Chapter 1

DIRECTED DERIVATIVES OF CONVEX COMPACT-VALUED MAPPINGS

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Abstract Convex compact sets can be embedded into the Banach space of directed sets. Directed sets allow a visualization as possibly non-convex, compact sets in $\mathbb{R}^n$ and hence, this space could be used to visualize differences of embedded convex compact sets. The main application is the visualization as well as the theoretical and numerical calculation of set-valued derivatives. Known notions of affine, semi-affine and quasi-affine maps and their derivatives are studied.

Keywords: directed sets, set-valued derivatives, differences of convex sets and their visualization, affine, semi-affine, quasi-affine maps, embedding of convex compact sets into a vector space, directed intervals

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1. INTRODUCTION

Directed sets in $\mathbb{R}^n$ presented in [1, 2, 3] form a Banach space $\mathcal{D}(\mathbb{R}^n)$ in which the convex cone of all compact, convex, nonempty sets can be embedded.

A directed set $\overrightarrow{A}$ is parametrized by unit vectors $l \in S_{n-1}$ and consists of two components, a describing continuous function $a_n(\cdot)$ and a $(n-1)$-dimensional directed set function $\overrightarrow{A_{n-1}(\cdot)}$ which is uniformly bounded. The hyperplane

$$H(l, a_n(\cdot)) = \{x \in \mathbb{R}^n \mid <l, x> = a_n(l)\} \tag{1.1}$$

is fixed by $a_n(\cdot)$ and contains the reprojection of the visualization of $\overrightarrow{A_{n-1}(l)}$. This image forms the boundary part of the visualized $\overrightarrow{A}$ in direction l.

For an embedded convex compact set C, the hyperplane is determined by the value of its support function in direction l and $\overrightarrow{A_{n-1}(l)}$ is the embedded projection of its supporting face C^l (seen as a $(n-1)$-dimensional set) into $\mathcal{D}(\mathbb{R}^{n-1})$. Generalized intervals in [6, 7, 9] are equivalent to one-dimensional directed sets and form the basis of the recursive definition of directed sets.

The proposed embedding is different to other approaches and is not minimal in the sense that Hörmander's embedding in [5] uses only the second component of our embedding. Nevertheless, we focus here on visualization of differences of convex, compact sets as eventually non-convex, compact sets in $\mathbb{R}^n$, not as real-valued functions with arguments in S_{n-1} resp. $\mathbb{R}^n$. For further references on embeddings and related articles see [2].

The visualization of directed sets consists of three parts, namely the convex part, the concave part (both are convex sets, one of them being empty, if they don't equal the same point) and the (non-convex) mixed-type part. The main application studies the visualization of derivatives of simple convex compact-valued mappings such as affine, semi-affine and quasi-affine maps.

Further on, the following notations are used throughout this paper:
The set of convex, compact, nonempty sets in $\mathbb{R}^n$ is denoted by $\mathcal{C}(\mathbb{R}^n)$. S_{n-1} denotes the unit sphere of the n-dimensional (closed) Euclidean ball $B_1(0)$.

The *support function* of $C \in \mathcal{C}(\mathbb{R}^n)$ and the *supporting face* of C in direction $l \in \mathbb{R}^n$ are denoted by

$$\delta^*(l, C) := \max_{c \in C} <l, c> \qquad \text{resp.} \quad C^l := \{c \in C \mid <l, c> = \delta^*(l, C)\}$$

and the *Hausdorff metric* on $\mathcal{C}(\mathbb{R}^n)$ as

$$\mathrm{d_H}(C, D) := \max_{l \in S_{n-1}} |\delta^*(l, C) - \delta^*(l, D)| \quad (C, D \in \mathcal{C}(\mathbb{R}^n)).$$

The *scalar multiplication* for $\lambda \geq 0$, the *Minkowski addition*, the *pointwise negative* (multiplication by -1) and the *algebraic difference* are defined as

$$\lambda C := \{\lambda c \,|\, c \in C\}, \;\; C + D := \{c + d \,|\, c \in C, d \in D\} \;\text{ for } C, D \in \mathcal{C}(\mathbb{R}^n),$$
$$\ominus C := \{-c \,|\, c \in C\}, \;\; C \ominus D := \{c - d \,|\, c \in C, \; d \in D\}.$$

The operation $\ominus$, frequently used also in interval analysis, does not even fulfill $C \ominus C = \{0_{\mathbb{R}^n}\}$ in general.

