Homogeneous control Lyapunov functions for homogeneous control systems *

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Abstract

We show that any asymptotically controllable homogeneous control system admits a homogeneous control Lyapunov function. As a consequence, we obtain that any such system admits a stabilizing discontinuous sampled feedback, where the sampling rate is bounded from below depending on the degree of the homogeneity. Furthermore, we show that asymptotic controllability of a homogeneous system implies local asymptotic controllability of an approximated system.

1 Introduction

In this paper we present an existence result for homogeneous control Lyapunov functions for homogeneous control systems, and derive some consequences from this result. Whereas the existence of homogeneous Lyapunov functions for asymptotically stable homogeneous ordinary differential equations has been proved some time ago [13], the corresponding result for controlled systems to be presented (under the analogous assumption of asymptotic null controllability) is more recent [8] and allows immediate applications to asymptotic controllability of homogeneously approximated systems and to feedback stabilization.

Stability and stabilization of homogeneous systems, as well as related Lyapunov functions have been investigated for a long time, see e.g. [10, 11, 12, 14, 15] to mention just a few references. Lyapunov functions are a standard tool when dealing with stability and stabilization of nonlinear systems, and homogeneous systems appear naturally as local approximations to nonlinear systems, cf. e.g. [9]. In order to make use of this approximation property in the context of stability one needs a compatibility between the structure of the approximation and the structure of the Lyapunov function, and this is exactly what is provided by homogeneous control Lyapunov functions.

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2 Homogeneous systems

We consider the class of systems

$$\dot{x}(t) = f(x(t), u(t)) \quad (1)$$

on $\mathbb{R}^n$ where $u(\cdot) \in \mathcal{U}$, and $\mathcal{U}$ denotes the space of measurable and locally essentially bounded functions from $\mathbb{R}$ to $U \subset \mathbb{R}^m$. We assume that the vector field g is continuous, $f(\cdot, u)$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ for each $u \in U$, and satisfies the following property.

Definition 2.1 *We call* f homogeneous *if there exist* $r_i > 0$, $i = 1, \ldots, n$, $s_j > 0$, $j = 1, \ldots, m$ *and* $\tau \in (-\min_i r_i, \infty)$ *such that*

$$f(\Lambda_\alpha x, \Delta_\alpha u) = \alpha^\tau \Lambda_\alpha f(x, u) \ \text{ for all } \ u \in U, \ \alpha \geq 0, \quad (2)$$

and we call f homogeneous-in-the-state *if*

$$f(\Lambda_\alpha x, u) = \alpha^\tau \Lambda_\alpha f(x, u) \ \text{ for all } \ u \in U, \ \alpha \geq 0. \quad (3)$$

Here

$$\Lambda_\alpha = \begin{pmatrix} \alpha^{r_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha^{r_n} \end{pmatrix}$$

and

$$\Delta_\alpha = \begin{pmatrix} \alpha^{s_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha^{s_m} \end{pmatrix}$$

are called dilation matrices. *With* $k = \min_i r_i$ *we denote the* minimal power *(of the state dilation) and the value* $\tau \in (-k, \infty)$ *is called the* degree *of the system.*

The use of dilation matrices instead of the usual dilation functions allows a more compact notation in what follows. Observe that if f is Lipschitz in the origin then $\tau \geq 0$ and if f is globally Lipschitz then $\tau = 0$, furthermore the definition implies $f(0,0) = 0$ for homogeneous systems and $f(0,u) = 0$ for all $u \in U$ for homogeneous-in-the-state systems.

Corresponding to the dilation matrix Λ_α we define a function $N : \mathbb{R}^n \to [0,\infty)$ which can be interpreted as a "dilated norm" w.r.t. Λ_α. Denoting $d = 2\prod_{i=1}^n r_i$ we define $N(x)$ by

$$N(x) := \left(\sum_{i=1}^n x_i^{\frac{d}{r_i}}\right)^{\frac{1}{d}} \tag{4}$$

implying $N(0) = 0$, $N(x) > 0$ if $x \neq 0$, and $N(\Lambda_\alpha x) = \alpha N(x)$.

We denote the trajectories of (1) by $x(t, x_0, u(\cdot))$ for each $x_0 \in \mathbb{R}^n$ and each $u(\cdot) \in \mathcal{U}$, where $x(0, x_0, u(\cdot)) = x_0$.

