SUBDIVISION TECHNIQUES FOR THE COMPUTATION OF DOMAINS OF ATTRACTIONS AND REACHABLE SETS

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Abstract: We present a set oriented subdivision technique for the numerical computation of reachable sets and domains of attraction for nonlinear systems. Using robustness properties of reachable sets for a suitable class of perturbed systems we state its convergence properties. Finally, we present results of numerical experiments which illustrate the performance of the algorithm.

Keywords: reachable set, domain of attraction, set oriented numerical method, robustness

1. INTRODUCTION

Reachable sets and domains of attractions play an important role in the analysis of nonlinear systems, see for instance (Colonius and Kliemann, 2000). Unfortunately, these objects are hardly ever analytically computable. Hence numerical methods form a natural part of a systematic analysis and during the last two decades a number of such methods has been developed, see, e.g., (Abu Hassan and Storey, 1981; Camilli *et al.*, 2000; Dang and Maler, 1998; Falcone *et al.*, 2000; Genesio *et al.*, 1985; Häckl, 1992–1993; Häckl, 1995; Szolnoki, 2001), to mention just a few examples.

In this paper we will present a subdivision algorithm for the computation of domains of attractions and reachable sets, or, more precisely, for the boundary of these sets. Subdivision techniques form a class of set oriented methods which are particularly successful in numerics for the long time behavior of dynamical systems, see, e.g., (Dellnitz and Junge, 1998).

2. PRELIMINARIES

In order to avoid too much technical overhead we restrict ourselves to discrete time systems given by

$$x(t+1) = F(x(t), u(t)), \;\; t \in \mathbb{N}_0 \qquad (1)$$

where $\Phi : \mathbb{R}^d \times U \to \mathbb{R}^d$ is a continuous map which is globally Lipschitz in x uniformly for each $u \in U$, and $U \subset \mathbb{R}^m$ is supposed to be compact. The set $\mathcal{U} := \{u : \mathbb{N}_0 \to U\}$ denotes the space of input sequences. For the treatment of continuous time systems, the map F could well be the (exact or numerically approximated) time–T map of such a system, cf. Remark 4.5, below. For any initial value $x_0 \in \mathbb{R}^d$ and any $u \in \mathcal{U}$ we denote the corresponding solution trajectory of (1) by $\Phi(t, x_0, u)$.

We recall that a continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{K}$ if it is monotone increasing with $\gamma(0) = 0$ and of class $\mathcal{K}_\infty$ if additionally it is unbounded. It is of class $\mathcal{L}$ if it is monotone decreasing to 0. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in the first and of class $\mathcal{L}$ in the second argument.

In order to measure distances between sets C, $D \subset \mathbb{R}^d$ we use the point–set distance

$$\|x\|_D := \inf_{x \in D} \|x - y\|,$$

the Hausdorff semi–distance

$$\operatorname{dist}(C, D) := \sup_{x \in C} \|x\|_D$$

and the Hausdorff distance

$$d_H(C, D) := \max\{\operatorname{dist}(C, D), \operatorname{dist}(D, C)\}.$$

With D^c we denote the complement of the set D.

A set $A \subset \mathbb{R}^d$ is called *locally asymptotically controllable* if there exists a neighborhood $B \supset A$ and a class $\mathcal{KL}$ function β such that for each $x \in B$ there exists a $u_x \in \mathcal{U}$ with

$$\|\Phi(t, x, u_x)\|_A \leq \beta(\|x\|_A, t) \text{ for all } t \in \mathbb{N}_0.$$

A set $A \subset \mathbb{R}^d$ is called *locally (robustly) asymptotically stable* if there exists a neighborhood $B \supset A$ and a class $\mathcal{KL}$ function β such that for each $x \in B$ and each $u \in \mathcal{U}$ the inequality

$$\|\Phi(t, x, u_x)\|_A \leq \beta(\|x\|_A, t) \text{ for all } t \in \mathbb{N}_0$$

holds.