The *geometric difference* and the *Demyanov difference*

$$C \overset{*}{-} D := \{x \in \mathbb{R}^n \,|\, \forall l \in S_{n-1} : \; <l, x> \leq \delta^*(l, C) - \delta^*(l, D)\},$$
$$C \dot{-} D := \overline{\mathrm{co}}\{C^l - D^l \,|\, l \in S_{n-1}, \; C^l \text{ and } D^l \text{ are singletons}\} \neq \emptyset$$

in [4, 10] resp. [11] are convex, compact sets. $C \stackrel{*}{-} D$ could be empty and is usually too small, whereas $C \dot{-} D$ is usually too big in the sense that

$$(C \stackrel{*}{-} D) + D \underset{\neq}{\subset} C \quad \text{resp.} \quad (C \dot{-} D) + D \underset{\neq}{\supset} C \tag{1.2}$$

is only satisfied in general. The last two differences have close connections to the visualization of differences of embedded convex compact sets (see (1.6)).

2. DIRECTED SETS

Most of the definitions presented here are recursive and recalled from [1, 2].

Definition 2.1 $\overrightarrow{A}$ *is called* ***directed interval/one-dimensional directed set****, if*

$$\overrightarrow{A} = (a_1(l))_{l=\pm 1}$$

with $a_1(l) \in \mathbb{R}$ *for* $l = \pm 1$. *Set* $\|\overrightarrow{A}\|_1 := \max\limits_{l=\pm 1} |a_1(l)|$.
The space of directed intervals is denoted by $\mathcal{D}(\mathbb{R})$.

For $n \geq 2$, $\overrightarrow{A}$ *is called* ***directed set****, if there exists a uniformly bounded function* $\overrightarrow{A_{n-1}} : S_{n-1} \to \mathcal{D}(\mathbb{R}^{n-1})$ *with respect to* $\| \cdot \|_{n-1}$ *and a continuous function* $a_n : S_{n-1} \to \mathbb{R}$. *Set* $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l \in S_{n-1}}$ *and*

$$\|\overrightarrow{A}\|_n := \max\{ \sup_{l \in S_{n-1}} \|\overrightarrow{A_{n-1}(l)}\|_{n-1}, \max_{l \in S_{n-1}} |a_n(l)| \}$$

and denote by $\mathcal{D}(\mathbb{R}^n)$ *the space of directed sets in* $\mathbb{R}^n$.

The definition is motivated by the fact that the supporting face of each convex compact set $C \in \mathcal{C}(\mathbb{R}^n)$ in direction $l \in S_{n-1}$ lies on the hyperplane given by the support function in this direction. To enable a recursive approach, the support function is saved separately from the supporting face and the latter is seen as a $(n-1)$-dimensional set.

Definition 2.2 *The* ***embedding*** $J_n : \mathcal{C}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ *is defined for* $n = 1$ *as*

$$J_1([\alpha_1, \alpha_2]) := (\delta^*(l, [\alpha_1, \alpha_2]))_{l=\pm 1} \quad \textit{for } [\alpha_1, \alpha_2] \in \mathcal{C}(\mathbb{R})$$

and for $n \geq 2$ *as*

$$J_n(C) := (J_{n-1}(\Pi_{n-1}^{l, \delta^*(\cdot, C)}(C^l)), \delta^*(l, C))_{l \in S_{n-1}} \quad \textit{for } C \in \mathcal{C}(\mathbb{R}^n)$$

embeds all convex, compact, nonempty sets into the space of directed sets. Hereby, $\Pi_{n-1}^{l, f(\cdot)}(X)$ *is the projection of a* $(n-1)$*-dimensional set* $X \subset \mathbb{R}^n$

contained in the hyperplane $H(l, f(\cdot))$ *in* (1.1)*, which is determined by the function* $f : \mathbb{R}^n \to \mathbb{R}$*, to* $\mathbb{R}^{n-1}$.

The embedding J_n is an isometric, injective map from $\mathcal{C}(\mathbb{R}^n)$ to $\mathcal{D}(\mathbb{R}^n)$ with respect to a new metric $\mathrm{d_V}(\cdot, \cdot)$ on $\mathcal{C}(\mathbb{R}^n)$ introduced in [2] which is equivalent to Demyanov's metric.

The operations introduced on $\mathcal{D}(\mathbb{R}^n)$ are defined recursively and act separately on both components of the directed set.