Note that the trajectories of (1) may tend to infinity in finite time if $\tau > 0$ and that uniqueness of the trajectory may not hold if $\tau < 0$, however it holds away from the origin. In the case of non-uniqueness of trajectories we implicitely assume the definitions below to be valid for *all* possible trajectories.

Observe that the homogeneity can also be expressed in terms of the trajectories: For homogeneous systems we obtain

$$x(t, \Lambda_\alpha x_0, \Delta_\alpha u(\alpha^\tau \cdot)) = \Lambda_\alpha x(\alpha^\tau t, x_0, u(\cdot)) \tag{5}$$

and similarly for homogeneous-in-the-state systems we get

$$x(t, \Lambda_\alpha x_0, u(\alpha^\tau \cdot)) = \Lambda_\alpha x(\alpha^\tau t, x_0, u(\cdot))$$

for all $x_0 \in \mathbb{R}^n$.

The connection between homogeneous and homogeneous-in-the-state systems is easily seen: Given some homogeneous system (1) satisfying

$$f(\Lambda_\alpha x, \Delta_\alpha u) = \alpha^\tau \Lambda_\alpha f(x,u)$$

a simple calculation shows that

$$g(x,u) := f(x, \Delta_{N(x)} u) \tag{6}$$

is homogeneous-in-the-state.

Homogeneous and homogeneous-in-the-state systems can be considerably simplified applying suitable coordinate and time transformations. We will make use of this procedure for homogeneous-in-the-state systems: Using the dilated norm N from (4) the function

$$P(x) := \Lambda_{N(x)}^{-1} x$$

defines a projection from $\mathbb{R}^n \setminus \{0\}$ onto $N^{-1}(1)$ satisfying $P(\Lambda_\alpha x) = P(x)$ for all $\alpha > 0$. We denote the $n-1$ dimensional embedded unit sphere $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ by $\mathbb{S}^{n-1}$. Then, since $N(tx)$ is strictly increasing in $t \geq 0$ the function $S : N^{-1}(1) \to \mathbb{S}^{n-1}$, $S(x) = x/\|x\|$ is a diffeomorphism between these two manifolds, thus we can define a coordinate transformation $y = \Psi(x)$ by

$$\Psi(x) = N(x)^k S(P(x)), \quad \Psi^{-1}(y) = \Lambda_{\sqrt[k]{\|y\|}} S^{-1}\left(\frac{y}{\|y\|}\right),$$

and $\Psi(0) = 0$, $\Psi^{-1}(0) = 0$, which is continuous on $\mathbb{R}^n$ and C^1 on $\mathbb{R}^n \setminus \{0\}$. This definition implies

$$\Psi(\Lambda_\alpha x) = \alpha^k \Psi(x), \quad \Psi^{-1}(\alpha^k y) = \Lambda_\alpha \Psi^{-1}(y)$$

and by differentiation of $\Psi(\Lambda_\alpha x)$ and $\alpha^k \Psi(x)$ one sees

$$D\Psi(\Lambda_\alpha x) = \alpha^k \Lambda_\alpha^{-1} D\Psi(x).$$

Thus defining

$$\tilde{f}(y,u) = D\Psi(\Psi^{-1}(y)) f(\Psi^{-1}(y), u)$$

we obtain (with $x = \Psi^{-1}(y)$)

$$\begin{aligned} \tilde{f}(\alpha^k y, u) &= D\Psi(\Lambda_\alpha x) f(\Lambda_\alpha x, u) \\ &= \alpha^k \Lambda_\alpha^{-1} D\Psi(x) \alpha^\tau \Lambda_\alpha f(x,u) \\ &= \alpha^\tau \alpha^k \tilde{f}(y,u) \end{aligned}$$

implying

$$\tilde{f}(\alpha y, u) = \alpha^{\gamma+1} \tilde{f}(y,u),$$

with $\gamma = \tau/k$, i.e. $\tilde{f}$ is homogeneous-in-the-state with respect to the standard dilation $\Lambda_\alpha = \alpha \mathrm{Id}$, with mimimal power $k = 1$, and with degree $\tau = \gamma$.

Furthermore setting $\bar{f}(y,u) = \tilde{f}(y,u)\|y\|^{-\gamma}$ (which defines a time transformation for $\tilde{f}$) we obtain a system with degree $\tau = 0$.