The *reachable set* for a locally asymptotically controllable set A is given by

$$\mathcal{R}(A) := \left\{ x \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \text{there exists } u_x \in \mathcal{U} \text{ with} \\ \lim\limits_{t \to \infty} \|\Phi(t, x, u_x)\|_A = 0 \end{array} \right\},$$

and the *(uniform robust) domain of attraction* for a locally robustly asymptotically stable set A is given by

$$\mathcal{D}(A) := \left\{ x \in \mathbb{R}^d \;\middle|\; \begin{array}{l} \text{there ex. } \delta_x \text{ of class } \mathcal{KL} \\ \text{with } \|\Phi(t, x, u)\|_A \leq \delta_x(t) \\ \text{for all } t \in \mathbb{N}_0,\ u \in \mathcal{U} \end{array} \right\}$$

In this paper we will concentrate on an algorithm for reachable sets. The necessary changes for domains of attraction, however, are straightforward, cf. Remark 5.2.

3. ROBUSTNESS PROPERTIES

In order to analyze the numerical algorithm, for any $\alpha > 0$ we consider the following *α–inflated* system.

$$x(t+1) = F(x(t), u(t)) + p[u](t), \;\; t \in \mathbb{N}_0. \quad (2)$$

The function p in (2) is from the set of *nonanticipating strategies* $\mathcal{P}_\alpha$. This is the set of all maps $p : \mathcal{U} \to \mathcal{W}_\alpha$ with the property that for each $k \in \mathbb{N}_0$ the equality $u_1(t) = u_2(t)$ for all $t = 0, 1, \ldots, k$ implies $p[u_1](t) = p[u_2](t)$ for all $t = 1, \ldots, k$, where $\mathcal{W}_\alpha := \{w : \mathbb{N}_0 \to W_\alpha\}$ and $W_\alpha := \{w \in \mathbb{R}^d \mid \|w\| \leq \alpha\}$. We denote the trajectories of (2) by $\Psi(t, x_0, u, p)$.

For the α–inflated system (2), $\alpha > 0$, we consider the following two objects.

A set A_α is called *locally robustly asymptotically controllable* if there exists a neighborhood B of A_α and a class $\mathcal{KL}$ function β such that for each $x \in B$ and each $p \in \mathcal{P}_\alpha$ there exists $u \in \mathcal{U}$ with

$$\|\Psi(t, x, u, p)\|_{A_\alpha} \leq \beta(\|x\|_{A_\alpha}, t).$$

The *robust reachable set* $\mathcal{R}_\alpha(A_\alpha)$ for a robustly asymptotically controllable set A_α is given by the set of all the points $x \in \mathbb{R}^d$ for which there exists a class $\mathcal{L}$ function δ_x such that for each $p \in \mathcal{P}_\alpha$ there exists $u \in \mathcal{U}$ with

$$\|\Psi(t, x, u, p)\|_{A_\alpha} \leq \delta_x(t).$$

The following theorem states robustness properties for attracting sets with respect to inflation.

Theorem 3.1. Consider a locally asymptotically controllable set A for the system (1) with reachable set $\mathcal{D}(A)$. Then there exists $\alpha_0 > 0$ and a set A_{α_0} with the following properties.

(i) The set A_{α_0} is locally robustly asymptotically controllable for each α–inflated system (2), $\alpha \in (0, \alpha_0]$, and satisfies $A \subseteq A_{\alpha_0} \subset \mathcal{D}(A)$.

(ii) There exists a class $\mathcal{K}_\infty$ function γ such that for all $\alpha \in (0, \alpha_0]$ the robust reachable set $\mathcal{R}_\alpha(A_{\alpha_0})$ for the α–inflated system (2) satisfies $d_H(\mathcal{R}(A)^c, \mathcal{R}_\alpha(A_{\alpha_0})^c) \leq \gamma(\alpha)$.

Proof: See (Grüne, 2001, Proposition 4.4.1 for (i) and Proposition 7.3.3 for (ii)). □

Unfortunately, for the design of an efficient algorithm these robustness properties are not sufficient. The reason for this is, that in the algorithm we will not only exploit the controllability property to A_{α_0} but we will also need a property of the complement $\mathcal{R}(A)^c$ of $\mathcal{R}(A)$. Observe, that $\mathcal{R}(A)^c$ is forward invariant in the sense that

$$\Phi(t, x, u) \in \mathcal{R}(A)^c$$

for all $t \geq 0$ and all $x \in \mathcal{R}(A)^c$. This forward invariance property, however, is not robust with respect to inflation. Hence we will make the following assumption.