Definition 2.3 *Consider* $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l \in S_{n-1}}$ *and* $\overrightarrow{B} = (\overrightarrow{B_{n-1}(l)}, b_n(l))_{l \in S_{n-1}} \in \mathcal{D}(\mathbb{R}^n)$*. Define the* ***operations*** *recursively (for* $n = 1$ *the first components of the directed sets do not exist) as*

$$\overrightarrow{A} + \overrightarrow{B} := (\overrightarrow{A_{n-1}(l)} + \overrightarrow{B_{n-1}(l)}, a_n(l) + b_n(l))_{l \in S_{n-1}}, \tag{1.3}$$

$$\lambda \cdot \overrightarrow{A} := (\lambda \cdot \overrightarrow{A_{n-1}(l)}, \lambda \cdot a_n(l))_{l \in S_{n-1}} \qquad (\lambda \in \mathbb{R}), \tag{1.4}$$

$$\overrightarrow{A} - \overrightarrow{B} := \overrightarrow{A} + ((-1) \cdot \overrightarrow{B}). \tag{1.5}$$

The space $\mathcal{D}(\mathbb{R}^n)$ has remarkable properties with these operations which were studied in [2]. E.g., $\| \cdot \|$ in Definition 2.1 is a norm on $\mathcal{D}(\mathbb{R}^n)$ for which $\mathcal{D}(\mathbb{R}^n)$ is a Banach space. Since we are interested in visualizations of differences of embedded elements of $\mathcal{C}(\mathbb{R}^n)$, we restrict our attention to the closed linear hull of $J_n(\mathcal{C}(\mathbb{R}^n))$ in most applications.

Proposition 2.4 ([2]) *The subspace*

$$\overrightarrow{\mathcal{C}}(\mathbb{R}^n) := \overline{\mathrm{lin}(J_n(\mathcal{C}(\mathbb{R}^n)))} \subset \mathcal{D}(\mathbb{R}^n)$$

for most applications is a Banach space as $\mathcal{D}(\mathbb{R}^n)$ *itself.*

The following result proven in [2] shows that the well-known and often used operations on $\mathcal{C}(\mathbb{R}^n)$ commute with the embedding.

Proposition 2.5 ([2]) *Consider* $C, D \in \mathcal{C}(\mathbb{R}^n)$ *and* $\lambda \geq 0$*. Then,*

$$J_n(C + D) = J_n(C) + J_n(D) \quad \textit{and} \quad J_n(\lambda C) = \lambda J_n(C).$$

The visualization for a directed set $\overrightarrow{A} \in \overrightarrow{\mathcal{C}}(\mathbb{R}^n)$ must incorporate the convex case, i.e. $\overrightarrow{A} \in J_n(\mathcal{C}(\mathbb{R}^n))$, the concave case, i.e. $\overrightarrow{A} \in -J_n(\mathcal{C}(\mathbb{R}^n))$, and the indefinite case, i.e. $\overrightarrow{A}$ is a difference of two embedded convex compact sets or the limit of such a sequence. Therefore, the visualization splits in three parts, the convex part, the concave part and the mixed-type part. In the definition of the mixed-type part, the boundary part is involved.

The projection involved in Definition 2.2 splits into an orthonormal, linear part $\mathcal{L}^l_{n-1}$ and a translation and allows an inverse reprojection on $H(l, a_n(\cdot))$, i.e.

$$\Pi^{l,a_n(\cdot)}_{n-1}(x) = \mathcal{L}^l_{n-1}(x - a_n(l)l), \quad Q^{l,a_n(\cdot)}_n(y) = (\mathcal{L}^l_{n-1})^{-1}(y) + a_n(l)l$$

for $x \in H(l, a_n(\cdot))$, $y \in \mathbb{R}^{n-1}$. It fulfills the inverse property

$$Q^{l,a_n(\cdot)}_n(\Pi^{l,a_n(\cdot)}_{n-1}(x)) = x \quad (x \in H(l, a_n(\cdot))).$$

Definition 2.6 *Let* $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l\in S_{n-1}} \in \mathcal{D}(\mathbb{R}^n)$. *Define the* ***convex (positive) part*** P_n*, the* ***concave (negative) part*** N_n *and the* ***visualization*** V_n *as*

$$P_n(\overrightarrow{A}) := \bigcap_{l\in S_{n-1}} \{x \in \mathbb{R}^n \mid <l, x> \leq a_n(l)\}, \quad N_n(\overrightarrow{A}) := \ominus P_n(-\overrightarrow{A}),$$
$$V_n(\overrightarrow{A}) := P_n(\overrightarrow{A}) \cup N_n(\overrightarrow{A}) \cup M_n(\overrightarrow{A}).$$