Note that the solutions $\tilde{x}$ and $\bar{x}$ corresponding to the vector fields $\tilde{f}$ and $\bar{f}$, respectively, are related via

$$\tilde{x}(t, x_0, u(\cdot)) = \bar{x}(\bar{t}(t), x_0, u(\tilde{t}(\cdot))), \tag{7}$$

where $\bar{t}(t)$ denotes the inverse of $\tilde{t}(t)$ which is given by

$$\bar{t}(t) = \int_0^t \|\tilde{x}(\tau, x_0, u(\cdot)))\|^\gamma d\tau.$$

3 Main Result

Recall that a continuous function $V : \mathbb{R}^n \to [0,\infty)$ is called *positive definite* if $V(x) = 0$ if and only if $x = 0$ and *proper* if $V(x) \to \infty$ as $\|x\| \to \infty$. Furthermore system (1) is called *asymptotically controllable (to the origin)*, if for each $x_0 \in \mathbb{R}^n$ there exists $u_{x_0}(\cdot) \in \mathcal{U}$ such that $\|x(t, x_0, u_{x_0}(\cdot))\| \to 0$ as $t \to \infty$.

A continuous, positive definite and proper function is called a *control Lyapunov function* if there exists a positive definite and proper function $W : \mathbb{R}^n \to \mathbb{R}_0^+$ such that for each bounded $G \subset \mathbb{R}^n$ there exists a bounded subset $U_G \subset U$ with

$$\inf_{v \in cof(x, U_G)} DV(x;v) \leq -W(x) \text{ for all } x \in G, \tag{8}$$

where

$$DV(x;v) := \liminf_{t \searrow 0,\, v' \to v} \frac{V(x + tv') - V(x)}{t}$$

is called the *lower directional derivative*.

Alternatively, one can define this property by assuming that V is a *viscosity supersolution* of

$$\sup_{u\in U_G}\{-DV(x)f(x,u)-W(x)\}=0$$

cf. [1].

It is known (see e.g. [16]) that the existence of a control Lyapunov function is equivalent to asymptotic controllability and also to stabilization via sampled discontinuous feedback with vanishing sampling rate. (Note, however, that for general nonlinear systems the definition of asymptotic controllability is slightly more complicated than the one above, as one also has to assume certain bounds on the trajectory and the controls which are not needed in the homogeneous setting.)

Using these concepts we can state our main result.

Theorem 3.1 *(a) Consider system* (1) *satisfying* (2) *with dilation matrices* Λ_α *and* Δ_α*, minimal power* $k>0$*, and degree* $\tau\in(-k,\infty)$*, and assume asymptotic controllability. Then there exists* $\mu>0$*,* $h>0$*, and a positive definite and proper control Lyapunov function* V *being Lipschitz on* $\mathbb{R}^n\setminus\{0\}$*, satisfying*

$$V(\Lambda_\alpha(x))=\alpha^{2k}V(x)$$

and there exists $\delta>0$ *such that for each* $x\in\mathbb{R}^n\setminus\{0\}$ *there is some control value* $u_x\in U_x$ *which depends homogeneously on* x *and satisfies*

$$V(x(t,x,u_x))-V(x)\le-2\mu tN^\tau(x)V(x)$$

for all $t\in[0,N(x)^{-\tau}\delta]$*, the function* N *from* (4) *and* $U_x=\Delta_{N(x)}U_0$ *for some suitable compact subset* $U_0\subset U$*.*

(b) If U *is compact the analogous result holds for system* (1) *satisfying* (3) *with* $U_x=U$ *for all* $x\in\mathbb{R}^n$*.*

Sketch of proof (See [8] for a detailled proof). First note that the inequality from the theorem immediately implies the control Lyapunov function property (8).

Now, observing that the transformation (6) does not destroy the property of asymptotic controllability Part (a) easily follows from Part (b).

