On the equivalence between asymptotic and exponential stability, and between ISS and finite H_∞ gain

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Abstract: We show that uniformly global asymptotic stability and input-to-state stability for a family of ordinary differential equations are equivalent to uniformly global exponential stability and a nonlinear H_∞ estimate, respectively, under suitable nonlinear changes of variables.

1 Introduction

When dealing with stability properties of differential equations and nonlinear control systems, the notion of asymptotic stability as introduced by Lyapunov is one of the fundamental concepts. For linear systems it is easily verified to be equivalent to the property of *exponential* stability, which requires decay estimates of the type "$\|x(t)\| \le ce^{-\lambda t}\|x(0)\|$." Usually motivated by linearization techniques, exponential stability is often also used for nonlinear systems, although it is well known that the equivalence from the linear case is no longer valid here. (See for instance [15] for detailed discussions of the comparative roles of asymptotic and exponential stability in control theory.)

In this paper we want to investigate the behaviour of those stability concepts under nonlinear changes of coordinates. Of course, in order to get nontrivial results one must define "coordinate change" with care, since under diffeomorphisms the character of the linearization at the equilibrium (which we take to be the origin) is invariant. However, if, in the spirit of both structural stability and the classical Hartman-Grobman Theorem we relax the requirement that the change of variables be *smooth at the origin*, more interesting things may happen. Thus, we ask that transformations be infinitely differentiable except possibly at the origin, where they are just continuously differentiable. Their respective inverses are continuous globally, and infinitely differentiable away from the origin.

Applying such a change of coordinates to an exponentially stable system the exponential stability may be lost, although asymptotic stability persists. On the other hand, with this notion of changes of coordinates it is in fact true that all asymptotically stable *linear* systems can be transformed to $\dot{x} = -x$; see e.g. [1]. The basic idea of the proof in [1] is based upon projections on the level sets of Lyapunov functions, which in the linear case can of course be taken to be quadratic (and hence have ellipsoids as level sets). It is natural to use these ideas also in the general nonlinear case, and Wilson's paper [31], often cited in control theory, remarked that level sets of Lyapunov functions are always homotopically equivalent to spheres. Indeed, it is possible to obtain, in great generality, a change of coordinates rendering the system exponentially stable, and several partial versions of this fact have appeared in the literature, especially in the context of generalized notions of homogeneity for nonlinear systems; see for instance [4, 21, 14, 23, 20].

In this paper, we show that for *all* differential equations evolving in finite-dimensional Euclidean spaces $\mathbb{R}^n$ (at least in spaces of dimensions $\neq 4, 5$) asymptotic stability in the sense of Lyapunov and exponential stability are one and the same under coordinate changes.

It is perhaps surprising that, at least for unperturbed systems, this full result seems not to have been observed before, as the proof is a fairly easy application of results from differential topology. (Those results are nontrivial, and are related to the generalized Poincaré conjecture and cobordism theory; in fact, the reason that we only make an assertion for $\neq 4, 5$ is closely related to the fact that the original Poincaré conjecture is still open.)

Note, however, that it has been common practice in the papers treating the nonlinear case to use the flow generated by the original system to define a change of coordinates transforming the system to $\dot{x} = -x$ (being exponentially stable), thereby reducing the regularity of the transformation to that of the system. Here we use the flow generated by the (normalized) Lyapunov function itself, which yields more regular transformations. In addition, and most importantly, our proof also allows for the treatment of perturbed systems for which the reduction to $\dot{x} = -x$ makes no sense. (Note, however, that this reduction is possible for all backward complete unperturbed systems, cf. [8]).

In particular, our result confirms that Lyapunov's notion is the appropriate generalization of exponential stability to nonlinear differential equations. For systems with inputs, the notion of *input to state stability (ISS)* introduced in [25] and developed further in [3, 7, 12, 13, 16, 17, 22, 24, 27, 28] and other references, has been proposed as a nonlinear generalization of the requirement of finite $\mathcal{L}^2$ gain or, as often also termed because of the spectral characterizations valid for linear systems, "finite nonlinear H^∞ gain" (for which see e.g. [2, 10, 11, 29]).

We also show in this paper that—in the same sense as asymptotic stability equals exponential stability—the two properties (ISS and finite H^∞ gain) coincide under coordinate changes now in both state and input variables (again, assuming dimension $\neq 4, 5$).

