# Convergence Rates of Perturbed Attracting Sets with Vanishing Perturbation

Lars Grüne
Fachbereich Mathematik
J.W. Goethe-Universität
Postfach 111932
60054 Frankfurt a.M., Germany
gruene@math.uni-frankfurt.de

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**Abstract:** We investigate the rate of convergence and the rate of attraction of perturbed attracting sets as the perturbation tends to zero. The perturbation model under consideration is a control system which contains the unperturbed system as zero dynamics. Necessary and sufficient conditions for certain rates of convergence are derived. Several applications especially to numerical approximations are given and a number of examples illustrates the results.

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## 1 Introduction

In the analysis of complex dynamic behaviour attracting sets form a fundamental concept. In many cases, all complicated dynamical behaviour appears inside attracting sets, like e.g. under suitable conditions all instable manifolds are contained inside the global attractor being just one special attracting set, cf. [7].

Therefore, the behaviour of attracting sets (or—more specifically—attractors) under perturbation has attracted notable attention during the last years, e.g. in the context of numerical approximation [10, 11, 2], control theory [6, 1], random dynamical systems [8], and differential inclusions [9]. Essentially, in all these works (semi-)continuity properties of (more or less specific) attracting sets are derived. For exponentially attracting sets, also the rate of convergence for vanishing perturbation is investigated in [11, 2].

In the present paper we are going to begin an analysis of the rate of convergence for attracting sets with arbitrary rate of attraction. The setup used here is a family of deterministic time varying perturbations of the given vectorfield which can be interpreted as a control system for which the original system is just the uncontrolled equation. This setup allows the definition of *strongly attracting* sets which—if existing—always contain an attracting set of the unperturbed system. As we will see, this setup ensures convergence of these sets without additional conditions on the attracting set. Furthermore it allows a number of implications to more specifically perturbed systems, like e.g. several types of numerical approximations.

The question under consideration is the following: Assume that the perturbation, measured e.g. by some real parameter $\alpha$, tends to 0. Then, how fast do the corresponding strongly attracting sets converge to an attracting set of the unperturbed system? Already simple examples show that this question cannot be answered in general, but that the answer strongly depends on properties of the unperturbed attracting set.

Here we introduce two conditions on the rate of attraction of attracting sets. The first is a condition on the *contractivity* of a suitable family of neighbourhoods of the attracting set, and will turn out to be equivalent to a related rate of convergence of the perturbed attracting sets as the perturbation vanishes. The second condition is on the *rate of attraction* of the attracted set. This condition is equivalent to a related rate of convergence of the perturbed attracted sets *plus* the persistence of the rate of attraction under perturbation, i.e. the perturbed attracting sets attract with the same rate as the unperturbed.

These conditions which can be seen as a generalization of the well known exponential attraction property can be regarded as one of the main contributions of the present paper. The fact that they are equivalent to certain convergence and attraction rates of the perturbed attracting sets indicates that these are exactly the right conditions for the given setup.

The results allow numerous applications, some of them will be discussed in this paper. For instance they allows estimates for the (semicontinuous) discretization error for one-step discretization of arbitrary attracting sets; in fact some results from [11] are improved even in the exponential case.

This paper is organized as follows: After defining the general setup in Section 2, we briefly discuss the behaviour of absorbing sets under perturbation in Section 3. In Section 4 and 5 we introduce the mentioned conditions and prove the equivalence results. Section 6 gives slightly simplified conditions, and Section 7 discusses the special case of exponentially attracting sets. In Section 8 we present a number of applications, and in Section 9 we provide a number of illustrative examples.

## 2 Setup and Definitions

We consider the ordinary differential equation

$$\dot{x}(t) = f(x(t)) \tag{2.1}$$

in $\mathbb{R}^n$, and assume that $f$ is globally Lipschitz with constant $L$. (Since we are only interested in the behaviour on compact subsets this is no real restriction.) We denote the solution by $\varphi_0(t,x)$, and for subsets $B \subset \mathbb{R}^n$ we set $\Phi_0(t,B) := \bigcup_{x\in B}\{\Phi_0(t,x)\}$.

In order to measure distances between sets we make use of the following functions.

**Definition 2.1** Let $C$, $D \subset \mathbb{R}^n$ be compact sets. We define the nonsymmetric Hausdorff distance by

$$\mathrm{dist}(C,D) := \max_{x\in C}\min_{y\in D} d(x,y),$$

the Hausdorff metric by

$$d_H(C,D) := \max\{\mathrm{dist}(C,D), \mathrm{dist}(D,C)\},$$

and, if $C \subseteq D$, we define the minimal distance $d_{\min}$ by

$$d_{\min}(C,D) := \inf_{x\notin D}\min_{y\in C} d(x,y).$$

**Remark 2.2** *Observe that for dist the following triangle inequality*

$$\mathrm{dist}(C,D) \le \mathrm{dist}(C,E) + \mathrm{dist}(E,D) \tag{2.2}$$

*holds for all compact sets* $C$, $D$, $E \subset \mathbb{R}^n$.

Now we define the meaning of an attracting set. Recall that a set $B \subset \mathbb{R}^n$ is called forward invariant if $\Phi_0(t,B) \subseteq B$ for all $t \ge 0$.

**Definition 2.3** Consider two compact forward invariant sets $A$, $B \subset \mathbb{R}^n$ with $A \subset \mathrm{int} B$.

We call $A$ *attracting* with *attracted neighbourhood* $B$, if

$$\mathrm{dist}(\Phi_0(t,B), A) \to 0$$

as $t \to \infty$.

We call $A$ *absorbing* with *absorbed neighbourhood* $B$, if there exists $T > 0$ with

$$\Phi_0(T,B) \subseteq A$$

**Remark 2.4** *Note that these are local definitions, in the sense that we require the attracting and absorbing property only in a neighbourhood of* $A$.

*Under our assumptions this definition of an attracting set is equivalent to the uniform asymptotic stability property as defined in [11, Definition 2.7.3]. It is slightly weaker than the notion of an attractor since no backward invariance of* $A$ *is assumed here, see [11, Theorem 2.7.4] for the precise relation.*

Associated to (2.1) we consider the family of perturbed systems

$$\dot{x}(t) = f(x(t)) + \alpha w(t) \tag{2.3$^\alpha$}$$

for some real parameter $\alpha \ge 0$ with $w(\cdot) \in \mathcal{W} := \{w : \mathbb{R} \to [-1,1]^n, \text{measurable}\}$. The systems (2.3$^\alpha$) can be interpreted as a family of control system with the original system as uncontrolled dynamics. We denote the solutions by $\varphi_\alpha(t,x,w)$, for $x \in \mathbb{R}^n$ we abbreviate $\Phi_\alpha(t,x) := \bigcup_{w\in\mathcal{W}}\{\varphi_\alpha(t,x,w)\}$, and for $B \subset \mathbb{R}^n$ we write $\Phi_\alpha(t,B) := \bigcup_{x\in B}\Phi_\alpha(t,x)$.

Note that the implication

$$B \subseteq C \quad\Rightarrow\quad \Phi_\alpha(t,B) \subseteq \Phi_\alpha(t,C) \tag{2.4}$$

is immediate for all $t \geq 0$.

**Definition 2.5** Fix some $\alpha > 0$. For the perturbed system ($2.3^\alpha$) we call a set $A_\alpha \subset \mathbb{R}^n$ *$\alpha$-strongly forward invariant* if $\Phi_\alpha(t,A_\alpha) \subseteq A_\alpha$ for all $t \geq 0$.

Observe that by definition of $\Phi_\alpha$ any $\alpha$-strongly forward invariant set also is an $\alpha'$-strongly forward invariant set for each $\alpha' \in [0,\alpha)$.

**Definition 2.6** Consider two compact $\alpha$-strongly forward invariant sets $A$, $B \subset \mathbb{R}^n$ with $A \subset \mathrm{int} B$.

We call $A$ *$\alpha$-strongly attracting* with *$\alpha$-strongly attracted neighbourhood* $B$, if

$$\mathrm{dist}(\Phi_\alpha(t,B),A) \to 0$$

as $t \to \infty$.