In order to prove Part (b) we use the coordinate transformations in space and time described at the end of Section 2. Observe that also these transformations do not affect the asymptotic controllability. Using the homogeneity of degree $\tau=0$ of the system defined by $\bar{f}$, we can conclude that it is asymptotically controllable if and only if it is exponentially controllable. Hence we obtain a characterization of asymptotic controllability in terms of the maximum (with respect to x) of the minimal (with respect to $u(\cdot)$) Lyapunov exponents, cp. [3], which in turn can be approximated by an infinite horizon discounted optimal value function on the unit sphere, see [5]. Lifting this value function to $\mathbb{R}^n$ (similar to [6]) and approximating it by a suitable inf-convolution then gives the desired function V (for the transformed problem), which can be retranslated by applying the inverses of the time and space transformations. The explicit estimate for t in the last inequality in the theorem is easily deduced from straightforward estimates on the time transformation (7). □

Recall that the existence of the control Lyapunov function V implies asymptotic controllability, hence we do in fact obtain equivalence between asymptotic controllability and the existence of V.

4 Applications

4.1 Homogeneous Approximations

In this section we indicate how Theorem 3.1 can be used for analyzing homogeneous approximations. We do this in the framework of homogeneous system; the similar statement can be made for homogeneous-in-the-state systems.

Assume we have a nonlinear system given by

$$\dot{y}=g(y,u)$$

with solution $y(t,x,u)$, which can locally approximated by a homogeneous system, i.e. there exists a homogeneous vectorfield f with degree τ such that for each $\varepsilon>0$ we find $\delta>0$ with

$$N(f(x,u)-g(x,u))\le\varepsilon N(x)^{\tau+1}$$

for all $x\in\mathbb{R}^n$ and all $u\in U$ with $\|x\|\le\delta$ and $\|u\|\le\delta$, where N denotes the dilated norm from (4). (This is satisfied, for instance, if $f-g$ is homogeneous with the same dilation as f but higher degree.)

Then by Gronwall's Lemma (taking into account the degree of f, which gives estimates for its Lipschitz constant depending on x) we find constants $C>0$, $\delta>0$ and $\delta_1>0$ such that

$$N(x(t,x,u)-y(t,x,u))\le Ct\varepsilon N(x)^{\tau+1} \qquad (9)$$

for all $\|x\|\le\delta$, all $u\in U$ with $\|u\|\le\delta$ and all times $t\in[0,N(x)^{-\tau}\delta_1]$.

Then the following corollary holds.

Corollary 4.1 *Consider the functions* f *and* g *from above. Assume the system given by* f *is asymptotically controllable to the origin. Then the function* V *from Theorem 3.1 is a local control Lyapunov function for* g *in a neighborhood of the origin. In particular, the system given by* g *is locally asymptotically controllable to the origin.*

Proof: Take the control Lyapunov function V from Theorem 3.1.

Using the homogeneity (5) of the trajectories of the homogeneous system and estimate (9) (making δ and δ_1 smaller, if necessary) we obtain the existence of a constant $\gamma\in(0,1)$ such that

$$N(\Delta^{-1}_{N(x)}x(t,x,u))\in[1-\gamma,1+\gamma]$$

and

$$N(\Delta^{-1}_{N(x)}y(t,x,u))\in[1-\gamma,1+\gamma]$$

for all $t \in [0, N(x)^{-\tau}\delta_1]$, all $x \in \mathbb{R}^n$ with $\|x\| \le \delta$ and all $u \in U$ and $\|u\| \le \delta$.

Now let L denote the Lipschitz constant of V on the compact set $\{x \in \mathbb{R}^n \mid N(x) \in [1-\gamma, 1+\gamma]\}$. Since V is homogeneous and N satisfies $N(x) \le K\|x\|$ for some suitable $K > 0$ we can conclude

$$\begin{aligned}
&|V(x(t,x,u)) - V(y(t,x,u))| \\
&= N(x)^{2k}|V(\Delta_{N(x)}^{-1}x(t,x,u)) - V(\Delta_{N(x)}^{-1}y(t,x,u))| \\
&\le N(x)^{2k}L\|\Delta_{N(x)}^{-1}x(t,x,u) - \Delta_{N(x)}^{-1}y(t,x,u)\| \\
&\le N(x)^{2k}LKN(\Delta_{N(x)}^{-1}x(t,x,u) - \Delta_{N(x)}^{-1}y(t,x,u)) \\
&= LKN(x)^{2k}N(x)^{-1}N(x(t,x,u) - y(t,x,u)) \\
&= LKN(x)^{2k-1}N(x(t,x,u) - y(t,x,u)).
\end{aligned}$$

Together with estimate (9) this implies

$$\begin{aligned}
&\inf_{u\in U_x} V(y(t,x,u)) \\
&\le \inf_{u\in U_x} V(x(t,x,u)) + LKN(x)^{2k-1}Ct\varepsilon N(x)^{\tau+1} \\
&\le V(x) - 2t\mu N^\tau(x)V(x) + LKCt\varepsilon N(x)^{\tau+2k} \\
&\le V(x) - t\mu N^\tau(x)V(x)
\end{aligned}$$

for all x with $\|x\|$ sufficiently small, and all times $t \in [0, N(x)^{-\tau}\delta_1]$.

