PATHWISE APROXIMATION OF RANDOM ORDINARY DIFFERENTIAL EQUATIONS *

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Abstract.

Standard error estimates for one–step numerical schemes for nonautonomous ordinary differential equations usually assume appropriate smoothness in both time and state variables and thus are not suitable for the pathwise approximation of random ordinary differential equations which are typically at most continuous or Hölder continuous in the time variable. Here it is shown that the usual higher order of convergence can be retained if one first averages the time dependence over each discretization subinterval.

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Key words: Random ordinary differential equation, Euler scheme, Heun scheme, time averaging.

1 Introduction.

A random ordinary differential equation (RODE)

$$\frac{dx}{dt} = f(t, x, \omega), \tag{1.1}$$

with the parameter or label ω denoting an element of the sample space Ω of an underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$, is pathwise an ordinary differential equation [1, 4]. Simple scalar examples are the RODE with additive noise

$$\frac{dx}{dt} = -x + \cos Z_t(\omega) \tag{1.2}$$

and the RODE with multiplicative noise

$$\frac{dx}{dt} = -x \cos Z_t(\omega), \tag{1.3}$$

where Z_t is some stochastic process such as a Brownian or fractional Brownian motion.

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Unlike Ito stochastic differential equations, which require stochastic calculus, RODE can be formulated and analysed pathwise completely in terms of deterministic calculus. Hence, in principle, an RODE can be solved numerically with a standard deterministic numerical scheme, such as the Euler or a Runge–Kutta scheme [2, 3]. The usual estimates of the discretization error of such schemes, however, require sufficient smoothness of the vector field function f in both the t variable as well as the x variables, but for an RODE (1.1) the function f is often only continuous or Hölder continuous in t for each fixed $\omega \in \Omega$. For example, if the driving process Z_t in the RODE (1.2) is a Brownian or fractional Brownian motion, then its sample paths are Hölder continuous but not differentiable in t, so the corresponding function$f$ is Hölder continuous but not differentiable in t for each fixed $\omega \in \Omega$. Hence, as will be shown below, the order of convergence of the Euler or a higher order Runge–Kutta scheme applied to equation (1.2) will be determined by the order of the Hölder continuity of the sample paths of the driving process, and is thus only of fractional order.

The aim of this paper is to show how the traditional convergence order of the Euler and Heun schemes can be retained by first averaging the noise process or time variable over each time step. We do this for a class of equations that is additively or multiplicatively separable in its t and x variables, thus including equations with additive noise as in (1.2) or with multiplicative noise as in (1.3), or with both types of noise together.

2 Estimate of the discretization error.

For conveniece we will henceforth omit the ω label in (1.1) and just consider a nonautonomous ordinary differential equation

$$\frac{dx}{dt} = f(t,x) \tag{2.1}$$

on $[0,T] \times I\!R^d$, which we assume to have a unique solution $x(t;t_0,x_0)$ for $t \in [t_0,T]$ with the initial condition $x(t_0;t_0,x_0) = x_0$ for each $t_0 \in [0,T)$ and $x_0 \in I\!R^d$. In particular, we assume that $(t,x) \mapsto f(t,x)$ is continuous on $[0,T] \times I\!R^d$ and locally Lipschitz in x uniformly in t in the interval $[0,T]$, that is, for each $R > 0$ there exists an L_R such that

$$|f(t,x) - f(t,y)| \le L_R |x-y|, \qquad \forall x,y \in B[0;R], t \in [0,T]$$

where $B[0;R] := \left\{ x \in I\!R^d \ : \ |x| \le R \right\}$. We assume that R has been chosen large enough so that the above solutions $x(t;t_0,x_0)$ remain in $B[0;R]$ for all $t \in [t_0,T]$. For later purposes we also define $M_R := \max_{t\in[0,T], x\in B[0;R]} |f(t,x)|$.