Assumption 3.2. Consider an asymptotically controllable set A for the system (1) with reachable set $\mathcal{R}(A)$. We assume that there exists $\alpha_0 > 0$ and a class $\mathcal{K}_\infty$ function $\tilde\gamma$ such that for all $\alpha \in (0, \alpha_0]$ there exists a set C_α satisfying

$$\Psi(t, x, u, p) \in C_\alpha$$

for all $t \geq 0$, $x \in C_\alpha$, $u \in \mathcal{U}$ and all $p \in \mathcal{P}_\varepsilon$, and

$$d_H(C_\alpha^c, \mathcal{R}(A)) \leq \tilde\gamma(\alpha).$$

It should be noted that Assumption 3.2 is equivalent to an assumption which can also be found in (Colonius and Kliemann, 2000; Häckl, 1995; Szolnoki, 2001), namely to the fact that the so called chain reachable set coincides with the reachable set, cf. (Grüne, 2001, Proposition 7.7.5). In particular, this implies that Assumption 3.2 is generically satisfied for parameterized families of systems satisfying a so called inner pair condition, cf. (Colonius and Kliemann, 2000, Theorem 4.5.7).

4. STATE SPACE DISCRETIZATION

Let us now consider a suitable abstract concept for a state space discretization of (1).

Definition 4.1. Consider a compact set $\Omega \subset \mathbb{R}^d$. A *cell covering* $\mathbf{Q} = (Q_i)_{i=1,\ldots,P}$ of Ω is a finite family of closed sets Q_i, $i = 1, \ldots, P$, $P \in \mathbb{N}$, with nonvoid interior such that $\operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1,\ldots,P} Q_i = \Omega$. The sets Q_i are called the *cells* of the discretization. With $\mathcal{C}_\mathbf{Q}$ we denote the family of all possible unions of cells in $\mathbf{Q}$, i.e., the family of all sets $C \subseteq \Omega$ such that $C = \bigcup_{i \in I} Q_i$ for some index set $I \subseteq \{1, \ldots, P\}, I \neq \emptyset$.

The value $\operatorname{diam}(Q_i) := \max_{x,y \in Q_i} \|x - y\|$ is called the *diameter* of Q_i, and the value $\operatorname{diam}(\mathbf{Q}) := \max_{i=1,\ldots,P} \operatorname{diam}(Q_i)$ is called the *maximal diameter* of $\mathbf{Q}$.

For the following definition for any set $B \subset \mathbb{R}^d$ we denote by $\mathcal{U}(\Omega)$ the set of all maps $\bar u : B \times \mathbb{N}_0 \to U$ satisfying $\bar u(x, \cdot) \in \mathcal{U}$ for all $x \in B$.

Definition 4.2. Consider a cell covering $\mathbf{Q}$ of some compact set $\Omega \subset \mathbb{R}^d$ and system (1) with right hand side F. Consider real numbers $\Delta_i \in \mathbb{R}^+$ for $i = 1, \ldots, P$. Then a set valued map $\widehat F : \mathcal{C}_\mathbf{Q} \times \mathcal{U}(\Omega) \to P(\mathbb{R}^n)$ is called a *cell-cell space discretization* of F with accuracy Δ_i if $\operatorname{diam}(Q_i) \leq \Delta_i$, for all $i = 1, \ldots, P$ and all $\bar u \in \mathcal{U}(\Omega)$ the map $\widehat F$ satisfies

$$\widehat F(Q_i, \bar u) \cap \Omega \in \mathcal{C}_\mathbf{Q} \cup \{\emptyset\}$$

and is such that for each $\bar u \in \mathcal{U}(\Omega)$ the inequality

$$\operatorname{dist}(\widehat F(Q_i, \bar u), F(Q_i, \bar u)) \leq \Delta_i$$

holds and for each $u \in \mathcal{U}$ and each $x \in Q_i$ there exists $\bar u \in \mathcal{U}(\Omega)$ such that

$$\operatorname{dist}(\{F(x, u)\}, \widehat F(Q_i, \bar u)) \leq \Delta_i \qquad (3)$$

holds and, conversely, for each $\bar u \in \mathcal{U}(\Omega)$ and each $x \in Q_i$ there exists $u \in \mathcal{U}$ such that (3) holds.

