cdot)|_{[0,t]}$ and $\bar{u}(\cdot)|_{(it,(i+1)t]} := u_{\varphi(it,y_0,\bar{u}(\cdot))}(\cdot)|_{(0,t]}$ for $i \in \mathbb{N}$. Clearly $\varphi(s,y,\bar{u}_y(\cdot)) \in B$ for all $s \geq 0$. By Lemma 3.7(i) this construction yields

$$J_0^{it}(y_0,\bar{u}(\cdot)) < \inf_{x\in B} \overline{v_0}(x,B) - \varepsilon$$

for all $i \in \mathbb{N}$ and by Lemma 3.7(ii) there exists $S > 0$ such that for all $s > S$ we obtain

$$J_0^s(y_0,\bar{u}) < \inf_{x\in B} \overline{v_0}(x,B) - \varepsilon/2$$

which contradicts the definition of $\overline{v_0}$. This proves the first assertion.

For the second assertion assume there exists a time $t > 0$ such that $\sup_{x\in B} \underline{v_0^t}(x,B) < \inf_{x\in B} \underline{v_0}(x,B)$. Then there exists $\varepsilon > 0$ such that for any $y \in B$ there exists a control function $u_y(\cdot)$ and a time $t_y \geq t$ satisfying $J_0^{t_y}(y,u_y(\cdot)) < \inf_{x\in B} \underline{v_0}(x,B) - \varepsilon$ and $\varphi(s,y,u_y(\cdot)) \in B$ for all $s \geq 0$. Starting in some arbitrary point $y_0 \in B$ we may again iteratively construct a control function $\bar{u}(\cdot)$ by $\bar{u}(\cdot)|_{[0,t_{y_0}]} := u_{y_0}(\cdot)|_{[0,t_{y_0}]}$ and $\bar{u}(\cdot)|_{(t_i,t_i+t_{\varphi(t_i,y_0,\bar{u}(\cdot))}]} := u_{\varphi(t_i,y_0,\bar{u}(\cdot))}(\cdot)|_{(0,t]}$, where $t_1 = t_{y_0}$ and $t_i = t_{i-1} + t_{\varphi(t_i,y_0,\bar{u}(\cdot))}$ for $i \geq 2$. Obviously $\varphi(s,y,\bar{u}_y(\cdot)) \in B$ for all $s \geq 0$. By Lemma 3.7(i) this construction yields

$$J_0^{t_i}(y_0,\bar{u}(\cdot)) < \inf_{x\in B} \underline{v_0}(x,B) - \varepsilon$$

for all $i \in \mathbb{N}$ which contradicts the definition of $\underline{v_0}$. This proves the second assertion. □

Proposition 5.3 Consider the system (2.1) and a compact viable set $K \subset M$. Then the following properties hold:

(i) For $\mathcal{K} := \{(u(\cdot),x) \in \mathcal{U} \times K \mid \varphi(t,x,u(\cdot)) \in K \text{ for all } t \geq 0\}$ the equality

$$\inf_{x\in K} \overline{v_0}(x,K) = \inf_{x\in K} \underline{v_0}(x,K) = \inf_{x\in K} \underline{v_0^\infty}(x,K) = \kappa^*(\mathcal{K})$$

is satisfied.

(ii) For any $\varepsilon > 0$ there exists $t > 0$ with

$$\overline{v_0^t}(x, K) \geq \underline{v_0^t}(x, K) \geq \inf_{x \in K} \overline{v_0}(x, K) - \varepsilon$$

for all $x \in K$

(iii) There exists a point $x \in K$ and a control function $u(\cdot) \in \mathcal{U}$ with $\varphi(t, x, u(\cdot)) \in K$ for all $t \geq 0$ satisfying

$$\overline{J_0^t}(x, u(\cdot)) \leq \inf_{x \in K} \underline{v_0^\infty}(x, K) \text{ for all } t \geq 0.$$

Proof: Observe that $\mathcal{K}$ is a compact forward invariant set for the control flow Φ. By the definition of $\underline{v_0^\infty}(x, K)$ and $\kappa^*(\mathcal{K})$ one obtains

$$\kappa^*(\mathcal{K}) = \inf_{x \in K} \underline{v_0^\infty}(x, K) \tag{5.1}$$

Now (iii) follows from Proposition 4.6, (i) from (iii) and (5.1), and (ii) follows from Proposition 4.7 and (i). □

6 Chain control sets

In this section we will interpret Theorem 4.15 in terms of the control system (2.1).

For this purpose we have to adapt the concept of chains to the control system, cp. Definition 4.4.

Definition 6.1 For $x, y \in M$ and $\varepsilon, T > 0$ a *(controlled)* (ε, T)*-chain* ζ is given by a number $n \in \mathbb{N}$ together with points in M

$$x_0 = x, x_1, \ldots, x_n = y, \text{ control functions } u_0(\cdot), \ldots, u_{n-1}(\cdot) \in \mathcal{U}$$

and times

$$t_0, \ldots, t_{n-1} \geq T$$

such that $d(\varphi(t_i, x_i, u_i(\cdot)), x_{i+1}) < \varepsilon$ for $i = 0, \ldots, n-1$. The *total time* of a chain is given by $T(\zeta) := \sum_{i=0}^{n-1} t_i$.