We also consider a one–step numerical scheme for (2.1)

$$x_{n+1} = x_n + h_n\, F(h_n,t_n,x_n) \tag{2.2}$$

with a sequence of discretization times $t_n \in [t_0,T]$ and step sizes $h_n = t_{n+1} - t_n > 0$, where the increment function F is continuous in all of its variables (h,t,x)

and satisfies the consistency condition

$$F(0,t,x) \equiv f(t,x), \qquad \forall t \in [0,T], x \in I\!R^d.$$

The modulus of continuity $\omega_f(h)$ of $f(\cdot,x)$ on $[0,T]$ uniformly in $x \in B[0;R]$ is defined by

$$\omega_f(h) := \omega_f(h;R,T) = \sup_{\substack{s,t\in[0,T]\\ 0\le|s-t|\le h}} \sup_{x\in B[0;R]} |f(t,x) - f(s,x)|$$

and the modulus of continuity $\omega_F(h)$ of $F(\cdot,t,x)$ on, say, $[0,1]$ uniformly in $(t,x) \in [0,T] \times B[0;R]$ is defined by

$$\omega_F(h) := \omega_F(h;R,T) = \sup_{0\le h_n\le h} \sup_{\substack{t\in[0,T]\\ x\in B[0;R]}} |F(h_n,t,x) - F(0,t,x)|.$$

(For convenience we omit the explicit mention of the dependence on T and R).

Note that $\omega_f(h) \to 0$ and $\omega_F(h) \to 0$ as $h \to 0$.

THEOREM 2.1. *The global discretization error of the numerical scheme (2.2) satisfies the estimate*

$$|x_n - x(t_n;t_0,x_0)| \le [\omega_f(h) + \omega_F(h) + L_R M_R h] \frac{1}{L_R} e^{2L_R T}. \tag{2.3}$$

on the set $[0,T] \times B[0;R]$.

The proof is given in the Appendix.

The increment functions F for the Euler and Heun schemes, respectively, are given by

$$F(h,t,x) = f(t,x), \qquad F(h,t,x) = \frac{1}{2}\left\{f(t,x) + f(t+h, x+hf(t,x))\right\}$$

and their moduli of continuity in h are, respectively,

$$\omega_F(h) \equiv 0, \qquad \omega_F(h) \le K_1\,\omega_f(h)$$

for some constant K_1 (which depends on T and R). Moreover, since we are considering functions f that are just continuous or at most Hölder continuous in t, we also have $h \le K_2\omega_f(h)$ for some constant K_2 when h is small. Then, for the Euler and Heun schemes (and similarly for higher order Runge–Kutta schemes), the discretization error bound has the form

$$|x_n - x(t_n;t_0,x_0)| \le K_{T,R}\,\omega_f(h) \tag{2.4}$$

for some constant $K_{T,R}$. The schemes thus both have "order" $\omega_f(h)$, although for smoother differential equations the Euler and Heun schemes have orders 1 and 2, respectively, that is, with h and h^2 instead of $\omega_f(h)$ in the bound (2.4).

3 The averaged Euler and Heun schemes.

We will now restrict attention to nonautonomous ordinary differential equation with the structure of a random ordinary differential equation with additive noise as in (1.2) or multiplicative noise as in (1.3), or both together. That is, we consider nonautonomous ordinary differential equation (2.1) with a separable vector field of the form $f(t,x) = G(t) + g(t)H(x)$, namely

$$\frac{dx}{dt} = G(t) + g(t)H(x), \tag{3.1}$$

where $g : [0,T] \to I\!R$ has modulus of continuity $\omega_g(\delta)$ on $[0,T]$, $G : [0,T] \to I\!R^d$ has modulus of continuity $\omega_G(\delta)$ on $[0,T]$ and $H : I\!R^d \to I\!R^d$ is at least 1 times continuously differentiable in the Euler case and at least 2 times continuously differentiable in the Heun case.

Our aim is to show how we can improve on the "order" $\omega_f(h)$ of the Euler and Heun schemes by first averaging the time dependent functions over each discretization subinterval. In particular, we will replace the function g (and with similar definitions, the function G) in the schemes by the singlely averaged function

$$\bar{g}^{(1)}_{h,\delta}(t) = \frac{1}{h}\sum_{j=0}^{N-1} g(t+j\delta, x)\,\delta = \frac{1}{N}\sum_{j=0}^{N-1} g(t+j\delta), \tag{3.2}$$

or the doublely averaged function

$$\bar{g}^{(2)}_{h,\delta}(t) = \frac{2}{h^2}\sum_{i=0}^{N-1}\sum_{j=0}^{i} g(t+j\delta, x)\,\delta^2 = \frac{2}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{i} g(t+j\delta). \tag{3.3}$$