Here we denote

$$F(Q_i, \bar u) := \bigcup_{x \in Q_i} \{F(x, \bar u(x, \cdot))\}.$$

The cell-cell space discretization $\widehat F(x, u)$ is called *rigorous*, if (3) holds with 0 instead of Δ_i.

For $C \in \mathcal{C}_\mathbf{Q}$ and some sequence $\bar{\mathbf{u}} = (\bar u_j)_{j \in \mathbb{N}_0}$ with $\bar u_j \in \mathcal{U}(\Omega)$, $j = 0, 1, \ldots$ we define the *iterate* $\widehat\Phi(t, C, \bar{\mathbf{u}})$, $t \in \mathbb{N}_0$, of $\widehat F$ (relative to Ω) inductively by $\widehat\Phi(0, C, \bar{\mathbf{u}}) := C$ and

$$\widehat\Phi(t+1, C, \bar{\mathbf{u}}) := \widehat F(\widehat\Phi(t, C, \bar{\mathbf{u}}) \cap \Omega, \bar u_i) \quad (4)$$

for $i \in \mathbb{N}_0$.

While we will see that rigorous state space discretizations are extremely useful from a theoretical point of view, in practice they are rather difficult to implement. We will therefore also use the following weaker property.

Definition 4.3. Consider a cell covering $\mathbf{Q}$ of some compact set $\Omega \subset \mathbb{R}^d$ and a cell–cell space discretization $\widehat F$ of system (1) on $\mathbf{Q}$.

Then we say that $\widehat F$ has *inner error* ε for some $\varepsilon > 0$ if (3) holds with $\min \varepsilon, \Delta_i$ instead of Δ_i.

Note that a rigorous discretization corresponds to an inner error $\varepsilon = 0$.

The following Lemma, whose proof is straightforward, shows the relation between $\widehat F$ and the inflated system (2).

Lemma 4.4. (i) For each $i = 1, \ldots, P$ and each $\bar u \in \mathcal{U}(\Omega)$ the inclusion

$$\widehat F(Q_i, \bar u) \subseteq \bigcup_{p \in \mathcal{P}_{\Delta_i}} \bigcup_{x \in Q_i} \{\Psi(h, x, \bar u(x, \cdot), p)\}$$

holds. (ii) If the space discretization has inner error $\varepsilon > 0$ then for each $i = 1, \ldots, P$, each $x \in Q_i$ and each $\bar u \in \mathcal{U}(\Omega)$ there exists $u \in \mathcal{U}$ and $p \in \mathcal{P}_\varepsilon$ such that

$$\Psi(h, x, u, p) \in \widehat F(Q_i, \bar u)$$

holds and for each $u \in \mathcal{U}$ there exists $\bar u \in \mathcal{U}(\Omega)$ and $p \in \mathcal{P}_\varepsilon$ such that this inclusion holds.

Due to space restrictions we will not discuss details about the implementation of $\widehat{F}$, and refer to (Grüne, 2001, Section 5.3) instead, see also (Junge, 2000*a*; Junge, 2000*b*) for the uncontrolled case. We just remark that the simplest way of implementing $\widehat{F}$ is by selecting a number of equidistributed test points x_i^j, $j=1,\ldots,k$, in each cell Q_i and defining $\widehat{F}(Q_i,\bar{u})$ to be the minimal set $C\in\mathcal{C}_{\mathbf{Q}}$ satisfying $F(x_i^j,u(x_i^j,0))\in C\cup\Omega^c$ for all $j=1,\ldots,k$.

Remark 4.5. If the map F is the numerical approximation of a time–T map of a continuous time system we can take into account the error caused by the time–discretization in the definition of the accuracy of $\widehat{F}$, cf. (Grüne, 2001, Remark 5.3.10).