2 Setup

We consider the family of differential equations

$$\dot{x}(t) = f(x(t), d(t)) \tag{2.1}$$

where $f : \mathbb{R}^n \times D \to \mathbb{R}^n$ is continuous and for $x \neq 0$ locally Lipschitz continuous in x, where the local Lipschitz constants can be chosen uniformly in $d \in D \subseteq \mathbb{R}^m$. Let $\mathcal{D}$ denote the set of measurable, locally essentially bounded functions from $\mathbb{R}$ to D. For any $x_0 \in \mathbb{R}^n$ and any $d(\cdot) \in \mathcal{D}$, there exists at least one maximal solution of (2.1) for $t \geq 0$, with $x(0) = x_0$. By abuse of notation, we denote any such solution, even if not unique, as $\phi(t, x_0, d(\cdot))$, $t \in I(x, d(\cdot))$, where $I(x, d(\cdot))$ is its existence interval. Throughout the paper, $\|\cdot\|$ denotes the usual Euclidean norm, and "smooth" means C^∞. For a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ the expression $L_{f_d}V(x)$ denotes the directional derivative $DV(x)f(x, d)$.

The general framework afforded by the model (2.1) allows us to treat simultaneously classical differential equations (the case when $D = \{0\}$) and more generally robust stability of differential equations subject to perturbations (when functions in $\mathcal{D}$ are seen as disturbances which do not change the equilibrium, as in parameter uncertainty), as well as systems with inputs in which elements of $\mathcal{D}$ are seen as exogenous tracking or regulation signals, or as actuator errors (in which case, the continuity properties of $(x, d) \mapsto \phi(\cdot, x, d)$ are of interest). In light of these applications, we now describe the appropriate stability concepts.

For the first, assume that D is compact and that $f(0, d) = 0$ for all $d \in D$. Then we say that the zero state is uniformly globally asymptotically stable (UGAS) if there exists a class $\mathcal{KL}$ function β such that, for each $d(\cdot) \in \mathcal{D}$, every maximal solution is defined for all $t \geq 0$ and

$$\|\phi(t, x, d(\cdot))\| \leq \beta(\|x\|, t) \tag{2.2}$$

for all $t \geq 0$. As usual, we call a function $\alpha : [0, \infty) \to [0, \infty)$ of class $\mathcal{K}$, if it satisfies $\alpha(0) = 0$ and is continuous and strictly increasing (and class $\mathcal{K}_\infty$ if it is unbounded), and we call a continuous function $\beta : [0, \infty)^2 \to [0, \infty)$ of class $\mathcal{KL}$, if it is decreasing to zero in the second and of class $\mathcal{K}$ in the first argument. (It is an easy exercise, cf. e.g. [18], to verify that this definition is equivalent to the requirements of uniform stability and uniform attraction stated in "$\varepsilon - \delta$" terms.) Note that while our general assumptions on the right hand side f do not guarantee uniqueness of solutions through zero, the added assumption of asymptotic stability implies that $\phi(t, 0, d) \equiv 0$ is the unique solution with initial condition $x = 0$, for all $d \in \mathcal{D}$. As a consequence, since away from zero we have a local Lipschitz condition, solutions are unique for each given initial state and $d \in \mathcal{D}$.

If the origin is no common fixed point for all values $d \in D$ then (2.2) is impossible. In this case, however, still a useful notion of stability is possible. We call the system (2.1) (globally) input-to-state stable (ISS), if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function α such that all solutions of (2.1) satisfy

$$\|\phi(t, x, d(\cdot))\| \leq \beta(\|x\|, t) + \alpha(\sup_{0 \leq \tau \leq t} \|d(\tau)\|) \tag{2.3}$$

for all $d(\cdot) \in \mathcal{D}$ and all $t \geq 0$. Formulation (2.3) is the most frequently used characterization of the ISS property. Note that with $\tilde{\beta} = 2\beta$ and $\tilde{\alpha} = 2\alpha$ inequality (2.3) immediately implies

$$\|\phi(t, x, d(\cdot))\| \leq \max\left\{\tilde{\beta}(\|x\|, t),\, \tilde{\alpha}(\sup_{0 \leq \tau \leq t} \|d(\tau)\|)\right\},$$

hence this "max" formulation can be used as an equivalent characterization.