over the interval $[t, t+h]$ for an appropriate sampling step size $\delta = h/N$. These averaged functions are just Riemann sum approximations of the single and double integrals, respectively,

$$\frac{1}{h}\int_t^{t+h} g(s)\,ds, \qquad \frac{2}{h^2}\int_t^{t+h}\int_t^{s} g(r)\,dr\,ds,$$

of the function g over the interval $[t, t+h]$ with approximation step size $\delta = h/N$. For computational purposes we note that $\bar{g}^{(2)}_{h,\delta}(t)$ can be rewritten in the convenient form

$$\bar{g}^{(2)}_{h,\delta}(t) = \frac{2}{N^2}\sum_{j=0}^{N-1}(N-j)g(t+j\delta).$$

3.1 The averaged Euler scheme.

The *averaged Euler* scheme with constant discretization step size h and sampling step size δ for the separable nonautonomous ordinary differential equation (3.1) is

$$x_{n+1} = x_n + h\,\bar{G}^{(1)}_{h,\delta}(t_n) + h\,\bar{g}^{(1)}_{h,\delta}(t_n)H(x_n). \tag{3.4}$$

We can rewrite this as

$$\begin{aligned} x_{n+1} &= x_n + \sum_{j=0}^{N-1} G(t_n + j\delta)\,\delta + \sum_{j=0}^{N-1} g(t_n + j\delta)\,H(x_n)\,\delta \\ &= \frac{1}{N}\sum_{j=0}^{N-1}\left\{x_n + hG(t_n + j\delta) + hg(t_n + j\delta)\,H(x_n)\right\}, \end{aligned}$$

which thus corresponds to averaging the conventional Euler scheme with discretization step size h over $N = h/\delta$ equally spaced time sampling points for the same x_n value.

As can be deduced from the details for the Heun case below, the local discretization error for the averaged Euler scheme is estimated by

$$L(h;t,x) \le K_{T,R}\, h\, \left(\omega_g(\delta) + \omega_G(\delta)\right).$$

If we choose the sampling step size $\delta > 0$ such that

$$\max\left\{\omega_g(\delta), \omega_G(\delta)\right\} = h,$$

the local discretization error will be of order 2. Hence the local discretization error and the averaged Euler scheme itself will be of order $p = 1$.

3.2 The averaged Heun scheme.

The *averaged Heun scheme* with constant discretization step size h and sampling step size δ for the separable nonautonomous ordinary differential equation (3.1) is defined by

$$(3.5)\qquad \begin{aligned} x_{n+1} &= x_n + h\,\bar{G}^{(1)}_{h,\delta}(t_n) + \frac{h}{2}\,\bar{g}^{(1)}_{h,\delta}(t_n)H(x_n) \\ &\quad + \frac{h}{2}\,\bar{g}^{(1)}_{h,\delta}(t_n)H\left(x + h\bar{G}^{(2)}_{h,\delta}(t_n) + h\bar{g}^{(2)}_{h,\delta}(t_n)H(x_n)\right), \end{aligned}$$

Note that the averaged functions involved with the second H term of the increment function here are evaluated at time t rather than $t+h$, since the latter would correspond to averaging over the next discretization subinterval $[t+h, t+2h]$.

We will show that the local discretization error of the averaged Heun scheme (3.5) satisfies the estimate

$$L(h;t,x) \le C_1\left(\left(h + h^2\right)\,\left(\omega_G(\delta) + \omega_g(\delta)\right) + h^3\right) \le K_{T,R}\, h^3$$

when the sampling stepsize size δ now chosen such that

$$(3.6)\qquad \max\left\{\omega_g(\delta), \omega_G(\delta)\right\} = h^2.$$

The averaged Heun scheme in this case will thus have order $p = 2$.