5. THE ALGORITHM

We do now give a formal description of the algorithm.

Algorithm 5.1. Let F be a discrete time system of type (1). Consider a compact set Ω, a cell covering $\mathbf{Q}^0$ of Ω with P^0 cells Q_i^0, $i=1,\ldots,P^0$, and a cell-cell space discretization $\widehat{F}$ on $\mathbf{Q}^0$ with accuracy $\Delta_i^0\le\varepsilon^0$ for all $i=1,\ldots,P^0$, and let $j=0$. To each cell Q_i^j associate a status $s(Q_i^j)$ which can take the values *in* (inside), *pin* (partially inside), *und* (undefined) and *out* (outside) and define the sets

$$\Omega_{in}^j=\bigcup_{i,\,s(Q_i^j)=in} Q_i^j,$$
$$\Omega_{pin}^j=\bigcup_{i,\,s(Q_i^j)=pin} Q_i^j\cup\Omega_{in}^j$$
and
$$\Omega_{out}^j=\bigcup_{i,\,s(Q_i^j)=out} Q_i^j\cup\Omega^c.$$

Consider some target set $\mathcal{S}\subset\operatorname{int}\Omega$ and set $s(Q_i^0):=in$ for all $Q_i^0\subseteq\mathcal{S}$, and $s(Q_i^0):=und$ else. Set $D^0:=\Omega_{in}^0$, $C^0:=\Omega_{out}^0$ and proceed iteratively

(1) **(Selection Step)**
For all Q_i^j with $s(Q_i^j)=und$ or $s(Q_i^j)=pin$ set

$s(Q_i^j):=pin$, if there exists $\bar{u}\in\mathcal{U}(Q_i^j)$ with $\widehat{F}^j(Q_i^j,\bar{u})\cap\Omega_{pin}^j\neq\emptyset$

$s(Q_i^j):=in$, if $Q_i^j\subseteq\mathcal{S}$ or if there exists $\bar{u}\in\mathcal{U}(Q_i^j)$ with $\widehat{F}^j(Q_i^j,\bar{u})\subseteq\Omega_{in}^j$

$s(Q_i^j):=out$, if $\widehat{F}^j(Q_i^j,\bar{u})\subseteq\Omega_{out}^j$ for all $\bar{u}\in\mathcal{U}(Q_i^j)$,

where we update the sets Ω_{in}^j, Ω_{pin}^j and Ω_{out}^j after each new assignment.
If some $s(Q_i^j)$ changed its value during this computation repeat this step.

(2) **(Status Update Step)**
Set $s(Q_i^j):=out$ for all Q_i^j with $s(Q_i^j)=und$.
Set $s(Q_i^j):=und$ for all Q_i^j with $s(Q_i^j)=pin$.
Update the sets Ω_{in}^j, Ω_{pin}^j and Ω_{out}^j and set $D^{j+1}:=\Omega_{in}^j$, $C^{j+1}:=\Omega_{out}^j$ and $E^{j+1}:=\Omega\setminus(D^{j+1}\cup C^{j+1})$.

(3) **(Refinement Step)**
Consider a new cell covering $\mathbf{Q}^{j+1}$ of Ω with P^{j+1} cells Q_i^j, $i=1,\ldots,P^{j+1}$, satisfying $\mathcal{C}_{\mathbf{Q}^j}\subset\mathcal{C}_{\mathbf{Q}^{j+1}}$ and a new cell–cell space discretization $\widehat{F}^{j+1}$ on $\mathbf{Q}^{j+1}$ with accuracy $\Delta_i^{j+1}\le\varepsilon^{j+1}$ for all $i=1,\ldots,P^{j+1}$ for some $\varepsilon^{j+1}<\varepsilon^j$. Set $s(Q_{i'}^{j+1})=s(Q_i^j)$ for all $Q_{i'}^{j+1}\subset Q_i^j$, $j:=j+1$ and continue with Step (1).

Note that in practice only the cells lying in E^{j+1} have to be refined in Step (3) since the cells in $D^{j+1}=\Omega_{in}^j$ and $C^{j+1}=\Omega_{out}^j$ remain unchanged in the future iterations.