The ***mixed-type part*** M_n *and the* ***boundary part*** B_n *are defined recursively:*

$$B_1(\overrightarrow{A}) := \partial P_1(\overrightarrow{A}) \cup \partial N_1(\overrightarrow{A}), \quad M_1(\overrightarrow{A}) := \emptyset \qquad \text{for } n = 1,$$
$$\left.\begin{array}{l} B_n(\overrightarrow{A}) := \bigcup\limits_{l\in S_{n-1}} Q^{l,a_n(\cdot)}_n(V_{n-1}(\overrightarrow{A_{n-1}(l)})), \\ M_n(\overrightarrow{A}) := B_n(\overrightarrow{A}) \setminus (\partial P_n(\overrightarrow{A}) \cup \partial N_n(\overrightarrow{A})) \end{array}\right\} \qquad \text{for } n \geq 2.$$

There is no mixed-type part for directed intervals, this part could appear only for dimensions $n \geq 2$. The convex and the concave part are both convex, compact sets, but could be empty as in one example in [3].

The directions $l \in S_{n-1}$ for which a boundary point lies on the hyperplane $H(l, a_n(\cdot))$ are elements of the orientation bundle.

Definition 2.7 *The* ***orientation bundle*** *of a directed set* $\overrightarrow{A}$ *depends on a boundary point and forms a nonempty subset of* S_{n-1}.
Let $n = 1$, $\overrightarrow{A} = (a_1(l))_{l=\pm 1} \in \mathcal{D}(\mathbb{R})$. *Then,*

$$\begin{aligned} \mathcal{O}_1(a_1(1), \overrightarrow{A}) &:= \{1\}, && \text{if } a_1(1) \neq -a_1(-1), \\ \mathcal{O}_1(-a_1(-1), \overrightarrow{A}) &:= \{-1\}, && \text{if } a_1(1) \neq -a_1(-1), \\ \mathcal{O}_1(a_1(1), \overrightarrow{A}) &:= \{-1, 1\}, && \text{if } a_1(1) = -a_1(-1). \end{aligned}$$

Let $n \geq 2$, $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l\in S_{n-1}} \in \mathcal{D}(\mathbb{R}^n)$ *and* $x \in B_n(\overrightarrow{A})$. *Then,*

$$\mathcal{O}_n(x, \overrightarrow{A}) := \{l \in S_{n-1} \mid x \in Q^{l,a_n(\cdot)}_n(V_{n-1}(\overrightarrow{A_{n-1}(l)}))\}$$

Let us collect special properties of the visualization parts and the boundary part which are proven in [3]. Especially, the boundary part and the visualization are never empty and the three parts of the visualization in Definition 2.6 are disjoint, except in the special case that the convex and the concave part equal to the same point.

Theorem 2.8 states that the visualized difference of two embedded convex compacts lies between the geometric and the Demyanov difference (compare (1.2)).

Theorem 2.8 (see [3]) *Let $\overrightarrow{A} = J_n(C) - J_n(D) \in \mathcal{D}(\mathbb{R}^n)$ with $C, D \in \mathcal{C}(\mathbb{R}^n)$. Then,*

$$P_n(\overrightarrow{A}) = C \overset{*}{-} D, \quad N_n(\overrightarrow{A}) = \ominus(D \overset{*}{-} C),$$
$$B_n(\overrightarrow{A}) = \partial(C \overset{*}{-} D) \cup (\ominus\partial(D \overset{*}{-} C)) \cup M_n(\overrightarrow{A}).$$

The formula for the mixed-type part is rather complicated and is stated in a recursive way in [3]. Furthermore,

$$C \overset{*}{-} D \subset V_n(\overrightarrow{A}) \subset C \dot{-} D, \;\; \ominus(D \overset{*}{-} C) \subset V_n(\overrightarrow{A}) \subset \ominus(D \dot{-} C) \qquad (1.6)$$

Theorem 2.8 shows that the visualization of an embedded convex compact set C is the set itself, the boundary part is the boundary of C (set $D = \{0_{\mathbb{R}^n}\}$). There is no mixed-type part in this situation. The theorem also informs on the visualization of the inverse of $J_n(D)$ (set $C = \{0_{\mathbb{R}^n}\}$). Although, the visualization is the pointwise negative set $\ominus D$, the operations (1.3)–(1.5) in $\mathcal{D}(\mathbb{R}^n)$ avoid this operation.

From Definition 2.7, $J_n(C)$ has outer normals and $-J_n(D)$ inner normals.

3. DERIVATIVES OF SET-VALUED MAPPINGS

In this section, notions of simple convex-valued mappings are recalled from [8, 12]. In the following, $F : [t_0, T] \Rightarrow \mathbb{R}^n$ is a set-valued mapping with images in $\mathcal{C}(\mathbb{R}^n)$. Derivatives of the following simple maps are studied in section 4.. Please see the above citations for a rich list of references.

