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# Directed Sets and Differences of Convex Compact Sets[3]

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## 1 Introduction

A linear normed and partially ordered space is introduced, in which the convex cone of all nonempty convex compact sets in $\mathbb{R}^n$ is embedded. This space of so-called "directed sets" is a Banach and a Riesz space for dimension $n \geq 2$ and a Banach lattice for $n = 1$.

We use essentially the specific parametrization of convex compact sets via their support functions and consider the supporting faces as lower dimensional convex sets. Extending this approach, we define a directed set as a pair of mappings that associate to each unit direction a $(n-1)$-dimensional directed set ("directed supporting face") and a scalar function determining the position of this face in $\mathbb{R}^n$. This method provides recursive definitions, constructions and inductive proofs as well as a visualization of differences of general convex sets with oriented boundary parts.

The basic differences of our approach to other existing embeddings are that there are no equivalence classes (as in [13], [15]) and secondly, that differences of directed convex sets in $\mathbb{R}^n$ are not real-valued functions of $n$ arguments as in [5], but higher-dimensional maps representable as oriented manifolds, e.g. oriented curves/surfaces in the cases $n = 2, 3$. For nonconvex polygons in $\mathbb{R}^2$ see [3] in which an interesting computational-geometric method of polygonal tracings is presented (this approach has been recently extended to polyhedrals in $\mathbb{R}^3$).

The approach is based on the notions of generalized ([6], [11]) or directed intervals ([8], [9]) in the one-dimensional case. In the $n-$dimensional case, there are essential differences, namely a mixed type part appears which does not exist in the case $n = 1$.

As an application we give an example of set-valued interpolation where nonconvex visualizations of directed sets appear as results.

**Basic Notations**

Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all convex, compact, nonempty subsets of $\mathbb{R}^n$. The following operations in $\mathcal{C}(\mathbb{R}^n)$ are well-known:

$$\begin{aligned} A + B &:= \{a + b \,|\, a \in A,\ b \in B\} && \text{(Minkowski addition)} \\ \lambda \cdot A &:= \{\lambda \cdot a \,|\, a \in A\} && \text{(scalar multiplication for } \lambda \in \mathbb{R}) \end{aligned} \tag{1}$$

Each convex, compact, nonempty set $A$ could be described via its support function $\delta^*(l, A) := \max\limits_{a \in A} < l, a >$ and reconstructed via the intersection of half-spaces with outer normal $l \in S_n$ ($S_n$ is the unit sphere in $\mathbb{R}^n$):

$$A = \bigcap_{l \in S_n} \{x \in \mathbb{R}^n \mid \ < l, x > \leq \delta^*(l, A)\}$$

The support function for $A \in \mathcal{C}(\mathbb{R}^n)$ is Lipschitz-continuous and fulfills

$$\delta^*(l, A+B) = \delta^*(l, A) + \delta^*(l, B), \qquad \delta^*(l, \lambda \cdot A) = \lambda \cdot \delta^*(l, A) \quad (\lambda \geq 0).$$

The Hausdorff-distance between two sets in $\mathcal{C}(\mathbb{R}^n)$ could be expressed via the difference of support functions:

$$\mathrm{d_H}(A, B) = \max_{l \in S_n} |\delta^*(l, A) - \delta^*(l, B)|$$

The supporting face (the set of supporting points) for the direction $l \in S_n$ is

$$Y(l, A) := \{y(l, A) \in A \,|\, <l, y(l, A)> = \delta^*(l, A)\}.$$

Some of the definitions of differences of sets which are known in the literature and which are not discussed in the beginning are listed below:

- algebraic difference $A - B := \{a - b \,|\, a \in A,\ b \in B\}$
  It is not useable in our context, since in general $A - A \underset{\neq}{\supset} \{0_{\mathbb{R}^n}\}$.
- differences of intervals
  Classical interval arithmetic uses the algebraic difference (cf. [10], ...), whereas the definition of the subtraction in the space of generalized intervals (cf. [6], [11]) resp. directed intervals (cf. [8], [9]) is specified by the subtraction of the corresponding end points of the intervals.
- Minkowski difference in [4], better known as geometric or star-shaped difference

  $$A \stackrel{*}{-} B := \{x \in \mathbb{R}^n \,|\, x + B \subset A\}$$

  This difference has the property that $A \stackrel{*}{-} A = \{0\}$, but may often be empty.
- Demyanov's difference in [14]

  $$A \dot{-} B := \overline{\mathrm{co}}\{y(l, A) - y(l, B) \,|\, l \in S_n,\ Y(l, A) \text{ and } Y(l, B) \text{ are singletons}\}$$

  The difference $A \ddot{-} B$ in [14] is always a superset of Demyanov's difference $A \dot{-} B$. There is a close connection between Demyanov's difference and the boundary mapping of the difference of directed sets (cf. Proposition 3.10).

# 2 Directed Intervals

In interval analysis, $\mathcal{I}(\mathbb{R})$ denotes the set of all real compact intervals

$$[a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}.$$

The operations $* \in \{+, -, \cdot, /\}$ known from $\mathbb{R}$ are generalized to the interval case by

$$[a, b] * [c, d] = \{x * y \,|\, x \in [a, b],\ y \in [c, d]\}.$$

Since the difference is the algebraic difference of intervals, $(\mathcal{I}(\mathbb{R}), +, \cdot)$ is only an Abelian semigroup and not a vector space.
In [6], [11] and [8], [9] generalized resp. directed intervals $\{[\alpha, \beta] \,|\, \alpha, \beta \in \mathbb{R}\}$ are studied for which the left end point could be greater than the right one. The isomorphism $[\alpha, \beta] \mapsto (\alpha, \beta) \in \mathbb{R}^2$ induces operations/definitions for generalized (directed) inter-

vals. The notion of directed intervals introduced here is in principle equivalent to the generalized intervals of Kaucher ([6]) and the directed intervals of Markov ([8], [9]). Our definition is slightly different, since it is based on support functions and its scalar multiples.

Every interval in $\mathcal{I}(\mathbb{R})$ is convex, compact with support function

$$a_1(l) := \delta^*(l, [a,b]) = \max\{l \cdot a, l \cdot b\} \qquad (l = \pm 1).$$

**Definition 2.1** *A directed interval* $\overrightarrow{A}$ *consists of a function* $a_1 : \{\pm 1\} \to \mathbb{R}$, *i.e.*

$$\overrightarrow{A} = (a_1(l))_{l=\pm 1} = (a_1(-1), a_1(1)) \in \mathbb{R}^2$$

*The notation* $\overrightarrow{[\alpha, \beta]} := (-\alpha, \beta)$, *where* $\alpha = a_1(-1)$, $\beta = a_1(1)$, *is often used. Let* $\mathcal{D}(\mathbb{R})$ *denote the set of all directed intervals. The operations in* $\mathcal{D}(\mathbb{R})$ *are defined as follows:*

| | | | |
|---|---|---|---|
| $\overrightarrow{A} + \overrightarrow{B}$ | $:=$ | $(a_1(l) + b_1(l))_{l=\pm 1}$ | addition |
| $\lambda \cdot \overrightarrow{A}$ | $:=$ | $(\lambda \cdot a_1(l))_{l=\pm 1} \quad (\lambda \in \mathbb{R})$ | scalar multiplication |
| $\overrightarrow{A} - \overrightarrow{B}$ | $:=$ | $(a_1(l) - b_1(l))_{l=\pm 1}$ | subtraction |
| $\Vert\overrightarrow{A}\Vert$ | $:=$ | $\max\limits_{l=\pm 1} \lvert a_1(l) \rvert$ | norm |
| $\overrightarrow{A} \leq \overrightarrow{B}$ | $:\Longleftrightarrow$ | $a_1(l) \leq b_1(l)$ for $l = \pm 1$ | partial ordering |
| $\sup\{\overrightarrow{A}, \overrightarrow{B}\}$ | $:=$ | $(c_1(l))_{l=\pm 1}$ with $c_1(l) = \max\{a_1(l), b_1(l)\}$ | supremum |
| $\inf\{\overrightarrow{A}, \overrightarrow{B}\}$ | $:=$ | $-\sup\{-\overrightarrow{A}, -\overrightarrow{B}\}$ | infimum |

Note that multiplication by negative scalars and subtraction are identical to the corresponding operations on vectors in $\mathbb{R}^2$ and differ from the standard interval operations (as in [10]). The space of directed intervals is isomorphic to the space of generalized intervals, so that according to [6] the following properties of $\mathcal{D}(\mathbb{R})$ could be stated.