Two apparently stronger formulations of these properties are obtained if we replace $\beta(\|x\|, t)$ by $ce^{-\lambda t}\|x\|$, more precisely we call the zero position of (2.1) uniformly globally *exponentially* stable (UGES), if there exist constants $c \geq 1, \lambda > 0$ such that

$$\|\phi(t, x, d(\cdot))\| \leq ce^{-\lambda t}\|x\| \tag{2.4}$$

holds for all $d(\cdot) \in \mathcal{D}$ and all $t \geq 0$, and we call the system input-to-state *exponentially* stable (ISES), if there exist a class $\mathcal{K}_\infty$ function α and constants $c \geq 1, \lambda > 0$ such that

$$\|\phi(t, x, d(\cdot))\| \leq \max\left\{ ce^{-\lambda t}\|x\|,\ \alpha(\sup_{0\leq\tau\leq t} \|d(\tau)\|) \right\} \tag{2.5}$$

for all $d(\cdot) \in \mathcal{D}$ and all $t \geq 0$. (As usual, these definitions use appropriate constants $c, \lambda > 0$. In this paper, however, we will see that we can always work with "normalized" versions choosing $c = 1, \lambda = 1$. For the (ISES) property we use the "max" formulation because it allows a further implication as stated in Theorem 5, below. Observe that (2.5) implies (2.3) with $\beta(\|x\|, t) = ce^{-\lambda t}\|x\|$.)

Extending the concepts in [1, p. 207] to our nonlinear setting, we will call a homeomorphism

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

a *change of variables* if $T(0) = 0$, T is C^1 on $\mathbb{R}^n$, and T is diffeomorphism on $\mathbb{R}^n \setminus \{0\}$ (i.e., the restrictions of T and of T^{-1} to $\mathbb{R}^n \setminus \{0\}$ are both smooth). Given a change of variables T and a system (2.1), we may consider the *transformed system*

$$\dot{y}(t) = \tilde{f}(y(t), d(t)), \tag{2.6}$$

where, by definition,

$$\tilde{f}(y, d) = DT(T^{-1}(y))f(T^{-1}(y), d).$$

In other words, system (2.6) is obtained from the original system by means of the change of variables $y = T(x)$. Observe that the new system again satisfies the general requirements: $\tilde{f}(y, d)$ is continuous, and it is locally Lipschitz on x for $x \neq 0$, uniformly on d.

It is our aim to show that for dimensions $n \neq 4, 5$ the following assertions are true. Given a system of the form (2.1) satisfying (2.2) or (2.3), respectively, there exists a transformed system that satisfies (2.4) or (2.5), respectively. In this sense, global asymptotic stability is equivalent to global exponential stability under nonlinear changes of coordinates. Furthermore, one may obtain transformed systems where the constants defining the exponential stability property can be chosen to be the special values $c = \lambda = 1$.

Furthermore we show that if system (2.1) is ISES (2.5) with $c = \lambda = 1$ then there exists a homeomorphism $R : \mathbb{R}^m \to \mathbb{R}^m$ on the input space with $R(0) = 0$ that is a diffeomorphism on $\mathbb{R}^m\setminus\{0\}$ such that the transformed system with $v = R(d)$

$$\dot{y}(t) = \bar{f}(x(t), v(t)), \quad \bar{f}(x, v) = f(x, R^{-1}(v)) \tag{2.7}$$

satisfies the following "L_2 to L_2" nonlinear H_∞ estimate:

$$\int_0^t \|\phi(s, x, v(\cdot))\|^2 ds \leq \|x\|^2 + \int_0^t \|v(s)\|^2 ds. \tag{2.8}$$

Since (2.8) in turn implies ISS (by [26, Theorem 1]), we obtain equivalence between ISS and the nonlinear H_∞ estimate (2.8) up to nonlinear changes of coordinates.

Remark 2.1 Since we are not requiring that the inverse of a change of variables be itself a change of variables (because one may, and in fact does in our constructions, have $DT(0) = 0$, in which case T^{-1} is not differentiable at the origin), the way to define a notion of "equivalence" is by taking the transitive and symmetric closure of the relation given by such changes of variables. That is, we could say that system (2.1) is *equivalent* to a system (2.6) if there exist $k \in \mathbb{N}$ and maps $f_0 = f, f_1, \ldots, f_k = \tilde{f} : \mathbb{R}^n \times D \to \mathbb{R}^n$, all satisfying the assumptions on f, with the following properties: For each $i = 0, \ldots, k-1$ there exists a change of variables T as above such that $f_l(y, d) = DT(T^{-1}(y))f_m(T^{-1}(y), d)$, where $l = i, m = i+1$ or $l = i+1, m = i$.