We call $A$ *$\alpha$-strongly absorbing* with *$\alpha$-strongly absorbed neighbourhood* $B$, if there exists $T > 0$ with

$$\Phi_\alpha(T,B) \subseteq A$$

**Remark 2.7** *If $A$ attracts a compact neighbourhood $B$ which is not $\alpha$-strongly forward invariant, then it also attracts its compact neighbourhood $\tilde{B} := \bigcup_{t\geq 0}\Phi_\alpha(t,B)$ which is $\alpha$-strongly forward invariant. Thus forward invariance of $B$ can be assumed without loss of generality.*

Observe that these Definitions for $\alpha = 0$ coincide with the corresponding Definitions for the unperturbed system. Keeping this in mind we will now introduce further concepts for both the unperturbed and the perturbed system by defining them for $\alpha \geq 0$.

**Definition 2.8** Let $\alpha \geq 0$. Consider two compact $\alpha$-strongly forward invariant sets $A$, $B \subset \mathbb{R}^n$ with $A \subset \mathrm{int} B$, where $A$ is $\alpha$-strongly attracting with attracted neighbourhood $B$

A family of compact, $\alpha$-strongly forward invariant sets $B_\vartheta$, $\vartheta \in \mathbb{R}_0^+$, which depend continuously (w.r.t. to the Hausdorff metric $d_H$) on $\vartheta$ and satisfy $B_0 = B$ is called an *$\alpha$-contracting family of neighbourhoods* if there exist $T > 0$ with

(i) $A = \bigcap_{\vartheta\in\mathbb{R}_0^+} B_\vartheta$

(ii) $B_{\vartheta'} \subseteq B_\vartheta$ for all $\vartheta, \vartheta' \in \mathbb{R}_0^+$, $\vartheta' \geq \vartheta$

(iii) $\Phi_\alpha(T, B_\vartheta) \subseteq B_{\vartheta+T}$ for all $\vartheta \in \mathbb{R}_0^+$.

**Remark 2.9** *(i) Any $\alpha$-strongly attracting set $A$ with attracted neighbourhood $B$ admits an $\alpha$-contracting family of neighbourhoods for arbitrary $T > 0$ given by $B_\vartheta = \Phi_\alpha(\vartheta, B) \cup A$, $\vartheta \in \mathbb{R}_0^+$. Conversely, if $A$ admits an $\alpha$-contracting family of neighbourhoods it is easily seen that it is attracting.*

*(ii) Using a suitable parametrization an $\alpha$-contracting family of neighbourhoods can also be constructed from the sublevel sets of some Lyapunov function for the attracting set $A$.*

# 3 Absorbing Sets under Perturbations

We start our analysis of the perturbation effects by considering the absorbing sets. Here the situation is rather simple, however, for sake of completeness we like to include it.

**Theorem 3.1** Consider system (2.1) and the perturbed family of systems ($2.3^\alpha$). Let $A$ be a compact forward invariant set for system (2.1). Then $A$ is an absorbing set for system (2.1) if and only if there exists an $\alpha_0 > 0$ such that $A$ is an $\alpha$-strongly absorbing set for system ($2.3^\alpha$) for each $\alpha \in [0, \alpha_0]$.

**Proof:** "$\Leftarrow$" follows immediately from the assumption. In order to see "$\Rightarrow$" let $A$ be an absorbing set for system (2.1), and let $B$ be its absorbed neighbourhood. Then there exists $T > 0$ such that $\Phi_0(T, B) \subset A$. Since both $A$ and $\Phi_0(T, B)$ are compact this implies $d_{\min}(\Phi_0(T, B), A) =: \varepsilon > 0$. Hence for each $\alpha < \varepsilon/(e^{LT}T)$ Lemma 10.1 implies $\Phi_\alpha(T, B) \subset A$, i.e. $A$ is $\alpha$-strongly absorbing. □

# 4 Attracting Sets under Perturbations

For general attracting sets $A$ of system (2.1) the situation is less simple. Here we cannot expect that $A$ also is an $\alpha$-strongly attracting set for small $\alpha > 0$. Instead, we can only ensure the existence of $\alpha$-strongly attracting sets $A_\alpha$ which are close to $A$. In order to estimate how close they are we introduce a measure for the contractivity of an $\alpha$-contracting family. Here we use a function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions.

$$\gamma \text{ is strictly increasing and continuous with } \gamma(0) = 0 \tag{4.1}$$

$$\text{for all } s_0,\, r > 0 \text{ there exist } \tilde{r},\, \bar{r} > 0 \text{ with } \tilde{r}\gamma(s) \leq \gamma(rs) \leq \bar{r}\gamma(s) \text{ for all } s \in [0, s_0] \tag{4.2}$$

Typical examples of $\gamma$ are $\gamma(s) = s^{1/k}$ or $\gamma(s) = s^{1/k}\ln(1/s)$, where $k \in \mathbb{N}$. Using this function $\gamma$ we can now define the rate of attraction.

**Definition 4.1** Let $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy (4.1)–(4.2), and let $\alpha \ge 0$. A $\alpha$-contracting family $B_\vartheta$ for some $\alpha$-strongly attracting set $A$ is called contracting with *contraction rate* $\gamma$, if for some constant $C > 0$ with $C\gamma(s) \ge s$ for all $s \in \mathbb{R}_0^+$ it satisfies

$$d_H(B_\vartheta, A) \le C\gamma(d_{\min}(\Phi_\alpha(T, B_\vartheta), B_\vartheta)) \text{ for all } \vartheta \in \mathbb{R}_0^+.$$

This definition requires that the rate of contraction of the sets $B_\vartheta$ can be estimated from above by the distance from the attracting set. Note that, although Remark 2.9(i) gives a constructive existence result for $\alpha$-contracting families of neighbourhoods, the neighbourhoods obtained there via $\Phi_\alpha$ might not be suitable for our purpose, cp. Example 9.1, below. Obviously, only the values of $\gamma$ for small $s$ are important in this definition.

A short computation reveals that Definition 4.1 implies (but is in general stronger than)

$$\text{dist}(\Phi_\alpha(T, B_\vartheta), A) \le \tilde\gamma(d_H(B_\vartheta, A))$$

with $\tilde\gamma(s) = s - \gamma^{-1}(s)/C$. Note that if $A$ is not absorbing this immediately implies $\tilde\gamma(s) > 0$ for all $s \in [0, d_H(B_0, A)]$, hence $\gamma(s) \ge s/C$, i.e. the assumption in Definition 4.1 is automatically satisfied.

Now we can state our first theorem on the perturbation of attracting sets.

**Theorem 4.2** Consider system (2.1) and the perturbed family of systems ($2.3^\alpha$). For system (2.1) consider compact and forward invariant sets $A$, $B \subset \mathbb{R}^n$ with $A \subset \text{intB}$. Let $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy (4.1)–(4.2). Then the following properties are equivalent.

(i) $A$ admits a 0-contracting family of neighbourhoods $B_\vartheta$ with $B_0 = B$ and contraction rate $\gamma$.

(ii) For suitable constants $K, \alpha^* > 0$ and all $\alpha \in (0, \alpha^*]$ there exist $\alpha$-strongly attracting sets $A_\alpha$ of ($2.3^\alpha$) with attracted neighbourhood $B$ and $A \subseteq A_\alpha$ which satisfy

$$d_H(A_\alpha, A) \le K\gamma(\alpha).$$

**Proof:** (i) $\Rightarrow$ (ii): Let $B_\vartheta$ be the 0-contracting family of neighbourhoods which satisfies $C\gamma(d_{\min}(\Phi_0(T, B_\vartheta), B_\vartheta)) \ge d_H(B_\vartheta, A)$ for all $i \in \mathbb{N}$. We define a new contracting family of neighbourhoods $\tilde B_\vartheta$ by setting

$$\tilde B_\vartheta := \bigcup_{t \in [0,T]} \Phi_{\alpha(\vartheta)}(t, B_\vartheta),$$

where $\alpha(\vartheta)$ is chosen such that $C\gamma(e^{LT}T\alpha(\vartheta)) = d_H(B_\vartheta, A)$. Observe that $\alpha(\vartheta)$ is monotone decreasing and continuous. The assumption on the $B_\vartheta$ and the choice of $\alpha(\vartheta)$ together with Lemma 10.1 ensure $\Phi_{\alpha(\vartheta)}(T, B_\vartheta) \subseteq B_\vartheta$, implying that each $\tilde B_\vartheta$ is $\alpha$-strongly forward invariant for all $\alpha \le \gamma^{-1}(d_H(\tilde{B}_\vartheta, A)/2C)/e^{LT}T$. Furthermore, the 0-forward invariance of $B_\vartheta$ and Lemma 10.1 imply $d_H(\tilde{B}_\vartheta, A) \le e^{LT}T\alpha(\vartheta) + d_H(B_\vartheta, A) \le 2d_H(B_\vartheta, A)$. Since

$$\Phi_0(2T, \tilde{B}_\vartheta) = \Phi_0(T, \underbrace{\Phi_0(T, \tilde{B}_\vartheta)}_{\subseteq B_\vartheta}) \subseteq \Phi_0(T, B_\vartheta) \subseteq B_{\vartheta+T} \subseteq \tilde{B}_{\vartheta+T}$$

these sets form a contracting family of neighbourhoods for $A$ with contraction rate $\gamma$, $\tilde{T} = 2T$ and $\tilde{C} = 2C$.