**Theorem 2.2** $(\mathcal{D}(\mathbb{R}), +, \cdot)$ *is a vector space with the inverse* $-\overrightarrow{A} = (-a_1(l))_{l=\pm 1}$ *and the subtraction defined in the table above. Furthermore, it is a a Banach space with the norm* $\Vert \cdot \Vert$, *a Banach lattice with the partial ordering "*$\leq$*" as well as a Riesz space.*

**Example 2.3** *Subtraction of embedded intervals gives* $\overrightarrow{[a,b]} - \overrightarrow{[c,d]} = \overrightarrow{[a-c, b-d]}$.

$$\overrightarrow{[-1,2]} - \overrightarrow{[-3,5]} = \overrightarrow{[2,-3]} \quad \textit{and} \quad \overrightarrow{[-3,5]} - \overrightarrow{[-1,2]} = \overrightarrow{[-2,3]}$$

*The results as well as an improper interval (the inverse of an embedded interval, also called proper interval) and an embedded scalar are visualized in Figure 2.1 resp. 2.2:*

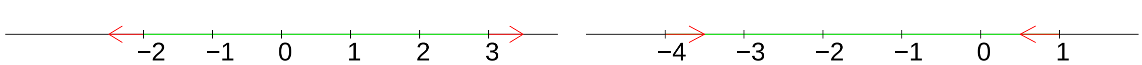


*Fig. 2.1: proper interval* $\overrightarrow{[-2,3]} \in \mathcal{D}(\mathbb{R})$ *resp. improper interval* $\overrightarrow{[1,-4]} \in \mathcal{D}(\mathbb{R})$


*Fig. 2.2: the inverse* $\overrightarrow{[2,-3]}$ *of* $\overrightarrow{[-2,3]}$ *resp. degenerate interval* $\overrightarrow{[1,1]}$

# 3 Directed Sets

We construct inductively the linear normed space $\mathcal{D}(\mathbb{R}^n)$ of directed sets in $\mathbb{R}^n$.

**Definition 3.1** $\overrightarrow{A}$ *is called a directed set*

*(i) in* $\mathbb{R}$*, if it is a directed interval and* $\|\overrightarrow{A}\|_1 := \max\limits_{l=\pm 1} |a_1(l)|$,

*(ii) in* $\mathbb{R}^n$, $n \geq 2$*, if there exists a continuous function* $a_n : S_n \to \mathbb{R}$ *and a uniformly bounded function* $\overrightarrow{A_{n-1}} : S_n \to \mathcal{D}(\mathbb{R}^{n-1})$ *with respect to* $\|\cdot\|_{n-1}$.

*Then, we denote* $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l\in S_n}$ *and define*

$$\|\overrightarrow{A}\| := \|\overrightarrow{A}\|_n := \max\{\sup_{l\in S_n} \|\overrightarrow{A_{n-1}(l)}\|_{n-1}, \max_{l\in S_n} |a_n(l)|\}, \qquad |\!|\!|\overrightarrow{A}|\!|\!| := \max_{l\in S_n} |a_n(l)|$$

*The set of all directed sets in* $\mathbb{R}^n$ *is denoted by* $\mathcal{D}(\mathbb{R}^n)$.

The definition above is motivated by describing the convex, compact, nonempty set $A$ for each direction $l \in S_n$ as a pair

$$(\text{"}Y(l,A) \text{ as } (n-1) \text{ dimensional (directed) set"}, \delta^*(l,A))_{l\in S_n}.$$

Each operation is defined recursively and works separately on both components.