Let $x(\cdot)$ be the solution with initial value $x(t) = x$ for a fixed t and x, so

$$x(\tau) = x + \int_t^\tau f(t, x(s))\, ds.$$

We compare this solution at time $\tau = t + h$ with a single iteration of (3.4) with $h_n = H$ and $x_n = x$. We estimate the local discretization error $L(h; t, x)$ of the averaged Heun scheme (3.5) as follows:

$$\begin{aligned}
L(h;t,x) & \\
:= \quad & \left| \int_t^{t+h} \{G(s) + g(s)H(x(s))\}\; ds - h\,\bar{G}^{(1)}_{h,\delta}(t) \right. \\
& \left. -\frac{h}{2}\,\bar{g}^{(1)}_{h,\delta}(t)\left\{H(x) + H\left(x + h\bar{G}^{(2)}_{h,\delta}(t) + h\bar{g}^{(2)}_{h,\delta}(t)H(x)\right)\right\}\right| \\
\leq \quad & \left|\int_t^{t+h}\left(G(s) - \bar{G}^{(1)}_{h,\delta}(t)\right)\, ds\right| \\
& + \left|\int_t^{t+h}\left(g(s)H(x(s)) - \frac{1}{2}\,\bar{g}^{(1)}_{h,\delta}(t)\left\{H(x)\right.\right.\right. \\
& \left.\left.\left. + H\left(x + h\bar{G}^{(2)}_{h,\delta}(t) + h\bar{g}^{(2)}_{h,\delta}(t)H(x)\right)\right\}\right)\, ds\right|
\end{aligned} \tag{3.7}$$

Then we use the Taylor expansions

$$\begin{aligned}
& H(x(s)) = H(x) + H'(x)(x(s) - x) + \frac{1}{2}H''(x)(x(s) - x)^2 + \cdots \\
= \quad & H(x) + H'(x)\int_t^s \{G(r) + g(r)H(x(r))\}\; dr + O(h^2) \\
= \quad & H(x) + H'(x)\int_t^s G(r)\, dr + H'(x)H(x)\int_t^s g(r)\, dr + O(h^2)
\end{aligned}$$

and

$$\begin{aligned}
& H\left(x + h\bar{G}^{(2)}_{h,\delta}(t) + h\bar{g}^{(2)}_{h,\delta}(t)H(x)\right) \\
& \qquad = H(x) + hH'(x)\left(\bar{G}^{(2)}_{h,\delta}(t) + \bar{g}^{(2)}_{h,\delta}(t)H(x)\right) + O(h^2) \\
& \qquad = H(x) + hH'(x)\bar{G}^{(2)}_{h,\delta}(t) + h\bar{g}^{(2)}_{h,\delta}(t)H'(x)H(x) + O(h^2)
\end{aligned}$$

Thus

$$g(s)H(x(s)) - \frac{1}{2}\,\bar{g}^{(1)}_{h,\delta}(t)\left\{H(x) + H\left(x + h\bar{G}^{(2)}_{h,\delta}(t) + h\bar{g}^{(2)}_{h,\delta}(t)H(x)\right)\right\}$$

$$\begin{aligned}
&= H(x)\left(g(s)-\bar{g}^{(1)}_{h,\delta}(t)\right) - H'(x)H(x)\left(g(s)\int_t^s g(r)\,dr - \frac{1}{2}h\bar{g}^{(1)}_{h,\delta}(t)\bar{g}^{(2)}_{h,\delta}(t)\right)\\
&\quad -H'(x)\left(g(s)\int_t^s G(r)\,dr - \frac{1}{2}h\bar{g}^{(1)}_{h,\delta}(t)\bar{G}^{(2)}_{h,\delta}(t)\right) + O(h^2)\\
&= H(x)\left(g(s)-\bar{g}^{(1)}_{h,\delta}(t)\right) - H'(x)H(x)\left(g(s)-\bar{g}^{(1)}_{h,\delta}(t)\right)\int_t^s g(r)\,dr\\
&\quad -H'(x)H(x)\bar{g}^{(1)}_{h,\delta}(t)\left(\int_t^s g(r)\,dr - \frac{1}{2}h\bar{g}^{(2)}_{h,\delta}(t)\right)\\
&\quad -H'(x)\left(g(s)-\bar{g}^{(1)}_{h,\delta}(t)\right)\int_t^s G(r)\,dr\\
&\quad -H'(x)\bar{g}^{(1)}_{h,\delta}(t)\left(\int_t^s G(r)\,dr - \frac{1}{2}h\bar{G}^{(2)}_{h,\delta}(t)\right) + O(h^2).
\end{aligned}$$