Definition 3.1 *F is an **affine map**, if there exists $F_0, F_1 \in \mathcal{C}(\mathbb{R}^n)$ with*

$$F(t) = F_0 + \frac{t - t_0}{T - t_0} F_1 \qquad (t \in I = [t_0, T]). \qquad (1.7)$$

*F is a **semi-affine map**, if there exists $F_0, F_1 \in \mathcal{C}(\mathbb{R}^n)$ with*

$$F(t) = (1 - \frac{t - t_0}{T - t_0}) F_0 + \frac{t - t_0}{T - t_0} F_1 \qquad (t \in [t_0, T]). \qquad (1.8)$$

*F is **quasi-affine**, if a positive homogeneous function $c : S_{n-1} \to \mathbb{R}$ exists with*

$$\delta^*(l, F(t)) = \mathrm{co}(\delta^*(l, F(t_0)) + (t - t_0)c(l)) \quad (l \in S_{n-1},\ t \in I). \quad (1.9)$$

*F is called **convex**, if for all $t_1, t_2 \in [t_0, T]$ and $\lambda \in [0, 1]$*

$$\lambda F(t_1) + (1 - \lambda)F(t_2) \subset F(\lambda t_1 + (1 - \lambda)t_2). \quad (1.10)$$

Since both affi ne and semi-affi ne maps are quasi-affi ne ([12, Proposition 2.2]) and quasi-affi ne maps are convex and continuous ([12, Lemma 2.7]), quasi-affi ne maps are an important class of simple set-valued mappings.

Definition 3.2 *Let $\overrightarrow{F} : [t_0, T] \to \mathcal{D}(\mathbb{R}^n)$ be a map with directed sets as images. $\overrightarrow{F}$ is a **finitely generated directed map**, if there exist $\overrightarrow{F_i} \in \mathcal{D}(\mathbb{R}^n)$, $i = 1, 2, ..., k$, and scalar functions $f_i : [t_0, T] \to \mathbb{R}$, $i = 1, 2, ..., k$, with*

$$\overrightarrow{F(t)} = \sum_{i=1}^{k} f_i(t)\overrightarrow{F_i} \quad (t \in [t_0, T]). \quad (1.11)$$

*$\overrightarrow{F}$ is called **directed differentiable in** $\mathbf{t} \in I = [t_0, T]$, if the limit*

$$\overrightarrow{DF(t)} := \lim_{\substack{h \to 0 \\ t+h \in I}} \underbrace{\frac{\overrightarrow{F(t+h)} - \overrightarrow{F(t)}}{h}}_{=:\Delta_h(\overrightarrow{F}, t)}$$

exists in $\mathcal{D}(\mathbb{R}^n)$. The k. derivative $\overrightarrow{F^{(k)}(t)}$ in $t \in I$ could be defined as the directed derivative of $\overrightarrow{F^{(k-1)}(\cdot)}$ for $k \geq 2$ with $\overrightarrow{F^{(1)}(\cdot)} = \overrightarrow{DF(\cdot)}$.

*F is called **directed differentiable in** $\mathbf{t} \in [t_0, T]$, if the function $t \mapsto J_n(F(t))$ is directed differentiable in t.*

*$\overrightarrow{F}$ is a **linear directed map**, if there exists $\overrightarrow{F_0}, \overrightarrow{F_1} \in \mathcal{D}(\mathbb{R}^n)$ with*

$$\overrightarrow{F(t)} = \overrightarrow{F_0} + t\overrightarrow{F_1} \qquad (t \in [t_0, T]). \quad (1.12)$$

Proposition 3.3 *Let F and $\overrightarrow{F}$ be given with $\overrightarrow{F(t)} := J_n(F(t))$, $t \in [t_0, T]$.*

(i) If $F(\cdot)$ is affine, then $\overrightarrow{F(\cdot)}$ is linear.

(ii) If $F(\cdot)$ is semi-affine, then $\overrightarrow{F(\cdot)}$ is linear.