We say that ζ *lies in* $B \subset M$ if $\varphi(t, p_i, u_i(\cdot)) \in B$ for all $t \in [0, t_i]$ and all $i = 0, \ldots, n-1$.

The *averaged value* of a controlled chain is given by

$$J_0(\zeta) := \frac{1}{T(\zeta)} \sum_{i=0}^{n-1} \int_0^{t_i} g(\varphi(\tau, x_i, u_i(\cdot)), u(\tau)) d\tau$$

For a subset $B \subset M$ we define the *infimal chain value over* B for $\varepsilon \to 0$ and $T \to \infty$ by

$$\kappa^*(B) := \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{there exist } \varepsilon_k \to 0,\, T_k \to \infty \text{ and controlled} \\ (\varepsilon, T)\text{-chains } \zeta_k \text{ in } B \text{ such that } \lim_{k\to\infty} J_0(\zeta_k) \to \mu \end{array} \right\}$$

Using the concept of chains we can define the chain control set as follows.

Definition 6.2 For then control system (2.1) a set E is called a *chain control set*, if

(i) For all $x,\, y \in E$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain from x to y

(ii) For all $x \in E$ there exists $u(\cdot) \in \mathcal{U}$ such that $\varphi(t,x,u) \in E$ for all $t \in \mathbb{R}$

(iii) E is maximal with the properties (i) and (ii)

We define the *lift* $\mathcal{M}(E) \subset \mathcal{U} \times M$ of E by

$$\mathcal{M}(E) := \{(u(\cdot), x) \in \mathcal{U} \times E \,|\, \varphi(t, x, u(\cdot)) \in E \text{ for all } t \in \mathbb{R}\}$$

The relation to the control flow is described by the following theorem.

Theorem 6.3 Consider the control system (2.1) and the corresponding control flow (4.1). Then the following properties hold

(i) Let $E \subset M$ be a chain control set for (2.1). Then its lift $\mathcal{M} = \mathcal{M}(E) \subset \mathcal{U} \times M$ is a maximal invariant chain transitive set for the flow (4.1)

(ii) Let $\mathcal{M} \subset \mathcal{U} \times M$ be a maximal invariant chain transitive set for (4.1). Then $E := \pi_M \mathcal{M}$ is a chain control set for (2.1)

(iii) In these cases the equality $\kappa^*(\mathcal{M}) = \kappa^*(E)$ holds for the values from the Definitions 4.4 and 6.1.

Proof: Assertions (i) and (ii) follow from [11, Theorem 4.8], assertion (iii) is immediately clear by the definition of the chain values. □

In particular if there are only finitely many chain control sets, then the chain recurrent set of (4.1) has only finitely many components, hence (cp. Remark 4.17) they are Morse sets corresponding to a sequence of attractors. Hence we can translate Theorem 4.15 to (2.1) using this relation. Here we define the *domain of attraction* of a chain control set E by

$$\mathbf{A}(E) := \pi_M \mathbf{A}(\mathcal{M}(E))$$

and define the order of the chain control sets to be the induced order of the Morse sets, i.e. for $E_i \neq E_j$ we have

$$E_i < E_j \iff \begin{cases} \text{there exist points } x_1, \ldots x_k \in M, \text{ control functions } u_1(\cdot), \ldots, \\ u_k(\cdot) \in \mathcal{U} \text{ and chain control sets } E_{i_0} = E_i,\, E_{i_1}, \ldots, E_{i_k} = E_j \\ \text{such that } \pi_M \omega^*(u_l(\cdot), x_l) \subset E_{i_{l-1}} \text{ and } \pi_M \omega(u_l(\cdot), x_l) \subset E_{i_l} \end{cases} \quad (6.1)$$

and for arbitrary E_i and E_j we define $E_i \leq E_j \;\; :\iff \;\; E_i = E_j$ or $E_i < E_j$.

Theorem 6.4 Consider the control system (2.1). Assume that there exist finitely many chain control sets $E_1, \ldots, E_d$ on M. Let $A_0 \subset A_1 \subset \ldots \subset A_d$ be a sequence of attractors for the flow (4.1) with $\pi_M A_0 = \emptyset$ and $\pi_M A_d = M$ such that the Morse sets $\mathcal{M}_{d-i} := A_{i+1} \cap A_i^*$ satisfy $\mathcal{M}_j = \mathcal{M}(E_j)$. Then the following properties hold for all $j = 1, \ldots d$

(i) For any $\varepsilon > 0$ there exists $t > 0$ such that

$$\underline{v_0^t}(x, E_j) \geq \kappa^*(E_j) - \varepsilon$$

for all $x \in E_j$

(ii) There exists $x \in E_j$ such that

$$\overline{v_0^t}(x) \leq \overline{v_0^t}(x, E_j) \leq \kappa^*(E_j)$$

for all $t \geq 0$.