Now choose $\alpha^* > 0$ with $\tilde{C}\gamma(e^{2L\tilde{T}}2\tilde{T}\alpha^*) \le d_H(B, A)$, and fix some $\alpha \in (0, \alpha^*]$. Since $d_H(\tilde{B}_\vartheta, A)$ depends continuously on $\vartheta$ and converges to 0 by the intermediate value theorem we find $\vartheta_\alpha > 0$ with $\tilde{C}\gamma(e^{L2\tilde{T}}2\tilde{T}\alpha) = d_H(\tilde{B}_{\vartheta_\alpha}, A)$ and $\tilde{C}\gamma(e^{2L\tilde{T}}2\tilde{T}\alpha) \le d_H(\tilde{B}_{\vartheta_\alpha}, A)$ for all $\vartheta \in [0, \vartheta_\alpha]$.

We claim that

$$A_\alpha := \tilde{B}_{\vartheta_\alpha}$$

satisfies assertion (ii): From the choice of $\vartheta_\alpha$ and (4.2) we know that

$$d_H(\tilde{B}_{\vartheta_\alpha}, A) = \tilde{C}\gamma(e^{2L\tilde{T}}2\tilde{T}\alpha) \le K\gamma(\alpha)$$

for some suitable constant $K > 0$. Furthermore, by the construction of the $\tilde{B}_\vartheta$ and the choice of $\vartheta_\alpha$ each set $\tilde{B}_\vartheta$, $\vartheta \in [0, \vartheta_\alpha]$ is $\alpha$-strongly forward invariant.

It remains to show that $A_\alpha$ is $\alpha$-strongly attracting with attracted neighbourhood $B$. By the choice of $\vartheta_\alpha$ and the rate of contraction we know that

$$d_{\min}(\Phi_0(2\tilde{T}, \tilde{B}_\vartheta), \tilde{B}_{\vartheta^*}) \ge e^{L2\tilde{T}}2\tilde{T}\alpha \ \text{ for all } \ \vartheta \in [0, \infty) \tag{4.3}$$

where $\vartheta^* = \min\{\vartheta + \tilde{T}, \vartheta_\alpha\}$. Thus from Lemma 10.1 we obtain

$$\Phi_\alpha(2\tilde{T}, \tilde{B}_\vartheta) \subseteq \tilde{B}_{\vartheta^*} \ \text{ for all } \ \vartheta \in [0, \infty).$$

By induction this implies $\Phi_\alpha(2\vartheta_\alpha, B) \subseteq \tilde{B}_{\vartheta_\alpha} = A_\alpha$, thus $A_\alpha$ is an $\alpha$-strongly absorbing set, hence also an $\alpha$-strongly attracting set.

(ii) $\Rightarrow$ (i): We may assume $A_{\tilde{\alpha}} \subseteq A_\alpha$ for all $0 < \tilde{\alpha} < \alpha \le \alpha^*$, otherwise we may use the sets $\tilde{A}_\alpha = \bigcap_{\beta \in [\alpha, \alpha^*]} A_\beta$, which are easily verified as $\alpha$-strongly attracting sets.

We inductively construct a positive sequence $\alpha_i$, a nonnegative sequence $\vartheta_i$, and a family of sets $B_\vartheta$ by setting $\alpha_0 = \alpha^*$, $B_0 = B$, $\vartheta_0 = 0$, and, for $i \in \mathbb{N}_0$

$$\begin{aligned}
\vartheta_{i+1} &:= \min\{\vartheta \in [\vartheta_i, \infty) \,|\, d_H(\Phi_{\alpha_i}(\vartheta - \vartheta_i, B_{\vartheta_i}), A_{\alpha_i}) \le \gamma(\alpha_i)\} \\
B_\vartheta &:= \Phi_{\alpha_i}(\vartheta - \vartheta_i, B_{\vartheta_i}) \cup A_{\alpha_i}, \ \vartheta \in (\vartheta_i, \vartheta_{i+1}] \\
\alpha_{i+1} &:= \frac{\alpha_i}{2}
\end{aligned}$$

Since by attractivity $\operatorname{dist}(\Phi_{\alpha_i}(t, B_{\vartheta_i}), A_{\alpha_i}) \to 0$ as $t \to \infty$ the time $\vartheta_{i+1}$ exists for all $i$ and thus we obtain $\alpha_i \to 0$ as $i \to \infty$. This construction implies

$$d_H(B_\vartheta, A_{2\alpha_i}) \le \gamma(2\alpha_i) \tag{4.4}$$

for each $\vartheta \in [\vartheta_i, \infty)$ with $i \geq 1$.

We claim that the $B_\vartheta$ form a 0-contracting neighbourhood for $A$ and system (2.1) as defined in Definition 2.8: In order to prove this fix some $i \in \mathbb{N}_0$ and some $\vartheta \in (\vartheta_i, \vartheta_{i+1}]$. Obviously the family $B_\vartheta$ is continuous w.r. to the Hausdorff metric. Then Property (i) follows from the inclusion $A \subseteq A_{\alpha_i} \subseteq B_\vartheta$, from $d_H(A_{\alpha_i}, A) \to 0$ as $i \to \infty$, and from (4.4). Property (ii) follows since by construction $B_\vartheta$ is $\alpha_i$-strongly forward invariant, hence also $\alpha_i/2$-strongly forward invariant, and Property (iii) follows directly from the construction.

It remains to show the rate of contraction, which we will in fact show for arbitrary $T > 0$. By construction, each $B_\vartheta$, $\vartheta \in (\vartheta_i, \vartheta_{i+1}]$ is $\alpha_i$-strongly forward invariant, thus by Corollary 10.3 we know that there exists $K_1 > 0$ such that

$$d_{\min}(\Phi_0(T, B_\vartheta), B_\vartheta) \geq K_1 \alpha_i \text{ for all } \vartheta \in (\vartheta_i, \vartheta_{i+1}]. \tag{4.5}$$

On the other hand, by (4.4) and by assumption on the $A_\alpha$ we know that

$$d_H(B_\vartheta, A) \leq d_H(B_\vartheta, A_{2\alpha_i}) + d_H(A_{2\alpha_i}, A) \leq \gamma(2\alpha_i) + K\gamma(2\alpha_i) \text{ for all } \vartheta \in (\vartheta_i, \vartheta_{i+1}] \tag{4.6}$$

and all $i \geq 1$ (i.e. $\alpha_i \neq \alpha^*$). If $i = 0$ by compactness of $B_0$ there exist constants $C_1, C_2 > 0$, independent of $\vartheta \in [0, \vartheta_1]$ with

$$d_H(B_\vartheta, A) \leq C_1 \leq C_2 \gamma(2\alpha_0). \tag{4.7}$$

Thus by (4.6), (4.7), and (4.2) there exists $K_2 \geq 0$ such that

$$d_H(B_\vartheta, A) \leq K_2 \gamma(\alpha_i) \text{ for all } i \in \mathbb{N}_0 \text{ and all } \vartheta \in (\vartheta_i, \vartheta_{i+1}]. \tag{4.8}$$

Putting (4.5) and (4.8) together and once more using (4.2) we obtain

$$\gamma(d_{\min}(\Phi_0(T, B_\vartheta), B_\vartheta)) \geq \gamma(K_1 \alpha_i) \geq \tilde{K}_1 \gamma(\alpha_i) \geq \frac{\tilde{K}_1}{K_2} d_H(B_\vartheta, A)$$

i.e. the assertion. □

**Remark 4.3** *(i) Note that in fact we have proven the existence of absorbing sets $A_\alpha$ for system* (2.3$^\alpha$).