This estimate immediately implies the assertion. $\square$

4.2 Sampled Feedbacks

The Lyapunov function from Theorem 3.1 allows the construction of a sampled feedback stabilizing the system. Here we only give the plain definition of a sampled feedback, for details and a discussion of this concept see e.g. [7, 16]. Again, we formulate the result for homogeneous systems; the analogous result is true for homogeneous-in-the-state systems.

A (discontinuous) *feedback* is any map $F : \mathbb{R}^n \to U$. F is called homogeneous if $F(\Lambda_\alpha x) = \Delta_\alpha F(x)$.

An infinite sequence $\pi = (t_i)_{i\in\mathbb{N}_0}$ of times satisfying

$$0 = t_0 < t_1 < t_2 < \ldots \quad \text{and} \quad t_i \to \infty \text{ as } i \to \infty$$

is called a *sampling schedule*. The values

$$\Delta t_i := t_{i+1} - t_i \quad \text{and} \quad d(\pi) := \sup_{i\in\mathbb{N}_0} \Delta t_i$$

is called the *intersampling times* and the *sampling rate*, respectively. For any sampling schedule π the corresponding *sampled* or *π-trajectory* $x_\pi(t, x_0, F)$ with initial value $x_0 \in \mathbb{R}^n$ at initial time $t_0 = 0$ is defined inductively by

$$x_\pi(t, x_0, F) = x(t - t_i, x_i, F(x_i)),$$

for all $t \in [t_i, t_{i+1}], i \in \mathbb{N}_0$, where $x_i = x_\pi(t_i, x_0, F)$ and $x(t, x_i, F(x_i))$ denotes the (open loop) trajectory of (1) with constant control value $F(x_i)$ and initial value x_i.

We now define a feedback $F : \mathbb{R}^n \setminus \{0\} \to U$ by choosing for each x the control value u_x from Theorem 3.1 and set

$$F(x) := u_x. \tag{10}$$

Recall that u_x depends homogeoeously on x, hence F is a homogeneous map, i.e. $F(\Lambda_\alpha x) = \Delta_\alpha F(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $\alpha > 0$. Note that when we set $F(0) = 0$ we even obtain continuity of F in the origin.

Using this F we obtain the following result from Theorem 3.1 by a standard Lyapunov function argument.

Corollary 4.2 *Assume asymptotic controllability of system* (1) *satisfying* (2)*. Then there exists $h > 0$ such that the sampled closed loop system using F from* (10) *is asymptotically stable if the intersampling times satisfy $\Delta t_i \le hN^{-\tau}(x_i)$.*

By the same arguments as used in Section 4.1 this feedback will also locally stabilize the nonlinear system given by g, provided g admits an asymptotically controllable homogeneous approximation.

Note that in contrast to the analogous result for general nonlinear systems in [2], here we can give a positive upper bound on the sampling rate depending on the degree τ of the system. In particular, if $\tau \ge 0$ we obtain semi-global asymptotic stability with fixed positive sampling rate, and if $\tau = 0$ we even obtain global asymptotic stability with fixed positive sampling rate, cf. also the more detailed discussions in [8] and [7].

Similar to what was done for semilinear systems in [4], also here we can obtain a numerical approximation of F, some numerical examples can be found in [8].

5 Conclusion

We have shown the existence of homogeneous control Lyapunov functions for asymptotically controllable homogeneous control systems.

This existence result implies local asymptotic controllability for nonlinear systems which can be approximated by asymptotically controllable homogeneous systems.

Furthermore, an additional property of the control Lyapunov functions allows the construction of a stabilizing homogeneous discontinuous sampled feedback. Depending on the degree of the system, we can give positive upper bounds on the sampling rate which ensure asymptotic stability of the sampled closed loop system.

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