(iii) For any compact set $K \subset M$ satisfying $K \cap \bigcup_{E_l \not\geq E_j} \mathbf{A}(E_l) = \emptyset$ and any $\varepsilon > 0$ there exists a time $t > 0$ such that

$$\underline{v_0^t}(x) \geq \min_{E_i \geq E_j} \kappa^*(E_i) - \varepsilon$$

for all $x \in K$.

Proof: First observe that any chain control set is a compact viable set.

(i) follows immediately from Proposition 5.3(i) and (ii) and Theorem 6.3(iii).

(ii) Taking the infimum over $u(\cdot)$ in Proposition 5.3(iii) and combining Proposition 5.3(i) with Theorem 6.3(iii) we obtain this inequality.

(iii) By Proposition 4.14 we may choose the attractor sequence in such a way that $\{E_i \mid E_i \geq E_j\} = \{E_i \mid i \geq j\}$ which implies $\{E_l \mid E_l \not\geq E_j\} = \{E_l \mid l < j\}$.

Thus $\bigcup_{E_l \not\geq E_j} \mathbf{A}(E_l) = \bigcup_{l=1,\ldots,j-1} \mathbf{A}(E_l)$ and we can use Lemma 4.11 in order to conclude $\bigcup_{l=1,\ldots,j-1} \mathbf{A}(E_l) = \pi_M \bigcup_{l=1,\ldots,j-1} \mathbf{A}(\mathcal{M}_l) = A_{d-j+1}^*$. (note that $A_d^* = \emptyset$).

Now from the definition of K it follows that $\mathcal{K} := \mathcal{U} \times K \cap A_{d-j+1}^* = \emptyset$ and hence $\mathcal{K}$ satisfies the assumption of Theorem 4.15. This implies the assertion. □

7 Control sets

In Theorem 6.4(iii) we have obtained a uniform *lower bound* of $\underline{v_0^t}$ for initial values in certain subsets of M. The main purpose of this section is to establish uniform *upper bounds* for $\overline{v_0^t}$ on certain subsets of M.

In addition we will give a criterion for the equality of $\overline{v_0^t}$ and $\underline{v_0^t}$ for $t \to \infty$ for trajectories staying in these subsets.

Our way to obtain these estimates is the exploitation of the controllability properties of (2.1). The basic concept in order to do this is given by the reachable and control sets.

Definition 7.1 For any point $x \in M$ we define the *positive reachable set up to the time* $T > 0$ by

$$O^+_{\leq T}(x) := \{y \in M \,|\, \text{there exists } y \in D,\ u(\cdot) \in \mathcal{U} \text{ and } t \in [0,T] \text{ such that } \varphi(t,x,u(\cdot)) = y\}$$

and the *positive reachable set by*

$$O^+(x) := \bigcup_{T>0} O^+_{\leq T}(x)$$

The *negative reachable* sets are defined the same way for the time reversed system.

A subset $D \subset M$ is called a *control set* if

(i) $D \subset \mathrm{cl}O^+(x)$ for all $x \in D$

(ii) For all $x \in D$ there exists a control function $u(\cdot) \in \mathcal{U}$ such that $\varphi(t,x,u(\cdot)) \in D$ for all $t > 0$

(iii) $D \subset M$ is maximal with respect to (i) and (ii)

The *domain of attraction* (or *negative reachable set*) $\mathbf{A}(D)$ of a control set $D \subset M$ is given by

$$\mathbf{A}(D) := \bigcup_{x \in D} O^-(x)$$

If we have a collection of control sets $D_1, \ldots, D_k$ we can define a partial order by

$$D_i \leq D_j \;\; :\Longleftrightarrow \;\; D_i \subset \mathbf{A}(D_j) \tag{7.1}$$

The maximality of the control sets guarantees that this order is well defined. In what follows we assume that the numbering of the control sets always corresponds to this partial order.

Note that the definition of control sets requires only approximate controllability. A convenient way to avoid technical assumptions on the speed of this controllability is to assume *local accessibility* of (2.1), i.e. we assume that the positive and negative orbit up to any time $T > 0$ have nonvoid interior for all $x \in M$. A sufficient analytic condition for this property is the following Lie algebraic assumption, cp. [22]: Let L denote the Lie-Algebra generated by the vector fields $f(x,u)$, $u \in U$. Let Δ_L be the distribution generated by L in TM. Then the condition

$$\mathrm{dim}\Delta_L(x) = \dim M \;\text{ for all }\; x \in M \tag{H}$$

ensures local accessibility of (2.1).

Under this condition we can cite the following lemma from [18].

Lemma 7.2 Consider the control system (2.1) satisfying (H).
Let $D \subset M$ be a control set and consider compact sets $K_1 \subset \mathbf{A}(D)$, $K_2 \subset \mathrm{int}D$. Then there exists a time $T > 0$ such that for every $x \in K_1$, $y \in K_2$ there exists a control function $u(\cdot) \in \mathcal{U}$ with $\varphi(t_0, x, u(\cdot)) = y$ for some $t_0 \leq T$.