*(ii) If in the second part of the proof we fix some $\tilde{\alpha} \in (0, \alpha^*)$ we can set $\alpha_{i+1} = (\alpha_i + \tilde{\alpha})/2$ instead of $\alpha_{i+1} = \alpha_i/2$, and obtain an $\alpha$-contracting family of neighbourhoods for $A_{\tilde{\alpha}}$. Then a straightforward modification of the arguments shows that this family is contracting with rate $\gamma$ and constant $C$ independent of $\tilde{\alpha}$.*

## 5 Attraction Rates and Perturbations

The contractivity condition from Definition 4.1 for each sufficiently small $\alpha > 0$ ensures both the existence of an attracting set for the perturbed system and the existence of an attracting neighbourhood with the same contraction rate, cf. Remark 4.3(ii).

However, often one is also interested in the rate of attraction of the attracting set, i.e. given an attracted neighbourhood $B$ one would like to know how fast $\Phi_\alpha(t, B)$ converges to $A_\alpha$. The contractivity condition from Definition 4.1 in general does not give any information about this rate. The following definition gives a stronger condition which can be used for this purpose.

**Definition 5.1** Let $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy (4.1)–(4.2), and let $\alpha \geq 0$. An $\alpha$-strongly attracting set $A$ is said to have *attraction rate* $\gamma$, if for some $T > 0$ it admits an $\alpha$-contracting family of neighbourhoods $B_\vartheta$ which for some constant $C > 0$ satisfies Definition 4.1, and

$$d_H(B_\vartheta, A) \leq C\gamma(d_{\min}(B_{\vartheta+T}, B_\vartheta)) \text{ for all } \vartheta \in \mathbb{R}_0^+ \text{ with } B_{\vartheta+T} \neq A.$$

This condition includes the contraction rate of Definition 4.1, but is slightly stronger since it also gives information about the distance between the $B_\vartheta$, i.e. we obtain

$$d_H(B_{\vartheta+kT}, A) \leq \tilde{\gamma}^k(B_\vartheta, A), \tag{5.1}$$

where $\tilde{\gamma}(s) = s - \gamma^{-1}(s)/C$, and $\tilde{\gamma}^k$ denotes the $k$-fold concatenation of $\tilde{\gamma}$. In particular this yields an explicit rate for the convergence $B_{kT} \to A$ as $k \to \infty$.

Similar to Theorem 4.2 we can now state a relation between the perturbed and the unperturbed systems.

**Theorem 5.2** Consider system (2.1) and the perturbed family of systems ($2.3^\alpha$). For system (2.1) consider compact and forward invariant sets $A$, $B \subset \mathbb{R}^n$ with $A \subset \text{intB}$. Let $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy (4.1)–(4.2). Then the following properties are equivalent.

(i) $A$ is attracting with attracted neighbourhood $B$ and attraction rate $\gamma$.

(ii) For suitable constants $K, \alpha^* > 0$ and all $\alpha \in (0, \alpha^*]$ there exist $\alpha$-strongly attracting sets $A_\alpha$ of ($2.3^\alpha$) with attracted neighbourhood $B$ and $A \subseteq A_\alpha$ which satisfy

$$d_H(A_\alpha, A) \leq K\gamma(\alpha)$$

and are attracting with rate $\gamma$.

The corresponding $\alpha$-contracting families $B_\vartheta^\alpha$ realizing this rate can be chosen in such a way that for each $\vartheta_0 > 0$ there exists $\tilde{\alpha}(\vartheta_0) > 0$ with

$$B_\vartheta^\alpha = B_\vartheta^{\tilde{\alpha}} \text{ for all } \vartheta \in [0, \vartheta_0] \text{ and all } \alpha \in (0, \tilde{\alpha}(\vartheta_0)], \tag{5.2}$$

and that the time $T$ in Definition 2.8 and the constant $C > 0$ in Definition 5.1 are independent of $\alpha$.

**Proof:** (i) $\Rightarrow$ (ii): Consider the 0-contracting family of neighbourhoods $B_\vartheta$ satisfying Definition 5.1.

Fixing some $\alpha \in (0, \alpha^*]$ we proceed as in the first part of the proof of Theorem 4.2, define the value $\vartheta_\alpha$ and the set $A_\alpha = \tilde{B}_{\vartheta_\alpha}$. Since Definition 5.1 includes Definition 4.1 we can again conclude that $A_\alpha$ is an $\alpha$-strongly attracting set which has the desired distance from $A$. In addition, the third inequality in Lemma 10.1 implies that passing from the $B_\vartheta$ to the $\tilde{B}_\vartheta$ preserves the rate of attraction from Definition 5.1 by suitably enlarging $C > 0$.

It remains to show the rate of attraction for $A_\alpha$. We will accomplish this by constructing a suitable $\alpha$-contracting family of neighbourhoods for $\bar{T} = 2\tilde{T}$. Consider the family of neighbourhoods given by $B^\alpha_\vartheta = \tilde{B}_{\vartheta/2}$ for $\vartheta < 2\vartheta_\alpha$, and $B^\alpha_\vartheta = \tilde{B}_{\vartheta_\alpha} = A_\alpha$ for $\vartheta \geq 2\vartheta_\alpha$. From the properties of the $\tilde{B}_\vartheta$ this family satisfies the inequality

$$d_H(B^\alpha_\vartheta, A_\alpha) \leq d_H(B^\alpha_\vartheta, A) \leq \bar{C}\gamma(d_{\min}(B^\alpha_{\vartheta+\bar{T}}, B^\alpha_\vartheta))$$

for all $\vartheta \in \mathbb{R}^+_0$ with $B^\alpha_{\vartheta+\bar{T}} \neq A_\alpha$, and some suitable constant $\bar{C} > 0$. Furthermore, translating inequality (4.3) to the $B^\alpha_\vartheta$ yields

$$d_{\min}(\Phi_0(\bar{T}, B^\alpha_\vartheta), B^\alpha_{\vartheta+\bar{T}}) \geq e^{L\bar{T}}\bar{T}\alpha \text{ for all } \vartheta \in [0, \infty).$$

Thus by Lemma 10.1 we obtain the inclusion

$$\Phi_\alpha(\bar{T}, B^\alpha_\vartheta) \subseteq B^\alpha_{\vartheta+\bar{T}} \text{ for all } \vartheta \in [0, \infty).$$

Obviously, these families $B^\alpha_\vartheta$ satisfy (5.2), hence the assertion follows.

(ii) $\Rightarrow$ (i): For each $\alpha \in (0, \alpha^*]$ denote the $\alpha$-contracting family of neighbourhoods by $B^\alpha_\vartheta$. We set

$$B_\vartheta := B^{\tilde{\alpha}(\vartheta)}_\vartheta$$

for $\tilde{\alpha}(\vartheta)$ from (5.2). Note that for each $\vartheta_0 > 0$ this implies

$$B_\vartheta = B^\alpha_\vartheta \text{ for all } \vartheta \in [0, \vartheta_0] \text{ and all } \alpha \in (0, \tilde{\alpha}(\vartheta_0)]. \tag{5.3}$$

Thus continuity of $B_\vartheta$ in $\vartheta$ follows, and the properties $A = \bigcap_{\vartheta \in \mathbb{R}^+_0} B_\vartheta$ and $B_\vartheta \subseteq B_{\vartheta'}$ for $\vartheta' \leq \vartheta$ are immediate from $A_\alpha = \bigcap_{\vartheta \in \mathbb{R}^+_0} B^\alpha_\vartheta$, $B^\alpha_\vartheta \subset B^\alpha_{\vartheta'}$ for each $\alpha \in (0, \alpha^*]$, and $d_H(A_\alpha, A) \to 0$ as $\alpha \to 0$.

Furthermore, for any $\vartheta > 0$ (5.3) implies

$$\Phi_0(T, B_\vartheta) = \Phi_0(T, B^\alpha_\vartheta) \subseteq \Phi_\alpha(T, B^\alpha_\vartheta) \subseteq B^\alpha_{\vartheta+T} = B_{\vartheta+T}$$

for $\alpha \in (0, \tilde{\alpha}(\vartheta + T)]$.