**Definition 3.2** *Let* $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l\in S_n}$, $\overrightarrow{B} = (\overrightarrow{B_{n-1}(l)}, b_n(l))_{l\in S_n}$.

$$\boxed{\begin{array}{rcl}
\overrightarrow{A} + \overrightarrow{B} & := & (\overrightarrow{A_{n-1}(l)} + \overrightarrow{B_{n-1}(l)}, a_n(l) + b_n(l))_{l\in S_n} \\
\lambda \cdot \overrightarrow{A} & := & (\lambda \cdot \overrightarrow{A_{n-1}(l)}, \lambda \cdot a_n(l))_{l\in S_n} \qquad\qquad (\lambda \in \mathbb{R}). \\
\overrightarrow{A} - \overrightarrow{B} & := & \overrightarrow{A} + (-\overrightarrow{B}) = (\overrightarrow{A_{n-1}(l)} - \overrightarrow{B_{n-1}(l)}, a_n(l) - b_n(l))_{l\in S_n} \\
\overrightarrow{A} \leq \overrightarrow{B} :\iff & & \left\{\begin{array}{lll} \text{(i)} & \forall l \in S_n : & a_n(l) \leq b_n(l) \\ \text{(ii)} & \text{if } \exists l \in S_n \text{ with} & a_n(l) = b_n(l), \\ & \qquad\qquad \text{then} & \overrightarrow{A_{n-1}(l)} \leq \overrightarrow{B_{n-1}(l)} \end{array}\right.
\end{array}}$$

$$\boxed{\begin{array}{l}
\sup\{\overrightarrow{A}, \overrightarrow{B}\} := (\overrightarrow{S_{n-1}(l)}, \max\{a_n(l), b_n(l)\})_{l\in S_n} \\
\qquad \overrightarrow{S_{n-1}(l)} := \left\{\begin{array}{ll} \overrightarrow{B_{n-1}(l)} & \text{if } a_n(l) < b_n(l) \\ \sup\{\overrightarrow{A_{n-1}(l)}, \overrightarrow{B_{n-1}(l)}\} & \text{if } a_n(l) = b_n(l) \\ \overrightarrow{A_{n-1}(l)} & \text{if } a_n(l) > b_n(l). \end{array}\right. \\
\inf\{\overrightarrow{A}, \overrightarrow{B}\} := -\sup\{-\overrightarrow{A}, -\overrightarrow{B}\}
\end{array}}$$

**Proposition 3.3** *$(\mathcal{D}(\mathbb{R}^n), +, \cdot)$ is a vector space with the zero element $\overrightarrow{0_{\mathcal{D}(\mathbb{R}^n)}} = (\overrightarrow{0_{\mathcal{D}(\mathbb{R}^{n-1})}}, 0_{\mathbb{R}})_{l \in S_n}$ and the inverse of $\overrightarrow{A}$, $-\overrightarrow{A} = (-\overrightarrow{A_{n-1}(l)}, -a_n(l))_{l \in S_n}$.*

**Proposition 3.4** *$(\mathcal{D}(\mathbb{R}^n), \|\cdot\|)$ is a Banach space and $\|\!|\cdot\|\!|$ is a semi-norm. It is even a lattice and a Riesz space with the ordering and supremum/infimum in Definition 3.2.*

One may interpret the supporting face $Y(l, A)$ as $(n-1)$-dimensional (directed) set, e.g. by the following procedure:

- translate the hyperplane which is orthogonal to $l$ and contains $Y(l, A)$ to the origin by the vector $\delta^*(l, A)l$
- rotate the result into the plane $\{x_n = 0\}$ until the attached orthogonal vector $l$ coincides with $e^n$
- project the rotated image of $Y(l, A)$ into $\mathbb{R}^{n-1}$
- embed the result in the space $\mathcal{D}(\mathbb{R}^{n-1})$

**Definition 3.5** *The set $A \in \mathcal{C}(\mathbb{R}^n)$ is embedded into the set $\mathcal{D}(\mathbb{R}^n)$ via $J_n : \mathcal{C}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$:*