Proof: See [18, Proposition 2.5]. □

The values of v_0 on the control sets can now be characterized in the following manner.

Proposition 7.3 Consider the system (2.1) satisfying (H). Consider $k \in \mathbb{N}$ control sets $D_1, \ldots, D_k \subset M$ with nonvoid interior with $D_i \leq D_j$ for all $1 \leq i \leq j \leq k$, i.e. the control sets are completely ordered by (7.1). Then for all $i, j \in \{1, \ldots, k\}$ with $i \leq j$ the inequalities

$$\overline{v_0}(x) \leq \overline{v_0}(y), \ \ \underline{v_0}(x) \leq \underline{v_0}(y) \text{ and } \underline{v_0^\infty}(x) \leq \underline{v_0^\infty}(y) \text{ for all } x \in \mathrm{int} D_i,\ y \in \mathrm{int} D_j$$

hold. In particular $\overline{v_0}|_{\mathrm{int} D_i} \equiv \mathit{const}$, $\underline{v_0}|_{\mathrm{int} D_i} \equiv \mathit{const}$ and $\underline{v_0^\infty}|_{\mathrm{int} D_i} \equiv \mathit{const}$ for all $i \in \{1, \ldots, k\}$.

Proof: Consider i, j and x, y as in the assumption. Then by Lemma 7.2 with $K_1 = \{y\}$, $K_2 = \{x\}$ there exists a control function $u(\cdot) \in \mathcal{U}$ with $\varphi(t, x, u(\cdot)) = y$ for some $t > 0$. Now fix $\varepsilon > 0$ and pick a control $u_y(\cdot) \in \mathcal{U}$ such that $\overline{J_0}(y, u_y(\cdot)) \leq \overline{v_0}(y) + \varepsilon$. By defining $u_x(s) := u(s)$ for all $s \leq t$ and $u_x(s) = u_y(s-t)$ for all $s > t$ we obtain by Lemma 3.7

$$\overline{J_0}(x, u_x(\cdot)) = \overline{J_0}(y, u_y(\cdot)) \leq \overline{v_0}(y) + \varepsilon$$

Since $\varepsilon > 0$ was arbitrary this proves the first assertion.

The second and third assertion follow by the same arguments. □

Again by using Lemma 7.2 we are now able to formulate the following result on the values of v_0^t on control sets with nonvoid interior.

Theorem 7.4 Consider the system (2.1) satisfying (H). Consider a control set $D \subset M$ with nonvoid interior. Then for any compact subset $K_1 \subset \mathrm{int}\mathbf{A}(D)$ and any $\varepsilon > 0$ there exists a time $t > 0$ such that

$$\overline{v_0^t}(x) \leq \overline{v_0}|_{\mathrm{int} D} + \varepsilon$$

and

$$\underline{v_0^t}(x) \leq \underline{v_0^\infty}|_{\mathrm{int} D} + \varepsilon$$

for all $x \in K_1$.

Furthermore for any compact subset $K_2 \subset \mathrm{int} D$ and any $\varepsilon > 0$ there exists a time $t > 0$ such that

$$|\overline{v_0^t}(x) - \overline{v_0^t}(y)| \leq \varepsilon \quad \text{and} \quad |\underline{v_0^t}(x) - \underline{v_0^t}(y)| \leq \varepsilon$$

for all x, $y \in K_2$.

Proof: Fix $\varepsilon > 0$. Pick an arbitrary point $x \in \mathrm{int} D$. Since $\overline{v_0}|_{\mathrm{int} D}$ is constant, there exists $u(\cdot) \in \mathcal{U}$ with $\overline{J_0}(x, u(\cdot)) < \overline{v_0}|_{\mathrm{int} D} + \varepsilon/4$, thus $\overline{J_0^t}(x, u(\cdot)) < \overline{v_0}|_{\mathrm{int} D} + \varepsilon/4$ for all $t \geq T_0$ for some $T_0 > 0$. Now by Lemma 7.2 there exists a time $T > 0$ such that for any point $y \in K_1$ there exists a control function $u_y(\cdot) \in \mathcal{U}$ such that $\varphi(t_0, y, u_y(\cdot)) = x$ for some $t \leq T$. Defining $\bar{u}_y(s) := u_y(s)$ for $s \in [0, t_0]$ and $\bar{u}_y(s) := u(s - t_0)$ for $s \geq t$ yields by Lemma 3.7(ii) $\overline{J_0^t}(y, \bar{u}_y) < v_0|_{\mathrm{int} D} + \varepsilon$ for all $t > T_1$ for some sufficiently large $T_1 > 0$ depending on T and T_0. This yields the first assertion.

For x, $y \in K_2$ the same construction yields $\overline{v_0^t}(y) < \overline{v_0^t}(x) + \varepsilon$ and also $\overline{v_0^t}(x) < \overline{v_0^t}(y) + \varepsilon$ which implies the third assertion.