Typically, the cells Q_i^j are chosen to be cuboids, which are refined by successive subdivision in each coordinate direction. This motivates the name "subdivision algorithm".

Remark 5.2. The corresponding algorithm for domains of attraction is obtained by by exchanging "there exists $\bar{u}\in\mathcal{U}(Q_i^j)$" and "for all $\bar{u}\in\mathcal{U}(Q_i^j)$" in the Selection Step.

Remark 5.3. We have formulated Algorithm 5.1 without using the iterates $\widehat{\Phi}(t,\cdot,\cdot)$ of $\widehat{F}$ as defined in (4), because we wanted to present a directly implementable version. Using these iterates, however, we can simplify the description of the algorithm. Denoting by $\mathbf{U}$ the set of all sequences $\bar{\mathbf{u}}=(\bar{u}_j)_{j\in\mathbb{N}_0}$ with $\bar{u}_j\in\mathcal{U}(\Omega)$, we can replace steps (1) and (2) by the following single step (1').

(1') **(Selection Step)**
Compute the sets

$$C^{j+1}=\left\{Q_i^j\in\mathbf{Q}^j\;\middle|\;\begin{array}{l}\widehat{\Phi}(t,x,\bar{\mathbf{u}})\cap D^j=\emptyset\\ \text{for all } t\in\mathbb{N}\\ \text{and all } \bar{\mathbf{u}}\in\mathbf{U}\end{array}\right\},\quad(5)$$

$$D^{j+1}=\left\{Q_i^j\in\mathbf{Q}^j\;\middle|\;\begin{array}{l}\text{there exist } \bar{\mathbf{u}}\in\mathbf{U},\\ t\in\mathbb{N} \text{ with}\\ \widehat{\Phi}(t,Q_i,\bar{\mathbf{u}})\subseteq D^j\end{array}\right\}\quad(6)$$

and set $E^{j+1}:=\Omega\setminus(D^{j+1}\cup C^{j+1})$.

A straightforward induction shows that step (1') indeed computes the same sets D^{j+1} and C^{j+1} as the steps (1) and (2) in the original description of Algorithm 5.1.

6. CONVERGENCE ANALYSIS

The following theorem shows the convergence properties of Algorithm 5.1 for a rigorous space discretization. For detailed proofs see (Grüne, 2001, Section 7.5).

Theorem 6.1. Consider a discrete time system F of type (1) and let A be a locally asymptotically controllable set with reachable set $\mathcal{R}(A)$. Let $\mathcal{S} \subset \mathcal{R}(A)$ be a neighborhood of A, let $\Omega \subset \mathbb{R}^n$ be a compact set containing $\mathcal{R}(A)$ and consider Algorithm 5.1 with rigorous space discretizations $\widehat{F}^j$. Then the inclusions $D^{j+1} \subseteq \mathcal{R}(A) \subseteq D^{j+1} \cup E^{j+1}$, $C^{j+1} \subseteq \mathcal{R}(A)^c \subseteq C^{j+1} \cup E^{j+1}$ and $\partial\mathcal{R}(A) \subseteq E^{j+1}$ hold.

Furthermore, for all $j \in \mathbb{N}_0$ with ε^j sufficiently small we have the following estimates:

(i) If γ of class $\mathcal{K}_\infty$ is such that Theorem 3.1(ii) is satisfied, then

$$d_H(C^{j+1} \cup E^{j+1}, \mathcal{R}(A)^c) \leq \gamma(\varepsilon^j) + \varepsilon^j.$$

(ii) If $\mathcal{R}(A)$ satisfies Assumption 3.2 for $\tilde{\gamma}$ of class $\mathcal{K}_\infty$, then

$$d_H(D^{j+1} \cup E^{j+1}, \mathcal{R}(A)) \leq \tilde{\gamma}(\varepsilon^j) + \varepsilon^j.$$

(iii) If the assumptions of (i) and (ii) hold, and $\hat{\gamma}$ is of class $\mathcal{K}_\infty$ and satisfies $\hat{\gamma} \geq \max\{\gamma, \tilde{\gamma}\}$, then

$$d_H(E^{j+1}, \partial\mathcal{R}(A)) \leq \hat{\gamma}(\varepsilon^j) + \varepsilon^j.$$

Sketch of Proof: Using the representations (5) and (6), the claimed inclusions follow from the fact that the discretization is rigorous.