*(i)* $J_1([a, b]) := \overrightarrow{[a, b]} = (-a, b)$ *for* $n = 1$

*(ii)* $J_n(\overrightarrow{A}) := (J_{n-1}(P_{n-1,l}(Y(l, A))), \delta^*(l, A))_{l \in S_n}$ *for* $n \geq 2$
$P_{n-1,l}(x) := \pi_{n-1,n} R_{n,l}(x - \delta^*(l, A)l)$ *and* $\pi_{n-1,n}$ *is the projection from* $\mathbb{R}^n$ *to* $\mathbb{R}^{n-1}$, $R_{n,l}$ *is a rotation matrix which satisfies for the unit vectors* $e^1, \ldots, e^n$

$$R_{n,l}(l) = e^n, \qquad R_{n,l}(span\{l\}^{\perp}) = span\{e^1, e^2, \ldots, e^{n-1}\} \tag{2}$$

*and must be uniquely defined for the embedding. A possible construction is skipped due to the lack of space, only the properties in (2) are used in the proofs.*

To define the visualization of a directed set, the convex and the concave part of a directed set are defined.

**Definition 3.6** *Let $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$. The definition of its convex and concave part are:*

$$\begin{aligned} P_n(\overrightarrow{A}) &:= \{x \in \mathbb{R}^n \mid \textit{for every } l \in S_n : \langle l, x \rangle \leq a_n(l)\}, \\ N_n(\overrightarrow{A}) &:= \{-x \in \mathbb{R}^n \mid \textit{for every } l \in S_n : \langle l, x \rangle \leq -a_n(l)\} \end{aligned}$$

At least one of the convex and the concave part of $\overrightarrow{A}$ is empty, except the case that both are equal and contain only one point. It could happen that both of them are empty and the set coincides with the mixed type part defined in Definition 3.7, but in the one-dimensional case, exactly one of $P_1(\overrightarrow{[a, b]})$ and $N_1(\overrightarrow{[a, b]})$ is empty, if $a \neq b$.

**Definition 3.7** *Let $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$. The visualization $V_n : \mathcal{D}(\mathbb{R}^n) \Rightarrow \mathbb{R}^n$ consists of three parts, the convex and concave part as well as the mixed type part $M_n(\overrightarrow{A})$. $M_n(\overrightarrow{A})$ collects all reprojected points from the visualization of the boundary parts $\overrightarrow{A_{n-1}(l)}$ which are not elements of the other two parts. Both sets are defined simultaneously:*

$$\begin{aligned}
M_1(\overrightarrow{A}) &:= \emptyset, \qquad V_1(\overrightarrow{A}) := P_1(\overrightarrow{A}) \cup N_1(\overrightarrow{A}) && (n = 1) \\
M_n(\overrightarrow{A}) &:= \bigcup_{l \in S_n} \{x \in Q_{n,l}(V_{n-1}(\overrightarrow{A_{n-1}(l)})) \,|\, x \notin P_n(\overrightarrow{A}) \cup N_n(\overrightarrow{A})\} && (n \geq 2) \\
V_n(\overrightarrow{A}) &:= P_n(\overrightarrow{A}) \cup N_n(\overrightarrow{A}) \cup M_n(\overrightarrow{A}) && (n \geq 2)
\end{aligned}$$

*with the reprojection $Q_{n,l}(y) := R_{n,l}^{-1}\pi_{n,n-1}(y) + a_n(l)l$, $y \in \mathbb{R}^{n-1}$. $\pi_{n,n-1}$ is the natural embedding of $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$.*
*The boundary mapping $B_n : \mathcal{D}(\mathbb{R}^n) \Rightarrow \mathbb{R}^n$ is defined as*

$$B_n(\overrightarrow{A}) := \partial P_n(\overrightarrow{A}) \cup \partial N_n(\overrightarrow{A}) \cup M_n(\overrightarrow{A}).$$

The "boundary" of a directed set consists of the boundary of the convex or concave part and the additional part of mixed type, which is outside the convex and the concave part. This mixed type part is always empty in the case $n = 1$ or if the set is an embedded convex set and usually nonempty otherwise. Each point $x \in \mathbb{R}^n$ from the reprojected image of the visualization of the $(n-1)$-dimensional boundary part $\overrightarrow{A_{n-1}(l)}$ for some "normal" direction $l \in S_n$ is a "boundary" point, i.e. $x \in B_n(\overrightarrow{A})$. All these directions $l$ are attached to $x$ and form its directions bundle $\mathcal{O}_n(x, \overrightarrow{A})$.