3 Construction of the coordinate transformation

The main tool for our construction of T is the use of an appropriate Lyapunov function V. In fact, we can obtain T for a whole class of functions as stated in the following proposition. Recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$, and *proper* if the set $\{x \mid V(x) \leq \beta\}$ is bounded for each $\beta \geq 0$.

The next result says that, given some class $\mathcal{K}_\infty$ function γ, any such function may look like $\gamma(\|x\|)$ under a coordinate change. This implies in particular that the level sets under coordinate change are spheres. It may therefore not come as a surprise that a basic ingredient of the proof is related to the question of whether level sets of Lyapunov functions in $\mathbb{R}^n$ are diffeomorphic to the sphere S^{n-1}. This question is solved except for the two special cases of dimensions $n = 4$ and $n = 5$, though in the case $n = 5$ it is at least known that the statement is true if only homeomorphisms are required. (For the case $n = 4$ this question is equivalent to the Poincaré conjecture; see [31].)

Proposition 3.1 Let $n \neq 4,5$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be a proper, positive definite C^1 function. Assume furthermore that V is smooth on $\mathbb{R}^n \setminus \{0\}$ with nonvanishing gradient. Then for each class $\mathcal{K}_\infty$ function γ which is smooth on $(0,\infty)$ there exists a homeomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(0) = 0$ such that

$$\tilde{V}(y) := V(T^{-1}(y)) = \gamma(\|y\|)\,.$$

T can be chosen to be a diffeomorphism on $\mathbb{R}^n \setminus \{0\}$, and, in particular, there exists a class $\mathcal{K}_\infty$ function γ which is smooth on $(0,\infty)$ and satisfies $\gamma(s)/\gamma'(s) \geq s$ such that T is C^1 with $DT(0) = 0$.

Proof: For the function V the right hand side of the *normed* gradient flow

$$\dot{x} = \frac{\nabla V(x)'}{\|\nabla V(x)\|^2}$$

is well defined and smooth for $x \neq 0$. Denote the solutions by $\psi(t,x)$. Then $V(\psi(t,x)) = V(x) + t$, and thus since V is proper and $\nabla V(x) \neq 0$ for $x \neq 0$ for a given initial value $x \in \mathbb{R}^n$ ψ is well defined for all $t \in (-V(x), \infty)$, thus also smooth (see e.g. [9, Corollary 4.1]).

Fix $c > 0$. We define a map $\pi : \mathbb{R}^n \setminus \{0\} \to V^{-1}(c)$ by

$$\pi(x) = \psi(c - V(x), x)\,.$$

Obviously π is smooth, and since the gradient flow crosses each level set $V^{-1}(a), a > 0$ exactly once it induces a diffeomorphism between each two level sets of V, which are C^∞ manifolds due to the fact that V is smooth away from the origin with nonvanishing gradient.

Now observe that the properties of V imply that $V^{-1}(c)$ is a homotopy sphere (cf. also [31, Discussion after Theorem 1.1]), which implies that $V^{-1}(c)$ is diffeomorphic to $\mathbb{S}^{n-1}$ for $n = 1,2,3$ (see e.g. [5]) and for $n \geq 6$ (by [19, §9, Proposition A]). Thus for all dimensions $n \neq 4,5$ we may choose a diffeomorphism $S : V^{-1}(c) \to \mathbb{S}^{n-1}$. (By [6] we could choose S to be at least a homeomorphism in the case $n = 5$.)

Let $Q := S \circ \pi$. The coordinate transformation T is now given by $T(0) = 0$ and

$$T(x) = \gamma^{-1}(V(x))\, Q(x)\,, \quad x \neq 0\,.$$

An easy computation verifies that $T^{-1}(0) = 0$ and

$$T^{-1}(y) = \psi\left(\gamma(y) - c, S^{-1}\left(\frac{y}{\|y\|}\right)\right)\,, \quad y \neq 0\,,$$

hence T is a diffeomorphism on $\mathbb{R}^n \setminus \{0\}$ (resp. a homeomorphism if $n = 5$). Since $V(0) = 0$, and $\psi(t, S^{-1}(y/\|y\|)) \to 0$ as $t \searrow -c$, both T and T^{-1} are homeomorphisms.