The second and fourth assertion follow by the same arguments. □

For trajectories staying inside some control set D we are now able to give a criterion for the equality of $\overline{v_0}|_{\mathrm{int}D}$ and $\underline{v_0}|_{\mathrm{int}D}$ and the uniform convergence of $\overline{v_0^t}$.

Corollary 7.5 Consider the system (2.1) satisfying (H). Consider a control set $D \subset M$ with nonvoid interior. Assume that $\inf_{x\in\mathrm{int}D} \overline{v_0}(x, D) = \inf_{x\in\mathrm{cl}D} \overline{v_0}(x, \mathrm{cl}D)$. Then

$$\overline{v_0^t}(\cdot, \mathrm{cl}D) \to \overline{v_0}(\cdot, D)|_{\mathrm{int}D} = \underline{v_0^\infty}(\cdot, D)|_{\mathrm{int}D}$$

uniformly on compact subsets of $\mathrm{int}D$ for $t \to \infty$.

Proof: This follows from Theorem 7.4 (which also applies if we require the trajectories to stay inside D) and Proposition 5.3 since $\mathrm{cl}D$ is a compact viable set. □

We end this section by considering the special case of an *invariant* control set. Note that under the assumption (H) any invariant control set has nonvoid interior, cf. [14]. Here the conditions from Corollary 7.5 are not necessary. Basically this result is a reformulation of [18, Theorem 2.11].

Corollary 7.6 Consider the system (2.1) satisfying (H). Consider an invariant control set $C \subset M$, i.e. a control set C satisfying $\varphi(t, x, u(\cdot)) \in C$ for all $x \in C$, all $u(\cdot) \in \mathcal{U}$ and all $t \geq 0$. Then

$$\sup_{x\in\mathrm{int}C} \overline{v_0^t}(x) \geq \overline{v_0}|_{\mathrm{int}C} \quad \text{and} \quad \sup_{x\in\mathrm{int}C} \underline{v_0^t}(x) \geq \underline{v_0^\infty}|_{\mathrm{int}C}$$

for all $t \geq 0$ and

$$\overline{v_0^t} \to \overline{v_0}|_{\mathrm{int}C} = \underline{v_0^\infty}|_{\mathrm{int}C}$$

uniformly on compact subsets of the interior of C for $t \to \infty$.

Proof: The invariance of C implies that also $\mathrm{int}C$ is invariant. Hence $\overline{v_0}(x, \mathrm{int}C) = \overline{v_0}(x)$ and $\overline{v_0^t}(x, \mathrm{int}C) = \overline{v_0^t}(x)$ for all $x \in \mathrm{int}C$ and all $t \geq 0$. Hence the first assertion follows from Proposition 5.2, and the second by the same arguments.

The convergence $\overline{v_0^t} \to \overline{v_0}|_{\mathrm{int}C}$ then is an easy consequence from the first assertion and Theorem 7.4.

It remains to show the equality $\overline{v_0}|_{\mathrm{int}C} = \underline{v_0^\infty}|_{\mathrm{int}C}$. From [9, Corollary 4.3] it follows that there exist periodic trajectories corresponding to periodic control functions that are approximately optimal with respect to $\underline{v_0^\infty}|_{\mathrm{int}C}$. Clearly, for periodic trajectories $\overline{J_0}$ and $\underline{J_0}$ coincide, hence the assertion follows. □

8 The complete picture

What we have obtained so far are

(i) A uniform lower bound for $\underline{v_0^t}$ related to chain control sets by Theorem 6.4

(ii) A uniform upper bound for $\overline{v_0^t}$ related to control sets by Theorem 7.4

(iii) A convergence result for trajectories staying inside control sets by the Corollaries 7.5 and 7.6

Here the estimate for the infimum and the supremum on viable sets from Proposition 5.3 and Proposition 5.2 have been used as ingredients for these results.

The aim of this section is now to combine the results (i)–(iii) in order to obtain uniform convergence results on a partition of the state space M.

For this purpose we establish a relation between the control sets and the chain control sets of the system (2.1). Clearly, for every control set D there exists a unique chain control set E such that $\mathrm{cl}D \subseteq E$ and $\mathrm{cl}\mathbf{A}(D) \subseteq \mathrm{cl}\mathbf{A}(E)$.

If $\mathrm{cl}D = E$ holds, Theorem 6.4 immediately implies

$$\inf_{x\in \mathrm{cl}D} \overline{v_0}(x, \mathrm{cl}D) = \kappa^*(E)$$

The following lemma shows how the desired partition can be constructed.

Lemma 8.1 Consider the control system (2.1). Assume that there exist finitely many chain control sets $E_1, \ldots, E_d$. Then the sets

$$B_i := \mathbf{A}(E_i) \setminus \left(\bigcup_{E_j \not\geq E_i} \mathbf{A}(E_j) \right) \quad \text{for all} \quad i \in \{1, \ldots, d\} \tag{8.1}$$

form a partition of M, i.e. they are pairwise disjoint and $\bigcup_{i=1,\ldots,d} B_i = M$.

Proof: Since $j < i$ implies $E_j \not\geq E_i$ it follows that $B_j \cap B_i = \emptyset$ for all $j < i$. This implies that the B_i are disjoint.