Each part of the boundary of the inverse of a directed set is the (pointwise) negative (according to (1)) of the boundary part of the directed set itself. The convex part of the inverse is the (pointwise) negative of the concave part of the original set. Therefore, the visualization of $-\overrightarrow{A}$ is the (pointwise) negative of the visualization of $\overrightarrow{A}$.

**Proposition 3.8** $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$. *It follows with the convention $-\emptyset = \emptyset$:*

$$P_n(-\overrightarrow{A}) = -N_n(\overrightarrow{A}), \quad N_n(-\overrightarrow{A}) = -P_n(\overrightarrow{A}), \quad V_n(-\overrightarrow{A}) = -V_n(\overrightarrow{A})$$

*Furthermore, the direction bundle of the "negative" points remains the same as of the corresponding "positive" points, i.e.*

$$\mathcal{O}_n(-x, -\overrightarrow{A}) = \mathcal{O}_n(x, \overrightarrow{A}) \quad (x \in B_n(\overrightarrow{A}) = -B_n(-\overrightarrow{A})). \tag{3}$$

**Example 3.9** *According to (3) the visualization of the inverse is formed by multiplying all boundary points of the original set with $-1$ and keeping their corresponding directions $l$. The outer normals $l \in S_n$ of the directed set $\overrightarrow{[0,2]^2}$ become inner normals of its inverse $-\overrightarrow{[0,2]^2}$ (see Figure 3.1).*

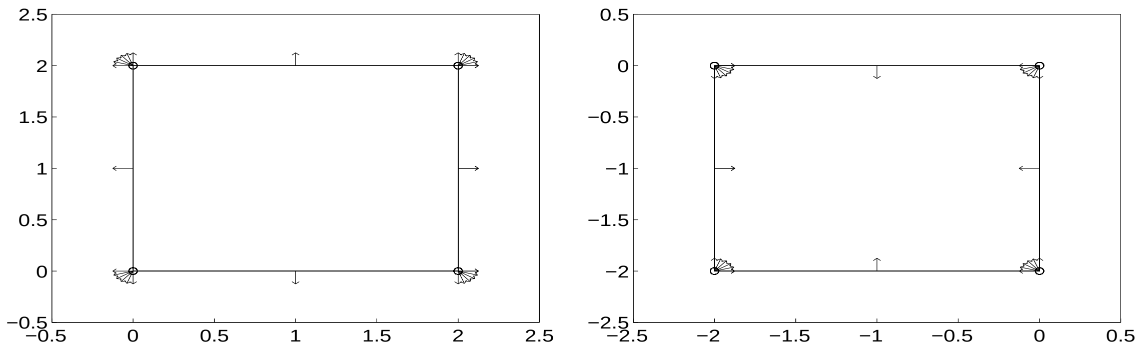

*Fig. 3.1:* $\overrightarrow{A}$ *resp.* $-\overrightarrow{A}$ *for* $A = [0,2]^2$

The visualization of differences of directed sets are strongly related to other differences.

**Proposition 3.10** $A, B \in \mathcal{C}(\mathbb{R}^n)$. *Then, the following is true:*

$$P_n(\overrightarrow{A} - \overrightarrow{B}) = A \overset{*}{-} B, \quad N_n(\overrightarrow{A} - \overrightarrow{B}) = -(B \overset{*}{-} A), \quad \overline{co}\, B_n(\overrightarrow{A} - \overrightarrow{B}) = A \dot{-} B,$$
$$\bigcup_{l \in S_n} \{y(l,A) - y(l,B) \,|\, Y(l,A),\, Y(l,B) \text{ are singletons}\} \subset B_n(\overrightarrow{A} - \overrightarrow{B})$$

The operations and definitions (addition, scalar multiplication, ordering, norm, ...) are generalizations to the ones known for convex sets.