Finally, we have that

$$\begin{aligned} V(T^{-1}(y)) &= V\left(\psi\left(\gamma(\|y\|) - c, S^{-1}\left(\frac{y}{\|y\|}\right)\right)\right) \\ &= \gamma(\|y\|) \end{aligned}$$

which finishes the proof of the first assertion.

For the second assertion define

$$L(s) := \sup_{V(x)=s} \|DQ(x)\|$$

and choose any class $\mathcal{K}$ function a which is C^∞ and satisfies

$$a(s) \leq \frac{s}{L(s)} \ \text{ for all } \ s \in (0,1]\,.$$

Then the function h given by

$$h(r) = \int_0^r a(s)ds$$

is smooth and of class $\mathcal{K}_\infty$, and by straightforward computations (see [8]) the desired properties are verified setting $\gamma := h^{-1}$. □

4 Main Results

Using the coordinate transformation T we can now prove our main results.

Theorem 4.1 Let $n \neq 4,5$ and consider any system (2.1) on $\mathbb{R}^n$ which is UGAS (2.2). We suppose that the set $D \subset \mathbb{R}^m$ is compact. Then, (2.1) can be transformed into a system (2.6) that is UGES (2.4).

In particular, the constants in (2.4) can be chosen to be $c = 1, \lambda = 1$.

Proof: Under our assumptions, by [18, Theorem 2.9, Remark 4.1][1] there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ for (2.1) such that

$$L_{f_d} V(x) \leq -\alpha_1(\|x\|) \tag{4.9}$$

for some class $\mathcal{K}_\infty$ function α_1. Furthermore, there exist class $\mathcal{K}_\infty$ functions α_2, α_3 such that

$$\alpha_2(\|x\|) \leq V(x) \leq \alpha_3(\|x\|)\,. \tag{4.10}$$

[1]To be precise, the results in that reference make as a blanket assumption the hypothesis that f is locally Lipschitz, not merely continuous, at $x = 0$. However, as noted in e.g. [30], the Lipschitz condition at the origin is not used in the proofs.

Now let α_4 be a C^1 function of class $\mathcal{K}_\infty$ which is smooth on $(0,\infty)$ and satisfies $\alpha_4'(0)=0$, such that $\alpha_4(a) \le \min\{a, \alpha_1 \circ \alpha_3^{-1}(a)\}$ for all $a \ge 0$.

Such a function can be obtained e.g. by a slight modification of the construction in [22, Proof of Lemma 11], see [8] for an explicit construction. Thus we obtain

$$L_{f_d} V(x) \le -\alpha_4(V(x)). \qquad (4.11)$$

Now define

$$\rho(a) := \exp\left(-\int_a^1 \alpha_4(\tau)^{-1} d\tau\right) \text{ for } a>0, \quad \rho(0):=0$$

Obviously ρ is smooth on $(0,\infty)$; furthermore ρ is of class $\mathcal{K}_\infty$ and by [22, Lemma 12] ρ is a C^1 function on $[0,\infty)$ with $\rho'(0)=0$. Thus defining

$$W(x) := \rho(V(x))$$

we obtain a C^1 Lyapunov function, which is smooth on $\mathbb{R}^n \setminus \{0\}$, for which an easy calculation shows that

$$L_{f_d} W(x) \le -W(x).$$

Applying Proposition 3.1 to W, using the class $\mathcal{K}_\infty$ function γ with $\gamma(s)/\gamma'(s) \ge s$ we obtain for each $d \in D$ and $y \ne 0$

$$\langle \tilde f(y,d), y\rangle = \frac{\|y\|}{\gamma'(\|y\|)} L_{\tilde f_d} \tilde W(y) \le -\|y\|^2 .$$

Clearly the overall inequality also holds for $y=0$ so that we obtain

$$\frac{d}{dt}\|y(t)\|^2 = 2\langle \tilde f(y(t), d(t)), y(t)\rangle \le -2\|y(t)\|^2$$

and hence $\|y(t)\|^2 \le e^{-2t}\|y(0)\|^2$, i.e. the desired exponential estimate. □

Theorem 4.2 Let $n \ne 4,5$ and suppose that the system (2.1) on $\mathbb{R}^n$ is ISS (2.3) with some class $\mathcal{K}_\infty$ function α and some class $\mathcal{KL}$ function β. Then (2.1) is can be transformed into a system (2.6) that is ISES (2.5) with constants $c=\lambda=1$.