It remains to show that any $x \in M$ lies in some B_i. Let $\mathcal{M}_i$ denote the lift of E_i. By Lemma 4.11 it follows that $\bigcup_{i=1,\ldots,k} \mathbf{A}(\mathcal{M}_i) = \mathcal{U} \times M$. Hence any x lies inside some $\mathbf{A}(E_i)$ for at least one $i \in \{1, \ldots, k\}$ and thus in some B_i. $\square$

We have defined the order of the chain control sets (6.1) using the order of the corresponding Morse sets and the order of the control sets (7.1) via their domains of attraction, hence these orders will not coincide in general. The following lemma shows how an equivalence of these orders can be established.

Lemma 8.2 Consider the control system (2.1) satisfying (H). Assume that there exist exactly finitely many chain control sets $E_1, \ldots, E_d$ and control sets $D_1, \ldots, D_d$ with nonvoid interior satisfying $\mathrm{cl}D_i = E_i$ and $\mathrm{cl}\mathbf{A}(D_i) = \mathrm{cl} \bigcup_{E_l \leq E_i} \mathbf{A}(E_l)$ for all $i \in \{1, \ldots, d\}$. Then

$$E_i < E_j \iff D_i < D_j$$

i.e. the orders of the chain control sets and the control sets coincide.

Proof: Clearly $D_i < D_j$ implies $E_i < E_j$ without any assumptions. Hence assume $E_i < E_j$. Clearly this implies $E_i \subset \bigcup_{E_l \le E_i} \mathbf{A}(E_l)$. Hence $\mathrm{int} D_i = \mathrm{int} E_i \subset \mathrm{int} \bigcup_{E_l \le E_i} \mathbf{A}(E_l) = \mathrm{int}\mathbf{A}(D_j)$ which yields $D_i < D_j$. $\square$

Remark 8.3 Note that $\mathbf{A}(D_i) \supset \bigcup_{D_l \le D_i} \mathbf{A}(D_l)$ follows from (7.1). Hence by looking at (6.1) the assumption of Lemma 8.2 demands that arbitrary small jumps at the boundaries of the chain control sets can be closed by trajectories. Note that this is much weaker than assuming that *every chain* can be closed, i.e. approximated by a trajectory.

We are now able to give the complete picture for the averaged functionals using the partition defined by the sets B_i.

Theorem 8.4 Consider the control system (2.1) satisfying (H). Assume that there exist exactly finitely many chain control sets $E_1, \ldots, E_d$ and control sets $D_1, \ldots, D_d$ with nonvoid interior satisfying $\mathrm{cl} D_i = E_i$ and $\mathrm{cl}\mathbf{A}(D_i) = \mathrm{cl} \bigcup_{E_l \le E_i} \mathbf{A}(E_l)$ for all $i \in \{1, \ldots, d\}$. Assume furthermore that

$$\inf_{x \in \mathrm{int} D_i} \overline{v_0}(x, D_i) = \kappa^*(E_i) =: \kappa_i^* \ \text{ for all } \ i \in \{1, \ldots, d\}$$

and let

$$\lambda_i = \min\{\kappa_l^* \mid l = i, \ldots d, \ D_l \ge D_i\}$$

for $i = 1, \ldots, d$ using the ordering (7.1) of the control sets.

Then the following properties hold for the sets B_i from (8.1) and all $i \in \{1, \ldots, d\}$.

$$\overline{v_0}|_{B_i} \equiv \underline{v_0}|_{B_i} \equiv \lambda_i \text{ on } \mathrm{int} B_i$$

and

$$\overline{v_0^t} \to \lambda_i \ \text{ and } \ \underline{v_0^t} \to \lambda_i$$

uniformly on compact subsets of $\mathrm{int} B_i$ for $t \to \infty$.

Proof: By Proposition 7.3 and the assumptions on $\overline{v_0}(x, D_i)$ it follows that $\overline{v_0}(x) \le \lambda_i$ for all $x \in \mathrm{int} D_i$. Now let $K \subset \mathrm{int} B_i$ be a compact set. Then by Theorem 7.4 for any $\alpha > 0$ there exists $t > 0$ such that

$$\overline{v_0^t}(x) \le \lambda_i + \alpha$$

for all $x \in K$. Conversely Theorem 6.4 implies for all sufficiently large $t > 0$

$$\underline{v_0^t}(x) \ge \lambda_i - \alpha$$

for all $x \in K$, since the orders of the chain control sets and the control sets agree by Lemma 8.2. These two inequalities imply the assertions. $\square$

Remark 8.5 The assumptions of Theorem 8.4 can be interpreted as *robustness conditions* on the values of $\overline{v_0}$ and $\overline{v_0}(\cdot, D)$ on the interior of the control sets D:

The first assumption is a condition on the control system itself; it is equivalent to the continuous dependence of the control sets and their domains of attraction under all arbitrarily small perturbations. If this is violated there will be at least one control set D for which the positive orbit $O^+(D)$ will change discontinuously (w.r.t. the Haussdorff metric) under suitable arbitrary small perturbations, and thus for appropriately chosen cost functions g also the value of $\overline{v_0}$ on $\mathrm{int}D$ will change discontinuously. So conversely if the values of $\overline{v_0}$ on the interior of all control sets D and for all cost functions g change continuously for all arbitrary small perturbations acting on the system the first assumption is implied.