**Proposition 3.11** $A, B \in \mathcal{C}(\mathbb{R}^n)$ *and* $\lambda \geq 0$. *Then, it is valid:*

| | |
|---|---|
| $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{A + B}$ | $V_n(\overrightarrow{A} + \overrightarrow{B}) = A + B$ |
| $\lambda \cdot \overrightarrow{A} = \overrightarrow{\lambda \cdot A}$ | $V_n(\lambda \cdot \overrightarrow{A}) = \lambda \cdot A$ |
| $\lVert\overrightarrow{A}\rVert = \lVert\overrightarrow{A}\rVert = \sup\limits_{a \in A} \lVert a \rVert_2$ | $\lVert V_n(\overrightarrow{A}) \rVert = \sup\limits_{a \in A} \lVert a \rVert_2$ |

$\lVert\cdot\rVert$ *defines a metric on* $J_n(\mathcal{C}(\mathbb{R}^n))$ *with* $\delta(\overrightarrow{A}, \overrightarrow{B}) := \lVert\overrightarrow{A} - \overrightarrow{B}\rVert = d_H(A,B)$.

**Example 3.12** *Let* $A = B_2(0)$, $B = [-1,1]^2$. *The boundary of* $\overrightarrow{A} - \overrightarrow{B}$ *consists of the boundary of* $A \overset{*}{-} B$ *(convex part), all other points of* $B_n(\overrightarrow{A} - \overrightarrow{B})$ *are elements of the mixed type part* $M_n(\overrightarrow{A} - \overrightarrow{B})$. *All differences of supporting points inside of* $A \dot{-} B$ *are elements of the boundary of* $\overrightarrow{A} - \overrightarrow{B}$ *(see Figure 3.2 and 3.3).*

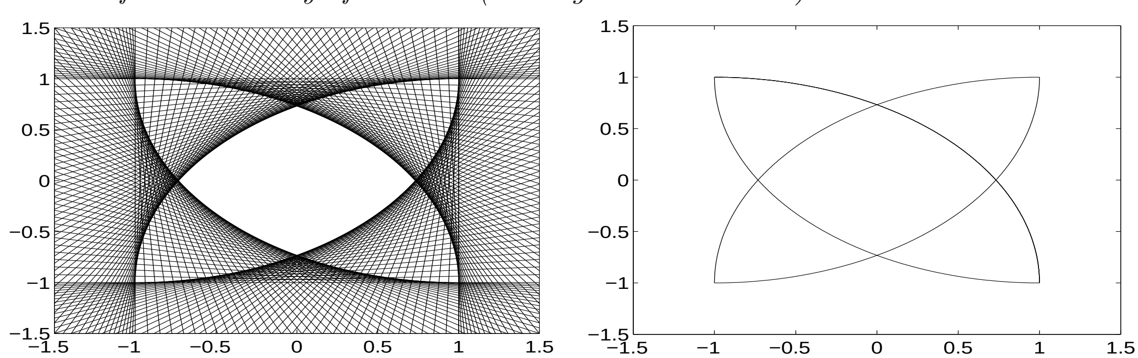

*Fig. 3.2: geometric difference* $A \overset{*}{-} B$ *resp. non-convexified part of* $A \dot{-} B$