Using a generic constant K that may change from line to line, we then obtain

$$\begin{aligned}
&\left|\int_t^{t+h}\left(g(s)H(x(s)) - \frac{1}{2}\,\bar{g}^{(1)}_{h,\delta}(t)\left\{H(x) + H\left(x + h\bar{G}^{(2)}_{h,\delta}(t) + h\bar{g}^{(2)}_{h,\delta}(t)H(x)\right)\right\}\right)\,ds\right|\\
&\quad\le K\left|\int_t^{t+h}\left(g(r)-\bar{g}^{(1)}_{h,\delta}(t)\right)\,dr\right| + K\left|\int_t^{t+h}\left(\int_t^s g(r)\,dr - \frac{1}{2}h\bar{g}^{(2)}_{h,\delta}(t)\right)\,ds\right|\\
&\qquad +K\left|\int_t^{t+h}\left(\int_t^s G(r)\,dr - \frac{1}{2}h\bar{G}^{(2)}_{h,\delta}(t)\right)\,ds\right| + Kh^3\\
&\quad\le Kh\omega_g(\delta) + Kh^2\left(\omega_g(\delta)+\omega_G(\delta)\right) + Kh\delta + Kh^3,
\end{aligned}$$

since

$$\begin{aligned}
\left|\int_t^{t+h}\left(g(r)-\bar{g}^{(1)}_{h,\delta}(t)\right)\,dr\right| &= \left|\sum_{j=0}^{N-1}\int_{t+j\delta}^{t+(j+1)\delta}\left(g(s)-g(t+j\delta)\right)\,ds\right|\\
&\le \sum_{j=0}^{N-1}\int_{t+j\delta}^{t+(j+1)\delta}\left|g(s)-g(t+j\delta)\right|\,ds\\
&\le \sum_{j=0}^{N-1}\int_{t+j\delta}^{t+(j+1)\delta}\omega_g(\delta)\,ds = N\delta\,\omega_g(\delta) = h\,\omega_g(\delta),
\end{aligned}$$

where we have used $N\delta = h$, and

$$\left| \int_t^{t+h} \left(\int_t^s g(r)\, dr - \frac{1}{2} h \bar{g}^{(2)}_{h,\delta}(t) \right) ds \right|$$

$$\begin{aligned}
&\leq \left| \sum_{i=0}^{N-1} \sum_{j=0}^{i} \int_{t+i\delta}^{t+(i+1)\delta} \int_{t+j\delta}^{t+(j+1)\delta} \left(g(r) - g(t+j\delta) \right) dr\, ds \right| \\
&\qquad + \left| \sum_{i=0}^{N-1} \int_{t+i\delta}^{t+(i+1)\delta} \int_{s}^{t+(i+1)\delta} g(r)\, dr\, ds \right| \\
&\leq \sum_{i=0}^{N-1} \sum_{j=0}^{i} \int_{t+i\delta}^{t+(i+1)\delta} \int_{t+j\delta}^{t+(j+1)\delta} \left| g(r) - g(t+j\delta) \right| dr\, ds \\
&\qquad + K \left| \sum_{i=0}^{N-1} \int_{t+i\delta}^{t+(i+1)\delta} \int_{s}^{t+(i+1)\delta} 1\, dr\, ds \right| \\
&\leq \sum_{i=0}^{N-1} \sum_{j=0}^{i} \int_{t+i\delta}^{t+(i+1)\delta} \int_{t+j\delta}^{t+(j+1)\delta} \omega_g(\delta)\, dr\, ds + K \left| \sum_{i=0}^{N-1} \int_{t+i\delta}^{t+(i+1)\delta} \int_{s}^{t+(i+1)\delta} 1\, dr\, ds \right| \\
&\leq \frac{1}{2} \left(N^2 - N \right) \delta^2 \omega_g(\delta) + KN\delta^2 \leq \frac{1}{2} \left(N^2 - N \right) \delta^2 \omega_g(\delta) + Kh\delta,
\end{aligned}$$

and similarly for the G terms.

Returning to line (3.7) and using $\delta \leq \omega_g(\delta)$ for h and hence δ small enough, we finally have

$$\begin{aligned}
L(h;t,x) &\leq \left| \int_t^{t+h} \left(G(s) - \bar{G}^{(1)}_{h,\delta}(t) \right) ds \right| + Kh\omega_g(\delta) \\
&\qquad + Kh^2 \left(\omega_g(\delta) + \omega_G(\delta) \right) + Kh^3 \\
&\leq K \left(h + h^2 \right) \left(\omega_G(\delta) + \omega_g(\delta) \right) + Kh^3 \leq Kh^3
\end{aligned}$$

with the above choice (3.6) of the sampling step size δ.