The second assumption is a condition on the value function and thus on the control system *and* the given cost function g. If it does not hold there exist arbitrary small perturbations of the system, such that the value $\overline{v_0}(\cdot, D)$ on the interior of D becomes equal or less that the value on the boundary of D and thus changes discontinuously. So this second assumption is implied by the continuous dependence of $\overline{v_0}(\cdot, D)$ for the given g for all arbitrarily small perturbations.

Note that these sufficient robustness conditions are are in general not necessary, i.e. the assumptions of the theorem are weaker.

Remark 8.6 If we assume that the set of admissible control values U is convex and $0 \in \mathrm{int}U$, we can consider

$$U^\rho := \{\rho u \mid u \in U\}$$

for all real values $\rho > 0$, and denote the corresponding sets of control functions by $\mathcal{U}^\rho$. This defines a family of control systems $(2.1)^\rho$. Then the assumptions of Theorem 8.4 are satisfied generically for $(2.1)^\rho$ (i.e. for all except at most countably many $\rho > 0$), provided $(2.1)^\rho$ satisfies a $\rho - \rho'$ inner pair condition as defined in [14] or [18]. This follows from the fact that under this condition the chain control sets, control sets and values $\overline{v_0}|_{\mathrm{int}D_i}$ depend monotonically on ρ, and hence can only exhibit at most countably many discontinuity points. A rigorous proof of this property can be obtained with exactly the same arguments as the proof for the relation between the Floquet, Lyapunov and Morse spectra in [13, Corollary 5.6] or [14].

We will briefly state the consequences of this central theorem for the discounted optimal value functions.

Corollary 8.7 Under the assumptions of Theorem 8.4 it holds that

$$v_\delta \to \lambda_i$$

uniformly on compact subsets of $\mathrm{int}B_i$ for $\delta \to 0$.

Proof: Follows immediately from Theorem 8.4 and Corollary 3.5. □

Remark 8.8 Note that v_δ is uniformly continuous for each $\delta > 0$. Hence uniform convergence can only be expected away from the boundaries of the B_i since the limit function $\overline{v_0}$ is discontinuous at these boundaries. From the point of view of uniform convergence Corollary 8.7 therefore gives the strongest possible result.

9 Restriction of the state space

In this section we will discuss a technique for the restriction of the state space. More precisely we are interested in the following problem: Assume that a control system (2.1) and a cost function g are given satisfying the assumption of Theorem 8.4 at least for some control sets.

For these control sets we consider the restriction to the intersection B of their positive and negative reachable sets. Then we are interested in $\overline{v_0^t}(\cdot, B)$, $\underline{v_0^t}(\cdot, B)$, $\overline{v_0}(\cdot, B)$ and $\underline{v_0^\infty}(\cdot, B)$ and in the relation of these functions, consequently also in $v_\delta(\cdot, B)$ for small $\delta > 0$.

However, for two reasons we do not want to introduce an explicit restriction of the state space. One the one hand we may not know B exactly but will only have a characterization of B in terms of control sets. On the other hand explicit restriction of the state space causes certain technical difficulties in the theory (see e.g. [26]) and in particular in the numerical calculation of optimal value functions (cp. [16]).

Hence we propose a penalization technique based on a transformation of the cost function g which is done as follows.

Definition 9.1 For a finite sequence $D_{l_1}, \dots D_{l_2}$ of control sets we define

$$B := \mathrm{cl}\left(\bigcup_{i=l_1,\dots,l_2} O^+(D_i)\right) \cap \mathrm{cl}\left(\bigcup_{i=l_1,\dots,l_2} \mathbf{A}(D_i)\right)$$

For a compact set K_1 with $B \subset \mathrm{int} K_1$ we denote by $\tilde{g}_{K_1} : M \times U \to \mathbb{R}$ a Lipschitz continuous function satisfying

$$\tilde{g}_{K_1}(x,u) = \begin{cases} g(x,u) & x \in B \\ \text{arbitrary in } [-M_g, M_g] & x \in K_1 \setminus B \\ M_g & x \notin K_1 \end{cases}$$

where $M_g := \sup_{x \in B, u \in U} |g(x,u)|$.

Again we define a partition of B similar to (8.1).

Definition 9.2 Let $E_{l_1}, \dots E_{l_2} \subset B$ be chain control sets with $\mathrm{cl} D_i = E_i$ for all $i = l_1, \dots, l_2$. Then we define $\mathbf{A}_B(E_i) := \mathbf{A}(E_i) \cap B$ for all $i = 1, \dots, d$ and

$$B_i := \mathbf{A}_B(E_i) \setminus \left(\bigcup_{E_j \not\succeq E_i} \mathbf{A}_B(E_j)\right) \quad \text{for all} \quad i \in \{1, \dots, d\}$$

Note that the structure of B implies that no trajectory can leave B and enter again, hence $\mathbf{A}_B(E_i)$ can be regarded as the domain of attraction of E_i relative to B.