In order to obtain the rate of attraction observe that $B_{\vartheta+T} \neq A$ implies $B_{\vartheta+T} \neq B_{\vartheta+T+t_\vartheta}$ for some sufficiently large $t_\vartheta > 0$. Thus $B^\alpha_{\vartheta+T} \neq B^\alpha_{\vartheta+T+t_\vartheta}$ for all $\alpha \in (0, \tilde{\alpha}(\vartheta + T + t_\vartheta)]$, and consequently $B^\alpha_{\vartheta+T} \neq A_\alpha$. Hence for all $\vartheta \in \mathbb{R}^+_0$ with $B_{\vartheta+T} \neq A$ and all $\alpha \in (0, \tilde{\alpha}(\vartheta+T+t_\vartheta)]$ we have

$$\begin{aligned} C\gamma(d_{\min}(B_\vartheta, B_{\vartheta+T})) &= C\gamma(d_{\min}(B^\alpha_\vartheta, B^\alpha_{\vartheta+T})) \\ &\geq d_H(B^\alpha_\vartheta, A_\alpha) \\ &\geq d_H(B^\alpha_\vartheta, A) - K\gamma(\alpha) \\ &= d_H(B_\vartheta, A) - K\gamma(\alpha) \end{aligned}$$

Thus, letting $\alpha \to 0$ the desired inequality follows. □

**Remark 5.3** *(i) Again we explicitely constructed absorbing sets for the perturbed system. This implies that for each fixed $\alpha$ we could indeed obtain arbitrary fast rates of convergence by choosing the constant $C > 0$ or the time $T > 0$ sufficiently large. Thus the important property in (ii) is the uniformity of $C > 0$ and $T > 0$ with respect to $\alpha$.*

*(ii) A particular (and rather natural) choice of $A_\alpha$ would be the* smallest *$\alpha$-strongly attracting set containing $A$ (in a differential inclusion setting a similar object is considered in [9] under the name "inflated attractor"). However, Example 9.3, below, shows that for this smallest set the assertion of Theorem 5.2 might not be true.*

# 6 A simplified criterion

An $\alpha$-contracting family of neighbourhoods meeting the conditions in the Definitions 4.1 and 5.1 might be difficult to construct. One reason for this is the continuity requirement for the family $B_\vartheta$ in $\vartheta$. This requirement, however, is not really necessary. In this section we indicate how such a continuous family can be constructed if we only have a discrete family of neighbourhoods.

Assume we are given a "discrete" $\alpha$-contracting family of neighbourhoods, i.e. a countable sequence of sets $\tilde{B}_i$, $i \in \mathbb{N}_0$, satisfying $\tilde{B}_0 = B$, and

(i) $A = \bigcap_{i\in\mathbb{N}_0} \tilde{B}_i$

(ii) $\tilde{B}_{i+1} \subseteq \tilde{B}_i$ for all $i \in \mathbb{N}_0$

(iii) $\Phi_\alpha(T, \tilde{B}_i) \subseteq \tilde{B}_{i+1}$ for all $i \in \mathbb{N}_0$.

If this family satisfies

$$d_H(\tilde{B}_i, A) \le C\gamma(d_{\min}(\Phi_\alpha(T, \tilde{B}_i), \tilde{B}_i)) \text{ for all } i \in \mathbb{N}_0 \tag{6.1}$$

or

$$d_H(\tilde{B}_i, A) \le C\gamma(d_{\min}(\tilde{B}_{i+1}, \tilde{B}_i)) \text{ for all } i \in \mathbb{N}_0, \tag{6.2}$$

respectively, then there also exists a $\alpha$-contracting family of neighbourhoods satisfying Definition 4.1 or 5.1, respectively.

This holds, since the family $\tilde{B}_i$ can always be extended to a continuous family $B_\vartheta$ depending on $\vartheta \in \mathbb{R}_0^+$. For this purpose we set

$$\tilde{B}_{i+\eta} := \bigcup_{t\in[0,T]} \Phi_\alpha(t, \{x \in \tilde{B}_i \,|\, (1-\eta)d(x, \partial\tilde{B}_i) \ge \eta d(x, \tilde{B}_{i+1})\})$$

for each $\eta \in [0, 1)$ and each $i \in \mathbb{N}_0$ which gives a continuous family of $\alpha$-strongly forward invariant sets with $\tilde{B}_i = B_i$ for $i \in \mathbb{N}_0$. If (6.2) holds we can inductively restrict these sets via

$$\tilde{B}_{i+\eta} := \{x \in \tilde{B}_{i+\eta} \,|\, C\gamma(d(\Phi_\alpha(t, x), \partial\tilde{B}_{i-1+\eta})) \ge d_H(\tilde{B}_{i-1+\eta}, A) \text{ for all } t \in [0, T]\} \tag{6.3}$$

for $\eta \in [0,1]$ and $i = 1, 2, \ldots$. Here (6.2) and the forward invariance imply that the sets $\tilde{B}_i$, $i \in B_0$ remain unchanged, thus this family is well defined and continuity in $\eta$ is ensured.

In both cases we set $B_\vartheta := \tilde{B}_{\vartheta/2T}$; if (6.2) holds the restriction (6.3) immediately implies Definition 5.1.

Furthermore

$$\Phi_\alpha(2T, \tilde{B}_{k+\eta}) \subseteq \Phi_\alpha(2T, \tilde{B}_k) \subseteq \tilde{B}_{k+2} \subseteq \tilde{B}_{k+1+\eta}$$

holds for all $k \in \mathbb{N}_0$ and all $\eta \in [0,1]$, hence

$$\Phi_\alpha(2T, B_\vartheta) = \Phi_\alpha(2T, \tilde{B}_{\vartheta/2T}) \subseteq \tilde{B}_{\vartheta/2T+1} = B_{\vartheta+2T}$$

Thus by using $2T$ instead of $T$ we obtain property (iii) of Definition 2.8 and—if (6.1) holds—the condition of Definition 4.1.

# 7 Exponentially Attracting Sets

In general the contraction or attraction properties in the Definitions 4.1 and 5.1 (even in the simplified version from Section 6) might be difficult to check. There exists, however, a special case allowing particularly nice estimates.

**Definition 7.1** Let $A$ be an $\alpha$-strongly attracting set. We call $A$ *exponentially attracting*, if there exist an attracted neighbourhood $B$ and constants $\beta \geq 1$, $\lambda > 0$ such that for each $x \in B$ and each $w \in \mathcal{W}$ the estimate

$$\mathrm{dist}(\varphi_\alpha(t, x, w), A) < \beta e^{-\lambda t} d(x, A)$$

holds.

**Remark 7.2** *Exponential attraction of an attracting set for the unperturbed system* (2.1) *is guaranteed e.g. if it is a compact hyperbolic set. See e.g. [2, Section 4] for a discussion of this fact in a discrete time setting (which applies here since we consider the time $T$-map).*

Exponential attraction has been investigated and used by a huge number of authors, see e.g. [2, Section 4], [11, Chapter 7], [3], [5], to mention just a few references. In the language of Definition 5.1 it translates into the following rate of attraction.

**Lemma 7.3** Let $A$ be an $\alpha$-strongly exponentially attracting set with attracted neighbourhood $B$. Then for each $C \in (\beta, \infty)$ there exists $T > 0$ such that $A$ is attracting with attraction rate $\gamma(s) = s$ and constant $C$. Conversely, if $A$ is $\alpha$-strongly attracting with attraction rate $\gamma(s) = s$ and some constant $C > 0$ then it is exponentially attracting.

**Proof:** Consider the balls $\mathcal{B}(\delta, A) := \{x \in \mathbb{R}^d \,|\, d(x, A) \leq \delta\}$, and fix some $\delta_0$ such that $\mathcal{B}(\delta_0, A) \subset B$. Set

$$T := \frac{-\ln \frac{C-\beta}{C\beta^2}}{\lambda}, \quad \delta(\vartheta) = \left(\frac{C-\beta}{C\beta}\right)^{\frac{\vartheta}{T}},$$

and $B_\vartheta := \bigcup_{t\geq 0}\Phi_\alpha(t,\mathcal{B}(\delta(\vartheta),A))$.