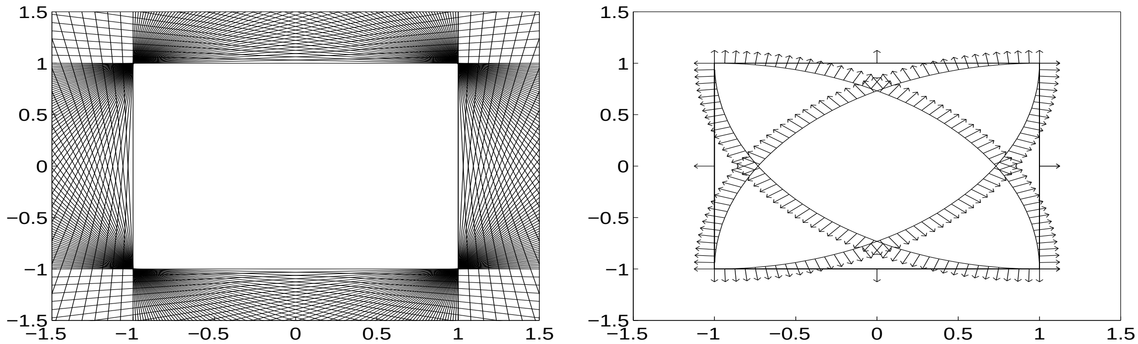


*Fig. 3.3: Demyanov's difference $A \dot{-} B$ resp. $V_n(\overrightarrow{A} - \overrightarrow{B})$*

# 4 Applications and Numerical Example

One example of set-valued interpolation is studied to show the visualization of directed sets in applications, especially, if negative weights appear in formulas. Further results and a more detailed research on applications (computation of reachable sets by extrapolation methods in [1], differentiable set-valued mappings in [2], the connection to minimal pairs in [12], error estimates, ...) must be postponed to a forthcoming publication.

Linear interpolation can be done in the space $\mathcal{C}(\mathbb{R}^n)$, but interpolation with a higher polynomial degree creates negative weights.

**Example 4.1** *Consider the quadratic interpolation of the set-valued mapping in [7]*

$$F(t) = \begin{pmatrix} (t+1)\cdot(t+2) & 1 \\ 0 & t^2+1 \end{pmatrix} B_1(0), \qquad (t \in [-3,3])$$

*with the prescribed sets $F(-3), F(0), F(3)$.*

*a) geometric difference*

$$P_2(t) := \{\, x \in \mathbb{R}^2 \mid \forall l \in S_2 : \,<l, x>\, \le p_2(l,t) := \sum_{i=0}^{2} L_i(t) \delta^*(l, F(t_i)) \,\} \qquad (4)$$

*with the Lagrange polynomials $L_i(t) = \prod\limits_{\substack{j=0,1,2 \\ j \neq i}} \frac{t - t_j}{t_i - t_j}$ and $t_i = -3 + i \cdot 3$ $(i = 0,1,2)$*

*Although, the prescribed sets are convex sets, $P_2(-1.5)$ is an empty set. $P_2(1)$ is convex and compact, but $p_2(\cdot, 1)$ is nonconvex which creates non-supporting hyperplanes in (4).*

*b) difference of directed sets*

$$\overrightarrow{P_2}(t) := \sum_{i=0}^{2} L_i(t) \overrightarrow{F(t_i)}$$

*$\overrightarrow{P_2}(-1.5)$ and $\overrightarrow{P_2}(1)$ are "mixed-type" directed sets (see Figure 3.4), $\overrightarrow{P_2}(-1.5)$ has an empty convex and concave part, $\overrightarrow{P_2}(1)$ has a nonempty convex part $P_2(1)$.*

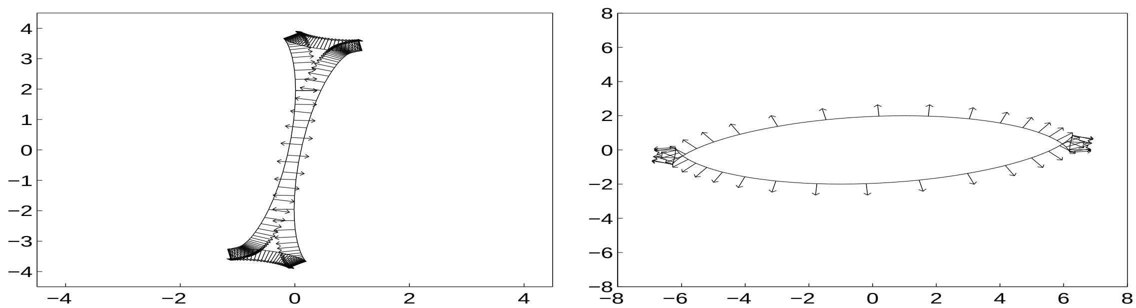


*Fig. 3.4:* $\overrightarrow{P_2}(-1.5)$ *resp.* $\overrightarrow{P_2}(1)$

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