REMARK 3.1. The total number of calculations required for the averaged schemes with discretization step size h and sampling step size δ lies between those of their traditional counterparts with discretization step sizes δ and h, respectively, since the same value of the iterate x_n is used on the right hand side of the schemes for the $N = h/\delta$ intermediate sampling calculations. Besides being possibly more stable numerically, this can often be a considerable saving in computational effort, particularly for vector differential equations, over the traditional schemes with the smaller discretization step size δ because the time functions that are to be averaged are often only scalar functions or scalar functions multiplied by a constant vector.

4 Numerical examples.

In this section we illustrate the schemes by two numerical examples, and compare them to the non-averaged Euler and Heun scheme.

The first example we consider is given by

$$\frac{dx}{dt} = -x + \cos W_t(\omega) \tag{4.1}$$

with $W_t(\omega)$ being a particular sample path of a Wiener process. We compute an approximation to the solution $x(t; t_0, x_0, \omega)$ with $t_0 = 0$, $t = 1$ and $x_0 = 1$, and compare it to the exact solution

$$x(t; t_0, x_0, \omega) = x_0 e^{-t} + e^{-t} \int_{t_0}^{t} e^s \cos W_s(\omega) ds,$$

which is also evaluated numerically but with high precision, i.e. as a Riemann sum with one million subintervals. The sample path of the Wiener process here is generated with the aid of the usual Box-Muller method, see [4].

Figure 4.1 shows the error at $t = 1$ for example (4.1).

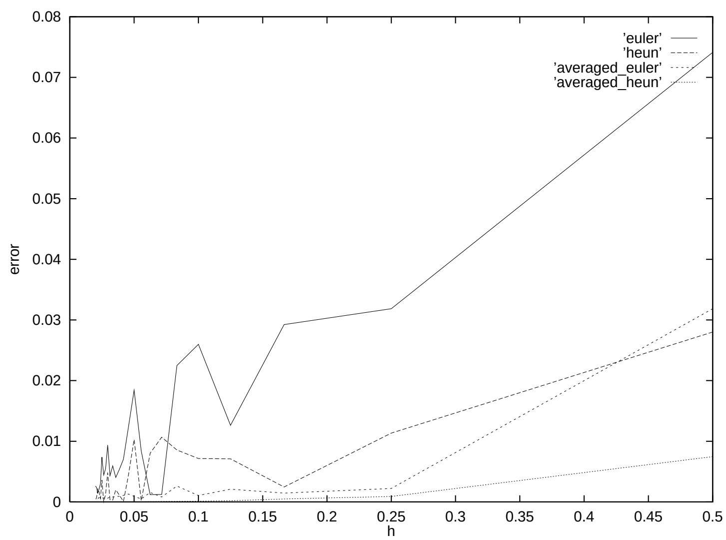

Figure 4.1: Example (4.1): Error at t=1 vs. timestep

The second example is

$$\frac{dx}{dt} = -x \cos 5W_t(\omega) \tag{4.2}$$

with exact solution

$$x(t;t_0,x_0,\omega) = x_0 e^{-\int_{t_0}^{t} \cos 5W_s(\omega)ds},$$

which again has been evaluated numerically with high precision in order to estimate the accuracy of the schemes approximating the solution with $t_0 = 0$, $t = 1$ and $x_0 = 1$.

Figure 4.2 shows the error at $t = 1$ with $x_0 = 1$ for example (4.2).