Then the family $B_\vartheta$ satisfies

$$C d_{\min}(B_\vartheta, B_{\vartheta+T}) \geq C(\delta(\vartheta)-\beta\delta(\vartheta+T)) = \beta\delta(\vartheta) \geq d_H(B_\vartheta, A)$$

and from the exponential estimate we can conclude

$$d_H(\Phi_\alpha(T,B_\vartheta),A) \leq \delta(\vartheta+T),$$

hence the desired inclusion

$$\Phi_\alpha(T,B_\vartheta)\subset B_{\vartheta+T}.$$

For the converse implication observe that the assumption immediately yields

$$\mathrm{dist}(\Phi_\alpha(kT,B_\vartheta),A) \leq (1-1/C)^k d_H(B_\vartheta,A)$$

which by Lemma 10.4 implies

$$\mathrm{dist}(\Phi_\alpha(t,B_\vartheta),A) \leq \tilde{\beta}e^{-\lambda t} d_H(B_\vartheta,A)$$

for suitable constants $\tilde{\beta}\geq 1$, $\lambda>0$ and all $t\geq 0$. Since furthermore the estimate

$$d_H(B_\vartheta,A) \leq C d_{\min}(B_\vartheta,B_{\vartheta+T}) \leq C d_{\min}(B_\vartheta,A)$$

holds by Definition 5.1 this implies the exponential estimate for each $x\in B$ with $\beta = C\tilde{\beta}$. □

The following corollary is now an easy consequence of Lemma 7.3 and Theorem 5.2.

**Corollary 7.4** Consider system (2.1) and the perturbed family of systems ($2.3^\alpha$).

For system (2.1) consider compact and forward invariant sets $A$, $B\subset\mathbb{R}^n$ with $A\subset \mathrm{intB}$. Then the following properties are equivalent.

(i) $A$ is exponentially attracting with attracted neighbourhood $B$.

(ii) For suitable constants $K,\alpha^*>0$ and all $\alpha\in(0,\alpha^*]$ there exist $\alpha$-strongly exponentially attracting sets $A_\alpha$ of ($2.3^\alpha$) with attracted neighbourhood $B$ and $A\subseteq A_\alpha$ which satisfy

$$d_H(A_\alpha,A)\leq K\alpha$$

The corresponding $\alpha$-contracting families $B_\vartheta^\alpha$ realizing this exponential rate can be chosen in such a way that for each $\vartheta_0>0$ there exists $\tilde{\alpha}(\vartheta_0)>0$ with

$$B_\vartheta^\alpha = B_\vartheta^{\tilde{\alpha}} \text{ for all } \vartheta\in[0,\vartheta_0] \text{ and all } \alpha\in(0,\tilde{\alpha}(\vartheta_0)],$$

and that the time $T$ in Definition 2.8 is independent of $\alpha$, furthermore also the constants $\beta$ and $C$ from Definition 7.1 are independent of $\alpha$.

# 8 Applications

In this section we will give a few straightforward applications of the Theorems 4.2 and 5.2. The purpose is to illustrate possible applications rather than elaborating these as far as possible. Stronger results might be possible but are beyond the scope of this paper.

## 8.1 Numerical Approximations by One-Step Schemes

The question of the behaviour of attracting sets under numerical discretization has first been investigated in [10], and then further elaborated by various authors, see [11] for a comprehensive description.

Assume we are given a numerical one step scheme described by a family of continuous maps $S_h : \mathbb{R}^n \to \mathbb{R}^n$, $h \in (0, H]$, satisfying the convergence property

$$\|S_h(t, x_0) - \varphi_0(t, x_0)\| \le C(t)h^p \text{ for all } t = hi,\ i \in \mathbb{N} \tag{8.1}$$

where $C(t)$ is a constant independent of $x_0$ and $h$. (In general such an estimate will only be valid as long as $S_h(t, x_0)$ and $\varphi_0(t, x_0)$ stay inside some compact subset of the state space; here we assume it without loss of generality on the whole $\mathbb{R}^n$ since one could always use a suitable cutoff technique for the vector field leaving the dynamics unchanged in a neighbourhood of the compact attracting set $A$.)

The precise relation between numerical one-step schemes and nonautonomous perturbations of system (2.1) is investigated e.g. in [4, 12]. For our purpose it is sufficient that from Corollary 10.3 and (8.1) for each $T > 0$ we can conclude the existence of a constant $M = M(T) > 0$ such that for each $h \in (0, H]$ and each $\alpha \ge Mh^p$ the inclusion

$$S_h(t, B) \subset \Phi_\alpha(t, B) \text{ for all } B \subset \mathbb{R}^n$$

follows for all $t = ih$, $i \in \mathbb{N}$ with $t \in [0, T]$. Thus any $\alpha$-strongly attracting set for system ($2.3^\alpha$) with $\alpha \ge Mh^p$ is an attracting set for the discrete time semidynamical system induced by $S_h$. This observation immediately implies the following Corollary of Theorem 4.2 and Corollary 7.4.

**Corollary 8.1** Let $S_h$, $h \in [0, H]$ be a family of numerical one-step approximations for system (2.1) satisfying (8.1). Let $A$ be an attracting set for (2.1) which admits a 0-contracting family of neighbourhoods $B_\vartheta$ with $B_0 = B$ and contraction rate $\gamma$. Then there exists a constant $K > 0$ such that for all $h > 0$ sufficiently small there exist attracting sets $A_h$ for the discrete time semidynamical systems induced by $S_h$ with attracted neighbourhood $B$, $A \subset A_h$, and

$$d_H(A_h, A) \le K\gamma(h^p).$$

If, in particular, $A$ is exponentially attracting then the estimate

$$d_H(A_h, A) \le Kh^p$$

holds.

**Remark 8.2** *In the case of exponential convergence this Corollary improves [11, Theorem 7.6.4]. In the general case, estimates for the rate of convergence—to the best of the authors knowledge—are not available at all in the literature.*

## 8.2 Iteration of perturbed time-$\tau$ maps

One way to approximate attractors $A$ (i.e. attracting sets which are invariant) of system (2.1) is by iterating the time-$\tau$ map $\Phi_0(\tau,\cdot)$, beginning with the attracted neighbourhood $B$. Assuming Definition 5.1 it immediately follows from (5.1) that for $\tau = kT$, $k \in \mathbb{N}$, the resulting sequence $B_0 = B$, $B_{i+1} = \Phi_0(\tau, B_i)$ satisfies

$$d_H(B_i, A) \leq \tilde{\gamma}^{ki}(d_H(B_0, A)),$$

where $\tilde{\gamma}(s) = s - \gamma^{-1}(s)/C$, and $\tilde{\gamma}^k$ denotes the $k$-fold concatenation of $\tilde{\gamma}$.

Any algorithmic implementation of this iteration, however, must use some discretization of the state space in order to represent the sets $B_i$ in a suitable data structure (see e.g. [2] for a discussion and implementation of a similar iteration). Denoting this representation by $\tilde{B}_i$ this results in the iteration $B_{i+1} = \Phi_0(\tau, \tilde{B}_i)$.

If we assume $B_i \subset \tilde{B}_i$ and $d_H(B_i, \tilde{B}_i) \leq \beta_i$, then we can "embed" this sequence into the following iteration

$$\bar{B}_0 = B, \quad \bar{B}_{i+1} = \Phi_{\alpha_i}(\tau, \bar{B}_i).$$

Corollary 10.3 ensures the existence of a constant $M > 0$ such that for $\alpha_i = M\beta_i$ the inclusion $\tilde{B}_i \subset \bar{B}_i$ holds for all $i \in \mathbb{N}$.

The following proposition gives an estimate for $d_H(\bar{B}_i, A)$, and by the preceding discussion it can also be used for estimating $d_H(\tilde{B}_i, A)$. For simplicity of exposition here we assume that $\tilde{\gamma}$ is monotone increasing (this is not in general guaranteed by the properties of $\gamma$ but a reasonable assumption, cp. the exponential case above).

**Corollary 8.3** Consider the system (2.1) and let $A \subset \mathbb{R}^n$ be a compact, forward invariant set which is attracting with attracted neighbourhood $B$ and attraction rate $\gamma$. Let $\alpha_i$, $i \in \mathbb{N}$ be a positive sequence with $\alpha_i \to 0$ as $i \to \infty$. Consider the sequence of sets $\bar{B}_i$, $i \in \mathbb{N}$ defined inductively by

$$\bar{B}_0 = B, \quad \bar{B}_{i+1} = \Phi_{\alpha_i}(\tau, \bar{B}_i),$$

where $\tau = kT$ for $k \in \mathbb{N}$ and the time $T$ from Theorem 5.2(ii). Assume $\alpha_1 \leq \alpha^*$ from Theorem 5.2(ii).

Then there exists a constant $K > 0$ independent of the sequence $\alpha_i$, $i \geq 1$, such that the estimate

$$\operatorname{dist}(\bar{B}_i, A) \leq \max_{l \in \{0,\ldots,i\}} \tilde{\gamma}^{k(i-l)}(K\gamma(\alpha_l)) \tag{8.2}$$

holds for all $i \in \mathbb{N}_0$ and $\alpha_0 = \gamma^{-1}(d_H(B, A)/K)$.