In order to prove estimate (i), observe that by Theorem 3.1(i) $\mathcal{S}$ contains a set A_{α_0} which is robustly attracting for the α–inflated system (2) for all $\alpha = \varepsilon^j$ sufficiently small. By the assumption in (i) the corresponding robust reachable set $\mathcal{R}_\alpha(A_{\alpha_0})$ satisfies $d_H(\mathcal{R}_\alpha(A_{\alpha_0})^c, \mathcal{R}(A)^c) \leq \gamma(\varepsilon^j)$. For ε^j sufficiently small one sees that D_j contains a neighborhood of A_{α_0}, hence the set of points x which can be steered to D_j for (2) contains $\mathcal{R}_\alpha(A_{\alpha_0})$. Thus from Lemma 4.4(i) we obtain that each cell $Q_i^j \subseteq E^{j+1}$ must intersect $\mathcal{R}_\alpha(A_{\alpha_0})^c$. Since $\operatorname{diam}(Q_i) \leq \varepsilon^j$ this implies $\operatorname{dist}(E^{j+1}, \mathcal{R}(A)^c) \leq \gamma(\varepsilon^j) + \varepsilon^j$ which together with $C^{j+1} \subseteq \mathcal{R}(A)^c \subseteq C^{j+1} \cup E^{j+1}$ implies (i).

Estimate (ii) follows by similar arguments and estimate (iii) by combining (i) and (ii). □

In other words, convergence of $C^{j+1} \cup E^{j+1}$ to $\mathcal{R}(A)^c$ is always guaranteed, while the additional Assumption 3.2 guarantees that the (iteratively refined) region E^{j+1} converges to the boundary of $\mathcal{R}(A)$.

Next we investigate the behavior of the algorithm for non–rigorous space discretizations.

Theorem 6.2. Consider a discrete time system F of type (1) and let A be a weakly asymptotically controllable set with reachable set $\mathcal{R}(A)$. Let $\mathcal{S} \subset \mathcal{R}(A)$ be a neighborhood of A, let $\Omega \subset \mathbb{R}^n$ be a compact set containing $\mathcal{R}(A)$ and consider Algorithm 5.1 with space discretizations $\widehat{F}^j$ with inner error $\varepsilon > 0$. Assume that γ of class $\mathcal{K}_\infty$ is such that Theorem 3.1(ii) and assume furthermore that $\mathcal{R}(A)$ satisfies Assumption 3.2 for this γ. Then for $\varepsilon > 0$ sufficiently small the following inequalities hold for $r^j := \max\{\gamma(\varepsilon^j) + \varepsilon^j, \gamma(\varepsilon)\}$.

(i) $\operatorname{dist}(C^{j+1} \cup E^{j+1}, \mathcal{R}(A)^c) \leq r^j$

(ii) $\operatorname{dist}(D^{j+1} \cup E^{j+1}, \mathcal{R}(A)) \leq r^j$

(iii) $\operatorname{dist}(E^{j+1} \cup (C^{j+1} \cap D^{j+1}), \partial\mathcal{R}(A)) \leq r^j$.

Sketch of Proof: The main difference to Theorem 6.1 is given by the fact that the inclusions stated in the first part of Theorem 6.1 might not hold. In order to circumvent this difficulty one shows that for sufficiently small $\alpha = \varepsilon$ the sets $C_* = C_\alpha$ from Assumption 3.2 and $D_* = \mathcal{R}_\alpha(A_\alpha)$ from Theorem 3.1 satisfy the properties $C_* \subseteq \mathcal{R}(A)^c$, $D_* \subseteq \mathcal{R}(A)$, $C^j \subseteq D_*^c$, $D^j \subseteq C_*^c$ for all $j > 0$, $d_H(C_*^c, \mathcal{R}(A)) \leq \gamma(\varepsilon)$ and $d_H(D_*^c, \mathcal{R}(A)^c) \leq \gamma(\varepsilon)$.