Proof: By [27, Theorem 1][2] there exists a C^1 function V which is smooth on $\mathbb{R}^n \setminus \{0\}$ and a class $\mathcal{K}_\infty$ function χ such that

$$\|x\| > \chi(\|d\|) \quad \Rightarrow \quad L_{f_d} V(x) \le -\alpha_1(\|x\|)$$

for some class $\mathcal{K}_\infty$ function α_1. Furthermore, there exist class $\mathcal{K}_\infty$ functions α_2, α_3 such that

$$\alpha_2(\|x\|) \le V(x) \le \alpha_3(\|x\|) .$$

[2] As with the UGAS proof, it is easy to verify that the assumption that the right-hand side is Lipschitz at zero is never actually used in [27]. The possible non-uniqueness of trajectories does not affect the argument used in Lemma 2.12 in that paper, which reduces the problem to one of UGAS.

As in the proof of Theorem 4.1 we find a function ρ which is class $\mathcal{K}_\infty$, C^1, and smooth on $\mathbb{R}^n \setminus \{0\}$, such that $W = \rho \circ V$ satisfies

$$\|x\| > \chi(\|d\|) \quad \Rightarrow \quad L_{f_d} W(x) \le -W(x) .$$

Now Proposition 3.1 yields a parameter transformation T such that $\tilde W(y) = W(T^{-1}(y)) = \gamma(\|y\|)$ and $\gamma(s)/\gamma'(s) \ge s$.

Now choose a class $\mathcal{K}_\infty$ function δ such that $\|T^{-1}(y)\| \ge \delta(\|y\|)$ and define $\tilde\alpha = \delta^{-1} \circ \chi$. Then a straightforward calculation yields

$$\|y\| > \tilde\alpha(\|d\|) \quad \Rightarrow \quad L_{\tilde f_d} \tilde W(y) \le -\tilde W(y). \qquad (4.12)$$

Similar to the proof of Theorem 4.1 this implies

$$\|\tilde\phi(t,y,d(\cdot))\| \le e^{-t}\|y\|$$

as long as $\|\tilde\phi(t,y,d(\cdot))\| > \tilde\alpha(\sup_{0\le\tau\le t}\|d(\tau)\|)$ which yields the desired estimate. □

Theorem 4.3 Consider the system (2.1) on $\mathbb{R}^n$ being ISES (2.5) with some class $\mathcal{K}_\infty$ function α and $c=\lambda=1$. Then there exists a homeomorphism $R: \mathbb{R}^m \to \mathbb{R}^m$ on the input space with $R(0)=0$, that is a diffeomorphism on $\mathbb{R}^m \setminus \{0\}$, such that the the transformed system (2.7) satisfies the nonlinear H_∞ estimate (2.8).

Proof: Consider the function $W(x) = \|x\|^2$. From (2.5) with $c=\lambda=1$ it is immediate that for any $d(\cdot) \in \mathcal{D}$, any $x \in \mathbb{R}^n$, and any $T>0$ we have

$$\|x\| \ge e^T \alpha(\sup_{0\le\tau\le T}\|d(\tau)\|)$$

$$\Rightarrow \quad W(\phi(t,x,d(\cdot))) \le e^{-2t} W(x) \text{ for all } t \in [0,T] .$$

In particular this estimate is valid for constant functions $d(\cdot) \equiv d \in D$, thus the mean value theorem (observe $W(\phi(0,x,d)) = W(x) = e^{-0}W(x)$) yields

$$\|x\| \ge \alpha(\|d\|) \quad \Rightarrow \quad L_{f_d} W(x) \le -2W(x) \le -W(x) .$$

Now defining

$$\tilde\alpha(r) = \sup_{\|x\|\le\alpha(r), \|d\|\le r} \langle f(x,d), x\rangle$$

we obtain a class $\mathcal{K}_\infty$ function $\tilde\alpha$ with

$$L_{f_d} W(x) \le -W(x) + \tilde\alpha(\|d\|).$$

Without loss of generality (one could take a larger $\tilde\alpha$), we may assume $\tilde\alpha$ to be smooth on $(0,\infty)$, and thus

$$R(d) := \frac{\tilde\alpha(\|d\|)^2 d}{\|d\|}$$

has the regularity properties as stated in the assertion. Now the transformation (2.7) yields

$$L_{\bar f_v} W(x) \le -W(x) + \|v\|^2.$$

Integrating this equation along a trajectory $x(\cdot)$ gives the desired estimate (2.8). □

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