Proof: (i) Let F be affi ne in the form (1.7). Then, Proposition 2.5 and Proposition 2.4, guaranteeing properties of a vector space, yield:

$$\begin{aligned}\overrightarrow{F(t)} &= J_n(F_0 + \frac{t - t_0}{T - t_0}F_1) = J_n(F_0) + \frac{t - t_0}{T - t_0}J_n(F_1) \\ &= \underbrace{(J_n(F_0) - \frac{t_0}{T - t_0}J_n(F_1))}_{=:\overrightarrow{F_0}} + t\,\underbrace{(\frac{1}{T - t_0}J_n(F_1))}_{=:\overrightarrow{F_1}}\end{aligned}$$

(ii) is proven similarly. ∎

Proposition 3.4 *Let us consider a linear directed map* $\overrightarrow{F}$ *in the form* (1.12). *Let* $\overrightarrow{C} \in \mathcal{D}(\mathbb{R}^n)$ *and* $F_0 \in \mathcal{C}(\mathbb{R}^n)$ *be arbitrarily chosen. If*

$$\overrightarrow{F_0} = J_n(F_0) - t_0\overrightarrow{C}, \quad \overrightarrow{F_1} = \overrightarrow{C}$$

and $P_n(\overrightarrow{F(t)})$ *is nonempty for all* $t \in [t_0, T]$*, then the function of the positive part of* $\overrightarrow{F}$*,* $P_n(\overrightarrow{F(\cdot)})$*, is quasi-affine with images in* $\mathcal{C}(\mathbb{R}^n)$*.*

Thus, positive parts of special linear directed maps are quasi-affi ne maps.

Proposition 3.5 *A finitely generated directed map* $\overrightarrow{F}$ *in the form* (1.11)*, where* $f_i(t)$*,* $i = 1, ..., k$*, are differentiable scalar functions with derivative* $f_i'(t)$*, is directed differentiable for all* $t \in I = [t_0, T]$ *with*

$$\overrightarrow{DF(t)} = \sum_{i=1}^{k} f_i'(t)\overrightarrow{F_i},$$

Proof: $\mathcal{D}(\mathbb{R}^n)$ is a normed vector space by Proposition 2.4, hence

$$\begin{aligned}\overrightarrow{DF(t)} &= \lim_{\substack{h\to 0\\ t+h\in I}} \frac{1}{h}\sum_{i=1}^{k}(f_i(t+h)\overrightarrow{F_i} - f_i(t)\overrightarrow{F_i})\\ &= \sum_{i=1}^{k}(\lim_{\substack{h\to 0\\ t+h\in I}} \frac{f_i(t+h) - f_i(t)}{h})\overrightarrow{F_i} = \sum_{i=1}^{k} f_i'(t)\overrightarrow{F_i}.\end{aligned}$$

∎

Proposition 3.3 states that affi ne and semi-affi ne convex-valued maps lead to linear directed maps which are special cases of fi nitely generated directed maps so that the next proposition can be easily derived from Proposition 3.5.

Proposition 3.6 *An affine* F *in the form* (1.7) *is directed differentiable with*

$$\overrightarrow{DF(t)} = \frac{1}{T-t_0}J_n(F_1) = J_n(\frac{1}{T-t_0}F_1) \quad \textit{for all } t \in [t_0, T].$$

A semi-affine map F *in the form* (1.8) *is directed differentiable with*

$$\overrightarrow{DF(t)} = \frac{1}{T-t_0}(J_n(F_1) - J_n(F_0)) \qquad \textit{for all } t \in [t_0, T].$$

Eclipsing maps introduced in [8] could be discontinuous (see [12]), so that directed differentiability for every eclipsing map could not be expected.

4. EXAMPLES OF DIRECTED DERIVATIVES

In all following examples, set $\overrightarrow{F(t)} := J_2(F(t))$. The numerical calculation of the derivative with difference quotients is neither very reliable nor very efficient and is chosen here only for simplicity of the presentation.

Example 3.7 *Consider $F : [0,1] \Rightarrow \mathbb{R}^2$ with*

$$F(t) = B_1(0) + t[0,2]^2 \quad (t \in [0,1]).$$

F is affine, hence $\overrightarrow{F}$ is linear and directed differentiable in $t \in [0,1]$ with

$$\overrightarrow{DF(t)} = \Delta_h(\overrightarrow{F}, t) = J_2([0,2]^2)$$

(see Proposition 3.6). The derivative is the embedded set $C = [0,2]^2 \in \mathcal{C}(\mathbb{R}^2)$, so that the visualization is simply C itself with outer normals (see Theorem 2.8 and Figure 1.1). Figure 1.1 shows the two values of $\overrightarrow{F}$ involved in $\Delta_1(\overrightarrow{F}, 0)$.