Now, just as in the proof of Theorem 6.1 we can prove that $\operatorname{dist}(E^{j+1}, \mathcal{R}(A)^c) \leq \gamma(\varepsilon^j) + \varepsilon^j$. Since $\operatorname{dist}(C^{j+1}, \mathcal{R}(A)^c) \leq \operatorname{dist}(D_*^c, \mathcal{R}(A)^c) \leq \gamma(\varepsilon)$ this implies (i).

Estimate (ii) follows by similar arguments and estimate (iii) by combining (i) and (ii). □

7. A NUMERICAL EXAMPLE

As an example we consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -b_1 x_3 \sin x_1 - b_2 x_2 + P \\
\dot{x}_3 &= b_3 \cos x_1 - b_4 x_2 + E + u
\end{aligned}$$

which models a synchronous generator. Using the parameters $b_1 = 34.29$, $b_2 = 0.0$, $b_3 = 0.149$, $b_4 = 0.3341$, $P = 28.22$ and $E = 0.2405$ this system, which was taken from (Ortega *et al.*, 2000), exhibits a locally stable equilibrium at $x^* = (1.12, 0.0, 0.914)$. In this reference the feedback law $u(x) = a_1((x_1 - x_1^*)b_4 + x_2)$ with feedback gain $a_1 > 0$ was proposed in order to enlarge the domain of attraction of x^*. Here we show three computations, where the discrete time map F is an approximation of the time–20 map of the system computed by an extrapolation method. For the computations the system was slowed down by replacing its right hand side $f(x, u)$ by $f(x, u)/(1 + \|f(x, u)\|^2)$.

We have computed $\mathcal{R}(\{x^*\})$ for three settings:

(i) for the uncontrolled system (i.e., $u \equiv 0$)

(ii) for feedback gain $a_1 = 0.45$

(iii) for time varying feedback gain $a_1(t) \in \{0, 0.15, 0.3, 0.45\}$, i.e., a_1 now plays the role of the control u in (1). Note that $a_1(t)$ here is the input of the discrete time system (1).

The corresponding reachable sets are shown in Figure 1(i)–(iii); the visualization was done with the graphics programming environment GRAPE (see `www.iam.uni-bonn.de/sfb256/grape/`).

The computation was done on the domain $\Omega = [0, \pi/2] \times [-5, 5] \times [0, 3]$ with a starting grid consisting of 4096 cubes, i.e., 16 cubes in each coordinate direction. Note that in this example $\mathcal{R}(\{x^*\})$ is not completely contained in Ω, in which case the algorithm gives the set of those points in $\mathcal{R}(\{x^*\})$ which can be controlled to x^* without leaving Ω. The cell–cell map $\widehat{F}$ was constructed according to the procedure described after Lemma 4.4 using 8 test points per cube. Figures 1(i) and (ii) were computed with 12 subdivision steps (4 in each coordinate direction), while for Figure 1(iii) only 9 subdivision steps (3 in each coordinate direction) were needed.

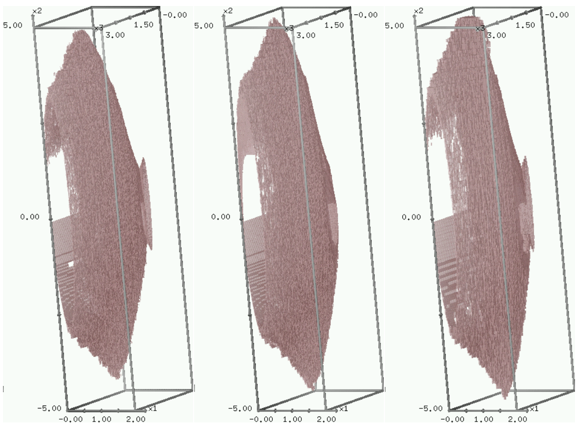

Fig. 1. (i) (ii) (iii)

The results show that the proposed feedback indeed increases the reachable set and that the time–varying choice of the feedback gain yields another significant enlargement.

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