Now we can formulate the result for the optimal control problem using $\tilde{g}$.

Theorem 9.3 Consider the system (2.1) satisfying (H) with cost function g on a compact manifold M. Assume that the system has finitely many chain control sets. Let $D_{l_1}, \ldots D_{l_2}$ be control sets with nonvoid interior and consider B from Definition 9.1. Assume that $D_{l_1}, \ldots D_{l_2}$ are all control sets in B and that there are exactly d chain control sets $E_{l_1}, \ldots, E_{l_2}$ with $E_i \subset B$ for all $i \in \{l_1, \ldots, l_2\}$ satisfying the assumptions from Theorem 8.4 with infimal chain values κ_i^*. Let

$$\lambda_i = \min\{\kappa_l^* \mid l = i, \ldots l_2,\ D_l \geq D_i\}$$

for $i = l_1, \ldots, l_2$ using the ordering (7.1) of the control sets.

Then there exists a compact set $K_1 \subset M$ with $B \subset \mathrm{int} K_1$, such that for the value functions $\tilde{v}$ for the cost function $\tilde{g}_{K_1}$ from Definition 9.1 the following properties hold for the sets B_i from Definition 9.2 and all $i \in \{l_1, \ldots, l_2\}$.

$$\overline{\tilde{v}_0}|_{B_i} \equiv \underline{\tilde{v}_0^\infty}|_{B_i} \equiv \lambda_i \text{ on } \mathrm{int} B_i$$

and

$$\overline{\tilde{v}_0^t} \to \lambda_i \text{ and } \underline{\tilde{v}_0^t} \to \lambda_i$$

uniformly on compact subsets of $\mathrm{int} B_i$ for $t \to \infty$. In particular this implies $\overline{\tilde{v}_0}|_{B_i} = \overline{v_0}(\cdot, B)|_{B_i}$ and $\underline{\tilde{v}_0^\infty}|_{B_i} = \underline{v_0^\infty}(\cdot, B)|_{B_i}$, hence the approximation of the restricted problem.

Proof: Fix a compact subset $K \subset \mathrm{int} B_i$. We prove the theorem by showing that for any $\alpha > 0$ there exists $t > 0$ such that the inequalities

$$\overline{\tilde{v}_0^t}(x) \leq \lambda_i + \alpha$$

and

$$\underline{\tilde{v}_0^t}(x) \geq \lambda_i - \alpha$$

hold for any $x \in K$.

"$\leq$" follows with the same arguments as in the proof of Theorem 8.4 for arbitrary compact sets K_1 with $B \subset \mathrm{int} K_1$ (note that B is viable).

"$\geq$" Consider the lifts $\mathcal{E}_j$ of the chain control sets E_j. Then there exists a unique smallest attractor A_{j_1} such that $\mathcal{E}_j \subset A_{j_1}$ for all $j = 1, \ldots, d$ and a unique smallest repeller A_{j2}^* with $\mathcal{E}_j \subset A_{j2}^*$ for all $j = 1, \ldots, d$.

Lemma 4.11 applied to the original and the time reversed system and the assumptions on the reachable sets imply $\mathcal{B} = A_{j_1} \cap A_{j_2}^*$ for the lift $\mathcal{B}$ of B.

Hence we may choose K_1 such that $\mathrm{cl} B \subset K_1$, $K_1 \cap \pi_M A_{j_1}^* = \emptyset$ and $K_1 \cap \pi_M A_{j_2} = \emptyset$. This implies that $\tilde{v}_0(\cdot, E) \equiv M_g$ on all chain control sets E with $E \cap B = \emptyset$.

Hence the inequality follows from Theorem 6.4. □

Remark 9.4 It is obvious that a change of the norm of f (i.e. a change of the speed of the trajectories) outside K_1 does not affect this result. In particular by choosing some compact set K_2 with $K_1 \subset \mathrm{int} K_2$ and some Lipschitz continuous function $\alpha : M \to \mathbb{R}$ with

$$\alpha(x) = \begin{cases} 1 & x \in K_1 \\ \text{arbitrary in } [0,1] & x \in K_2 \setminus K_1 \\ 0 & x \notin K_2 \end{cases}$$

we can replace $f(x,u)$ by $\alpha(x)f(x,u)$ and the convergence result remains valid. Here K_2 becomes a compact invariant set.

Remark 9.5 For systems on a non-compact state space M this enables us to focus on a compact subset $K_2 \subset M$, provided there exists a repeller for the corresponding control flow playing the role of A^*_{j2} from the proof above. (The results of Hurley [21] for discrete time flows suggest that in general such a repeller exists.) Under this condition all results remain valid for systems with non compact state space. However, in that case the considered chain control sets must be compact and their number must be finite. A corresponding theory for non-compact or infinitely many chain control sets has not yet been developed.

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