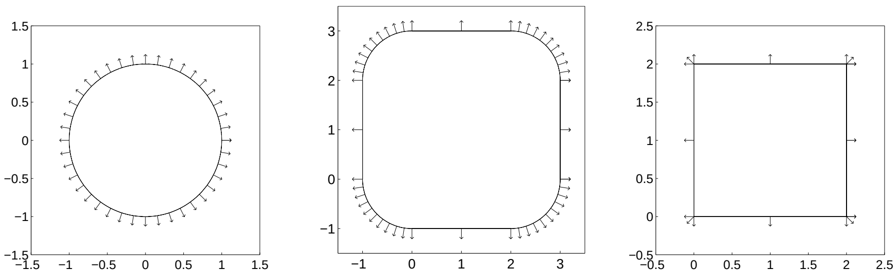

Figure 1.1 $J_2(F(0))$, $J_2(F(1))$ and $\overrightarrow{DF(0)} = \Delta_1(\overrightarrow{F}, 0)$

Example 3.8 *Consider $F : [0,1] \Rightarrow \mathbb{R}^2$ with*

$$F(t) = (1-t)[0,2]^2 \quad (t \in [0,1]).$$

F is semi-affine, hence $\overrightarrow{F}$ is linear and directed differentiable in $t \in [0,1]$ with

$$\overrightarrow{DF(t)} = \Delta_h(\overrightarrow{F}, t) = -J_2([0,2]^2)$$

(see Proposition 3.6). In this case, the derivative is the inverse of the embedded set $C = [0,2]^2 \in \mathcal{C}(\mathbb{R}^2)$ visualized in Figure 1.1, so that the visualization is simply $\ominus C$ with inner normals (see Theorem 2.8 and Figure 1.2).

Example 3.9 *Consider $F : [0,1] \Rightarrow \mathbb{R}^2$ with*

$$F(t) = (1-t)B_{0.5}(0) + t[-1,1]^2 \quad (t \in I = [0,1]).$$

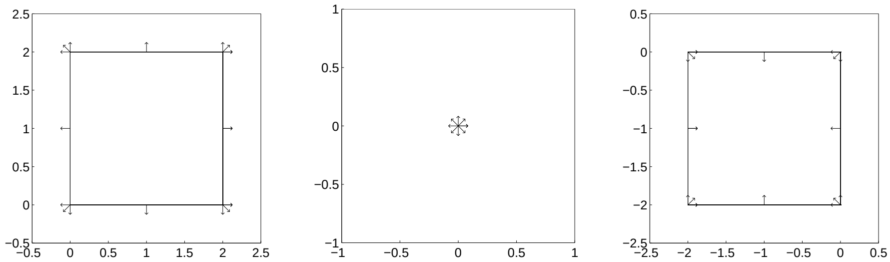

Figure 1.2 $J_2(F(0))$, $J_2(F(1))$ and $\overrightarrow{DF(0)} = \Delta_1(\overrightarrow{F}, 0)$

F is also semi-affine, hence $\overrightarrow{F}$ is linear and directed differentiable with

$$\overrightarrow{DF(t)} = \Delta_h(\overrightarrow{F}, t) = J_2([-1,1]^2) - J_2(B_{0.5}(0)) \quad (t \in I)$$

(see again Proposition 3.6). In this example, the derivative is the difference of the two embedded sets $C = [-1,1]^2$ and $D = B_{0.5}(0) \in \mathcal{C}(\mathbb{R}^2)$, so that the positive part is $C \overset{*}{-} D$ with outer normals and a nonempty, non-convex mixed-type part appears (see Theorem 2.8 and Figure 1.3).

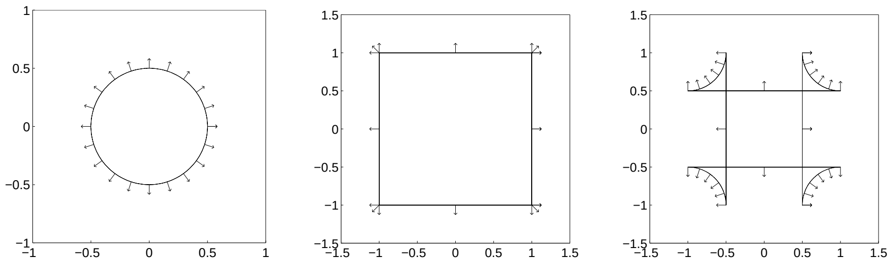

Figure 1.3 $J_2(F(0))$, $J_2(F(1))$ and $\overrightarrow{DF(0)} = \Delta_1(\overrightarrow{F}, 0)$

Example 3.10 *Consider the convex map $F : [-1,1] \Rightarrow \mathbb{R}^2$ (see (1.10)) with*

$$F(t) = (-t^2 + 1)B_1(0) \quad (t \in I = [-1,1]).$$

($t \mapsto -t^2 + 1$ is concave and $B_1(0)$ is absorbing). Proposition 3.5 and direct calculations show that $\overrightarrow{F}$ is finitely generated and directed differentiable with

$$\overrightarrow{DF(t)} = -2tJ_2(B_1(0)), \quad \Delta_h(\overrightarrow{F}, t) = -(2t+h)J_2(B_1(0)) \quad (t \in I).$$

Here, the difference quotients $\Delta_h(\overrightarrow{F}, 0.5)$ approach the inverse of the embedded set $B_1(0)$ with inner normals (see Theorem 2.8 and Figures 1.4, 1.5).