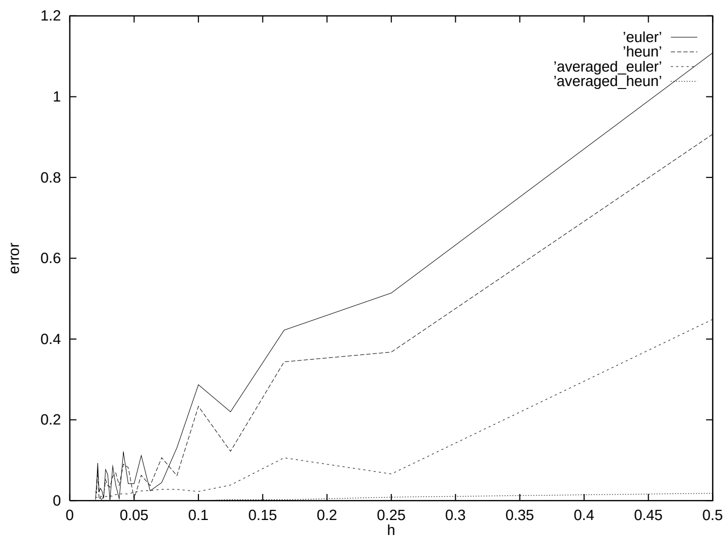

Figure 4.2: Example (4.2): Error at t=1 vs. timestep

Note that in both examples not only a smaller error is achieved for the averaged schemes, but also a much more stable behaviour of these algorithms for small time steps.

5 Appendix: Proof of Theorem 2.1.

We can rewrite the differential equation (2.1) as an integral equation. In particular, for the sequence of discretization times t_n and stepsizes $h_n = t_{n+1} - t_n > 0$ we have

$$x(t_{n+1};t_0,x_0) = x(t_n;t_0,x_0) + \int_{t_n}^{t_{n+1}} f(t, x(t;t_0,x_0))\, dt.$$

In future we just write $x(t)$ for this solution $x(t_0;t_0,x_0)$. By the Mean Value Theorem there exists $\tau_n \in [0,1]$ such that

$$x(t_{n+1}) = x(t_n) + h_n\, f(t_n + \tau_n h_n, x(t_n + \tau_n h_n)).$$

Thus

$$x(t_{n+1}) - x_{n+1} = x(t_n) - x_n + h_n\left[f(t_n + \tau_n h_n, x(t_n + \tau_n h_n)) - F(h_n, t_n, x_n)\right]$$

Now suppose that the time steps $h_n = t_{n+1} - t_n$ satisfy $0 < h_n \leq h$ for some $h > 0$. Then the global discretization error $E_n := |x(t_n) - x_n|$ is estimated by

$$\begin{aligned}
E_{n+1} &\leq E_n + h_n\left|f(t_n + \tau_n h_n, x(t_n + \tau_n h_n)) - F(h_n, t_n, x_n)\right| \\
&\leq E_n + h_n\left|f(t_n + \tau_n h_n, x(t_n + \tau_n h_n)) - f(t_n + \tau_n h_n, x(t_n))\right| \\
&\qquad + h_n\left|f(t_n + \tau_n h_n, x(t_n)) - f(t_n, x(t_n))\right| \\
&\qquad + h_n\left|f(t_n, x(t_n)) - f(t_n, x_n)\right| \\
&\qquad + h_n\left|F(0, t_n, x_n) - F(h_n, t_n, x_n)\right| \qquad \text{by consistency} \\
&\leq E_n + h_n L_R\left|x(t_n + \tau_n h_n) - x(t_n)\right| + h_n\omega_f(h) \\
&\qquad + h_n L_R\left|x(t_n) - x_n\right| + h_n\omega_F(h) \\
&= (1 + h_n L_R)E_n + h_n L_R\left|\int_{t_n}^{t_n + \tau_n h_n} f(s, x(s))\,ds\right| \\
&\qquad + h_n\omega_f(h) + h_n\omega_F(h) \\
&\leq (1 + h_n L_R)E_n + h_n^2 L_R M_R + h_n\omega_f(h) + h_n\omega_F(h) \\
&\leq (1 + L_R h)E_n + h\left[\omega_f(h) + \omega_F(h) + L_R M_R h\right].
\end{aligned}$$

With the initial value $E_0 = 0$ the above difference inequality yields

$$\begin{aligned}
E_n &\leq h\left[\omega_f(h) + \omega_F(h) + L_R M_R h\right]\frac{(1 + L_R h)^n - 1}{(1 + L_R h) - 1} \\
&\leq \left[\omega_g(h) + \omega_F(h) + L_R M_R h\right]\frac{1}{L_R}e^{L_R n h} \\
&\leq \left[\omega_g(h) + \omega_F(h) + L_R M_R h\right]\frac{1}{L_R}e^{2L_R T},
\end{aligned}$$

that is, the desired estimate (2.3).

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