In particular if $K\gamma(\alpha_i) \leq \tilde{\gamma}^k(K\gamma(\alpha_{i-1}))$ for all $i \in \mathbb{N}$ the estimate

$$\operatorname{dist}(\bar{B}_i, A) \leq \tilde{\gamma}^{ik}(d_H(\bar{B}_0, A))$$

holds, and if $K\gamma(\alpha_i) \geq \tilde{\gamma}^k(K\gamma(\alpha_{i-1}))$ for all $i \in \mathbb{N}$ the estimate

$$\mathrm{dist}(\bar{B}_i, A) \leq K\gamma(\alpha_i)$$

is implied.

**Proof:** Consider the contracting family of neighbourhoods $B_\vartheta$ of $A$, and recall the construction of the families $B_\vartheta^\alpha$ and the attracting sets $A_\alpha$ in Theorem (5.2)(ii). Let $K > 0$ be the constant from this Theorem.

By the construction for each $i \in \mathbb{N}_0$ there exists $\vartheta_i \geq 0$ such that $\bar{B}_i \subset B_{\vartheta_i}$, where, for $i \geq 1$, either $B_{\vartheta_i} = A_{\alpha_i}$ or $B_{\vartheta_i} = B_{\vartheta_{i-1}+kT}$. Thus Theorem 5.2 together with (5.1) yields the assertion.

The two special cases immediately follow from the monotonicity of $\tilde{\gamma}$. □

**Remark 8.4** *(i) If we assume the set $A$ to be an attractor (i.e. an invariant attracting set) we get the same estimates with the Hausdorff metric $d_H$ instead of the nonsymmetric distance dist.*

*(ii) The embedding of the $\tilde{B}_i$ into the $\bar{B}_i$ will in general not produce the best possible constants for this iteration. For a sharper evaluation of the constant $K$ in estimate* (8.2)*, the proof of Theorem 5.2 could be performed using the map $\tilde{\Phi}_\alpha(\cdot,\cdot) := \mathcal{B}(\beta_i, \Phi_0(\cdot,\cdot))$ instead of $\Phi_\alpha(\cdot,\cdot)$. The asymptotic rate of convergence, however, cannot be improved.*

*(iii) The similar estimate can be obtained if we consider an arbitrary Lipschitz map $F(\cdot)$ instead of the time-$\tau$ map $\Phi_0(\tau,\cdot)$, and construct $\tilde{\Phi}_\alpha$ as in (ii).*

### 8.3 Control Sets

In mathematical control theory one basic property is controllability, i.e. the property that for each two points $x$, $y$ in the state space there exists a controlled trajectory from $x$ to $y$. In general, this will only be possible in certain subsets of the state space, the so called *control sets*, see [1] for a comprehensive exposition.

One situation in which the existence of control sets can be shown is when the uncontrolled system exhibits an attractor on which it is chain recurrent. Under some conditions then there exists a control set around this attractor, cp. [1, Corollary 4.7.2]. This property may be used e.g. in order to approximate this attractor numerically, see [6] or [1, Chapter 13 and Appendix C].

Using Theorem 4.2 we can give an estimate about the distance between the attractor and the control set.

**Corollary 8.5** Let $\tilde{f}(x,u)$, $u \in U^\rho$ be a control system on $\mathbb{R}^n$ satisfying

$$\|\tilde{f}(x,u) - f(x)\| \leq C\rho \text{ for all } x \in \mathbb{R}^n,\ u \in U^\rho$$

for some $C > 0$ and all $\rho \in (0, \rho^*]$ for some $\rho^* > 0$. Let $A$ be a connected attracting set for system (2.1) on which the system is chain recurrent and let $\tilde{f}$ satisfy the conditions of [1, Corollary 4.7.2]. Assume furthermore that $A$ admits a 0-contracting family of neighbourhoods with contraction rate $\gamma$.

Then there exists a constant $K > 0$ such that for all $\rho > 0$ sufficiently small there exists an (invariant) control set $D^\rho$ containing $A$ and satisfying

$$d_H(D^\rho, A) \leq K\gamma(\rho).$$

**Proof:** The existence of the control set $D^\rho$ follows from [1, Corollary 4.7.2].

Under our assumption each $C\alpha$-strongly attracting set for system ($2.3^\alpha$) is a strongly attracting set for the control system given by $\tilde{f}$. Since in particular no controlled trajectory can leave a strongly attracting set, the control set $D^\rho$ around $A$ must be contained in this set. Thus Theorem 4.2 implies the assertion. $\square$

## 9 Examples

In this section we provide a number of examples illustrating the previous results.

We start with a rather simple example which, however, shows that a suitable 0-contracting family of neighbourhoods has to be chosen with care.

**Example 9.1** Consider the differential equation in $\mathbb{R}^2$ given by

$$\begin{aligned} \dot{x}_1 &= -2x_1 \\ \dot{x}_2 &= -x_2. \end{aligned}$$

The solution of this equation is given by

$$\varphi_0(t, x) = \begin{pmatrix} e^{-2t}x_1 \\ e^{-t}x_2 \end{pmatrix}.$$

Obviously, the set $A = \{0\}$ is an exponentially attracting set; in particular the family of neighbourhoods $B_\vartheta = \mathcal{B}(e^{-\vartheta}, \{0\})$ satisfies Definition 5.1 with $T > 0$ arbitrary, $\gamma(s) = s$ and $C = 1/(1 - e^{-T})$. Consequently, by Theorem 5.2 we expect the existence of a family of $\alpha$-stongly attracting sets $A_\alpha$ converging linearly to $A = \{0\}$. It is easily verified that the sets $A_\alpha = \{4x_1^2 + x_2^2 \leq \alpha^2\}$ form such a family, in fact this is the smallest possible choice of $A_\alpha$.

Following Remark 2.9(i), a family $\tilde{B}_\vartheta$ can also be obtained by setting

$$\tilde{B}_\vartheta := \Phi(\vartheta, \mathcal{B}(1, \{0\})) = \{x \in \mathbb{R}^2 \,|\, e^{-4\vartheta}x_1^2 + e^{-2\vartheta}x_2^2 \leq 1\}.$$

This family, however, meets neither Definition 4.1 nor Definition 5.1 with $\gamma(s) = s$, since

$$d_H(\tilde{B}_\vartheta, A) = e^{-\vartheta}, \quad \text{and} \quad d_{\min}(\tilde{B}_\vartheta, \tilde{B}_{\vartheta+T}) = e^{-2(\vartheta+T)},$$

thus $d_H(\tilde{B}_\vartheta, A) = e^{\vartheta+2T} d_{\min}(\tilde{B}_\vartheta, \tilde{B}_{\vartheta+T})$, and $e^{\vartheta+2T} \to \infty$ as $\vartheta \to \infty$.

Observe that this corresponds to the fact that the (minimal) $\alpha$-strongly attracting sets $A_\alpha$ in this example have minimal distance $\alpha/2$ from $A = \{0\}$. If the $\tilde{B}_\vartheta$ met Definition 4.1 the construction in the proof of Theorem 4.2 would yield sets $A_\alpha$ with $d_{\min}(A_\alpha, A) \le K\alpha^2$ for some constant $K > 0$ which is not possible.

Summarizing, this example shows that the use of the flow $\Phi_0$ itself to generate the family $B_\vartheta$ will in general not yield the desired properties—even in very simple exponentially attracting systems.

With the second example we illustrate the results for slower than exponential attraction.

**Example 9.2** Consider the differential equation in $\mathbb{R}^2$ given by

$$\begin{aligned} \dot{x}_1 &= x_2 - \frac{x_1}{\|x\|}(\|x\| - 1)^{1+k} \\ \dot{x}_2 &= -x_1 - \frac{x_2}{\|x\|}(\|x\| - 1)^{1+k} \end{aligned}$$

for some $k \in \mathbb{N}$.