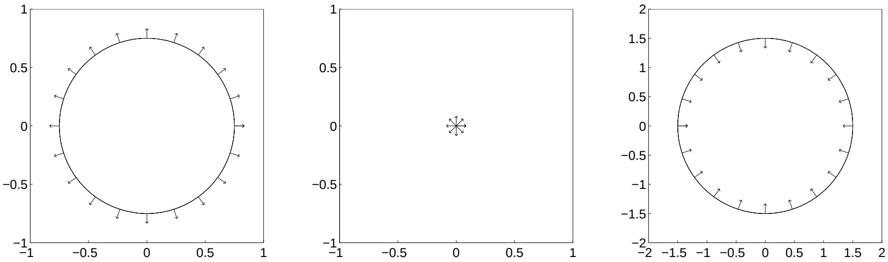

Figure 1.4 $J_2(F(0.5))$, $J_2(F(1))$ and $\Delta_{0.5}(\overrightarrow{F}, 0.5)$

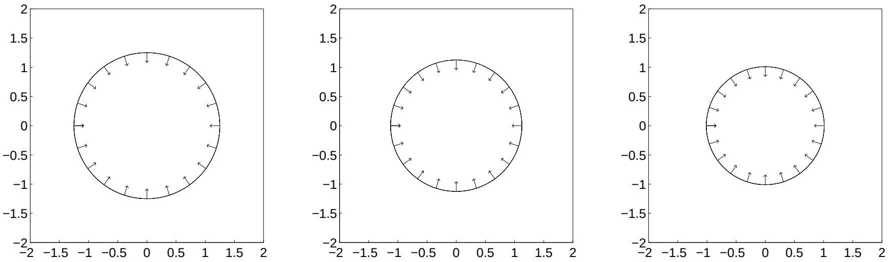

Figure 1.5 $\Delta_{0.25}(\overrightarrow{F}, 0.5)$, $\Delta_{0.125}(\overrightarrow{F}, 0.5)$ and $\Delta_{0.01}(\overrightarrow{F}, 0.5)$

Example 3.11 *Consider the quasi-affine map* $F : [0, 3] \Rightarrow \mathbb{R}^2$ *with*

$$F(t) = \{x \in \mathbb{R}^2 \,|\, \forall\, l \in S_1 : <l, x> \leq \delta^*(l, C) + t \cdot (\delta^*(l, D) - \delta^*(l, C))$$

and $C = \{0\} \times [-1, 1]$, $D = B_1(0)$ *(see* (1.9)*).* F *equals the positive part of the linear directed map* $\overrightarrow{G(t)} = J_2(C) + t \cdot (J_2(D) - J_2(C))$ *in Proposition 3.4. The approximated derivative for* $t = 2$ *is visualized in Figure 1.6.*

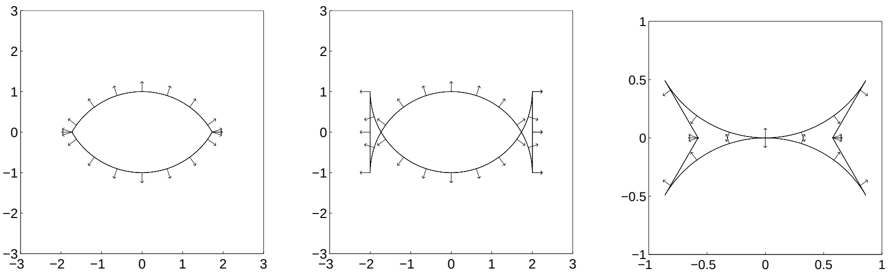

Figure 1.6 $J_2(F(2))$, $\overrightarrow{G(2)}$ and $\overrightarrow{DF(2)} = \Delta_{0.01}(\overrightarrow{F}, 2)$

5. SUMMARY

Embedded affine and semi-affine maps are linear directed maps. The situation for quasi-affine maps is not so easy, but specially chosen linear directed maps have positive parts which, seen as a convex-valued maps, are quasi-affine.

The concept of linear directed maps tries to unify the different notions of simple mappings with convex, compact images in one hand and allows on the other hand to study their differences by visualizing their derivative. Directed sets offer a convenient tool for studying the possible directed differentiability of set-valued mappings with the help of the visualization of difference quotients.

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