Denoting $r(t) = \|x(t)\| - 1$ a straightforward computation yields that any solution satisfies

$$r(t) = \frac{r(0)}{\sqrt[k]{ktr(0)^k + 1}},$$

i.e. in particular the set $A = \mathcal{B}(1, \{0\})$ is an attracting set. A 0-contracting family for $A$ with $T = 1$ is given by $B_\vartheta = \mathcal{B}(\delta(\vartheta) + 1, \{0\})$ with

$$\delta(\vartheta) = \frac{1}{\sqrt[k]{k\vartheta + 1}}.$$

Since $\delta(\vartheta+1) = \delta(\vartheta)/\sqrt[k]{k\delta(\vartheta)^k + 1}$, and since by Taylor expansion one gets $\sqrt[k]{k\delta(\vartheta)^k + 1} = 1 + \delta(\vartheta)^k + O(\delta(\vartheta)^{k+1})$ we obtain

$$\delta(\vartheta) - \delta(\vartheta + 1) = \frac{\delta(\vartheta)\sqrt[k]{k\delta(\vartheta)^k + 1} - \delta(\vartheta)}{\sqrt[k]{k\delta(\vartheta)^k + 1}} = \delta(\vartheta)^{k+1} + O(\delta(\vartheta)^{k+2}),$$

and consequently for some suitable constant $C > 0$ depending on $k$ but not on $\vartheta$

$$C d_{\min}(B_\vartheta, B_{\vartheta+1})^{1/(k+1)} = C(\delta(\vartheta + 1) - \delta(\vartheta))^{1/(k+1)} \ge \delta(t) = d_H(B_\vartheta, A).$$

On the other hand, it is immediately seen that the boundary of the (smallest) $\alpha$-strongly attracting set $A_\alpha$ containing $A$ is given by

$$x \in \mathbb{R}^2 \text{ with } (\|x\| - 1)^{1+k} = \alpha,$$

hence $A_\alpha = \mathcal{B}(1 + \alpha^{1/(k+1)}, \{0\})$. Again we obtain the result expected from Theorem 5.2

Observe that by minimality of the $A_\alpha$ this also yields that the rate of attraction $1/(1 + k)$ is optimal, and that in this example the rate of attraction is immediately seen from the vectorfield.

In the following example we illustrate the fact, that the assertion of Theorem 5.2 might be wrong if the $A_\alpha$ are chosen as the *smallest* $\alpha$-strongly attracting set.

**Example 9.3** Consider the differential equation in $\mathbb{R}^1$ given by

$$\dot{x} = f(x), \quad f(x) = \begin{cases} -2^{k+1} - (x-2^k)^2 2^{1-k}, & x \in [2^k, 2^{k+1}], \quad k \in \mathbb{Z} \\ 2^{k+1} + (x-2^k)^2 2^{1-k}, & x \in [-2^{k+1}, -2^k], \quad k \in \mathbb{Z} \end{cases}$$

A short computation reveals $\dot{x} \le -5x/3$ for $x \ge 0$ and $\dot{x} \ge -5x/3$ for $x \le 0$, hence $A = \{0\}$ is an exponentially attracting set. Setting $B_\vartheta = [-2^{-\vartheta}, 2^{-\vartheta}]$ for $\vartheta \in \mathbb{R}_0^+$ we obtain a 0-contracting family of neighbourhoods for $T = 1$ with $d_H(B_\vartheta, A) = 2^{-\vartheta} = d_{\min}(B_\vartheta, B_{\vartheta+1})$.

For any $\alpha > 0$ we obtain the set $A_\alpha = \{-\alpha, \alpha\}$ as $\alpha$-strongly attracting set, which has the same rate of attraction as $A$ for the unperturbed system. For $\alpha = 2^{-i}$, $i \in \mathbb{N}$ the *smallest* $\alpha$-strongly attracting set is $A_\alpha = [-\alpha/2, \alpha/2]$. Here, however, we obtain the (local) estimates $f(x) + \alpha \ge -(x-\alpha/2)^2 2^{1-k}$ for $x \ge \alpha$, $x$ close to $\alpha$, and $f(x) - \alpha \le (x-\alpha/2)^2 2^{1-k}$ for $x \le -\alpha$, $x$ close to $\alpha$.

Thus for any set $B = [-\beta_1, \beta_2]$, $\beta_1$, $\beta_2 > \alpha/2$, $\beta_1$, $\beta_2$ close to $\alpha/2$, we obtain

$$d_{\min}(\Phi_\alpha(B), B) \le \min_{i=1,2} T(\beta_i - \alpha/2)^2 \le T d_H(B, A_\alpha)^2,$$

and consequently, here a rate of attraction $\gamma$ with $\gamma(s) = s$ is impossible.

## 10 Appendix: Relation between $\Phi_0$ and $\Phi_\alpha$

In this appendix we provide some technical but straightforward lemmas which allow us to estimate the difference between $\Phi_0(t, B)$ and $\Phi_\alpha(t, B)$ from above and below. Furthermore, we state immediate consequences from Gronwall's Lemma on the continuity of $\Phi_\alpha$.

**Lemma 10.1** For each two compact sets $B \subset C \subset \mathbb{R}^n$ and all $t$, $\alpha > 0$ the inequalities

$$d_H(\Phi_0(t, B), \Phi_\alpha(t, B)) \le e^{Lt} t\alpha,$$

$$d_H(\Phi_\alpha(t, B), \Phi_\alpha(t, C)) \le e^{Lt} d_H(B, C)$$

and

$$d_{\min}(\Phi_\alpha(t, B), \Phi_\alpha(t, C)) \ge e^{-Lt} d_{\min}(B, C)$$

hold.

**Proof:** Immediately from Gronwall's Lemma. □

**Lemma 10.2** For each $x_0 \in \mathbb{R}^n$, each $T > 0$ and each $x \in \mathbb{R}^n$ with $\|x - \varphi_0(T, x_0)\| \le \alpha/(L + 1/T)$ there exists $w \in \mathcal{W}$ with $\varphi_\alpha(T, x_0, w) = x$.

**Proof:** Fix $T$, $x_0$ and $x$ as in the assumption and define a function $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F(t, y) = f(\varphi_0(t, x_0)) - f(y).$$

Obviously $\|F(t, y)\| \le L\|y - \varphi_0(t, x_0)\|$, furthermore $F$ is Lipschitz in $y$ uniformly in $t$ and continuous in $t$. Hence the time varying perturbed differential equation

$$\dot{y}(t) = f(y) + F(t, y) + u(t) \tag{10.1}$$

has unique solutions on $[0, T]$ for any bounded and measurable function $u : \mathbb{R} \to \mathbb{R}^n$. For initial time $t_0 = 0$ and initial value $y_0$ we denote these by $\psi(t, y_0, u)$. Now denote $\Delta x := x - \varphi_0(T, x_0)$ and consider $u \equiv \Delta x/T$. Then the solution of (10.1) satisfies

$$\begin{aligned} \psi(t, x_0, u) &= x_0 + \int_0^t f(\psi(t, x_0, u)) + F(t, \psi(t, x_0, u)) + \Delta x/T\, dt \\ &= x_0 + \int_0^t f(\varphi_0(t, x_0)) + \Delta x/T\, dt \end{aligned}$$

for each $t \in [0, T]$. On the other hand, consider the function $g(t) := \varphi_0(t, x_0) + t\Delta x/T$. This function satisfies

$$g(t) = x_0 + \int_0^t f(\varphi_0(t, x_0)) + \Delta x/T\, dt$$

thus we can conclude $\psi(t, x_0, u) = g(t)$, hence in particular $\psi(T, x_0, u) = x$. Setting $w(t) := (F(t, \psi(t, x_0, u)) + \Delta x/T)/\alpha$ we obtain $\varphi_\alpha(t, x, w) = \psi(t, x_0, u)$, i.e. the desired solution, and

$$\|w(t)\| \le (L\|t\Delta x/T\| + \|\Delta x/T\| \le (L + 1/T)\|\Delta x\|)/\alpha \le 1$$

implying $w \in \mathcal{W}$. $\square$

**Corollary 10.3** For each compact $B \subset \mathbb{R}^n$ and each $T > 0$ the inequality

$$d_{\min}(\Phi_0(T, B), \Phi_\alpha(T, B)) \ge \alpha/(1/T + L)$$

holds.

**Proof:** Immediately from Lemma 10.2. $\square$

**Lemma 10.4** Consider a compact $A \subset \mathbb{R}^n$ which is forward invariant for system (2.1) and a compact set $B \supset A$. Then for each $T > 0$ the estimate

$$\mathrm{dist}\Big(\bigcup_{t \in [0,T]} \Phi_0(t, B),\ A\Big) \le e^{LT} d_H(B, A)$$

holds.

**Proof:** Immediately from Gronwall's Lemma. $\square$

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