Gain preserving Lyapunov functions for perturbed and controlled systems

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Abstract: We give Lyapunov function characterizations for variants of the input–to–state stability property, which do not only imply the qualitative properties but also represent the robustness gains and attraction rates.

1 Introduction

Lyapunov functions are an important tool for stability analysis and stabilization of nonlinear systems. They are useful in many ways, e.g., for the design of (robustly) stabilizing feedback laws, for the analysis of the system's behavior and, last but not least, as a technical tool for many proofs involving stability properties of nonlinear systems.

Most stability concepts for nonlinear systems can be characterized by suitable Lyapunov functions, where characterized means, that a system has a certain stability property if and only if an associated Lyapunov functions exists. For ordinary differential equations the basic concept was introduced more than a century ago by Lyapunov [12] as a sufficient condition, while the "if and only if" results date back to the middle of the last century, with the works of Kurzweil [10], Massera [13], Yoshisawa [20] and Zubov [21]. For controlled and perturbed systems these results were generalized subsequently see, e.g., Sontag [15], Rifford [14] or Kellet and Teel [9] for asymptotic controllability and Lin, Sontag and Wang [11] for robust stability, or, more specifically, the generalization of Zubov's construction to controlled and perturbed systems by Camilli, Wirth and the author [3, 7]. For perturbed systems where asymptotic stability cannot be guaranteed, suitable generalizations of these concepts like, e.g., input–to–state stability (ISS) and integral input–to–state stability (iISS) may still hold, and again complete characterizations of these properties by suitable Lyapunov functions is possible, see Sontag and Wang [18] and Angeli, Sontag and Wang [1].

These results deal with *qualitative* characterizations of stability or controllability properties, in the sense that the corresponding statements are of the type "the system is asymptotically stable if and only if a Lyapunov function exists". In many applications, however, also *quantitative* information is important, like, e.g., in the analysis of coupled systems, where the stability property of a coupled system does not only depend on the qualitative behavior of the single subsystems but also on the actual size of the robustness gains, cf. [8, Theorem 2.1] for a result using ISS techniques. Another application is the use of robustness properties for the analysis of the long time behavior of numerical approximation, see, e.g., [4] and [5]. Yet another example is H_∞ control, where the purpose of the controller is to minimize the H_∞ gain, which is a quantitative problem by its very nature. It is the aim of the present paper to give Lyapunov function characterizations for several types of asymptotic stability and controllability, which do not only provide a qualitative description but also an exact representation of the related gains.

Note that not all of the results mentioned above provide smooth Lyapunov functions. In fact, for asymptotic controllability a characterization by a smooth control Lyapunov function cannot be expected in general, since it is known that the existence of a smooth control Lyapunov function is a strictly stronger property than asymptotic controllability, see, e.g., [16]. However, even in the case of robust stability there might be situations where nonsmooth Lyapunov functions provide more information, though in this case the differences are typically considerably smaller and do only concern the quantitative aspects, like, e.g., in the case of H_∞ control, see [17].

Of course, if a Lyapunov function V is not smooth, one has to give a meaning to the partial differential inequality $DV(x)f(x) \le C(x)$, which in one way or the other appears in any Lyapunov type characterization (here DV stands for the derivative of V with respect to x). One way is to replace this inequality by a suitable inequality for V along the solution trajectories. This approach provides a large flexibility, in the sense that it allows to choose V from a very large class of functions, in particular it could be even discontinuous. The disadvantage with this approach is, that inequalities along solutions are typically very difficult to check for a given function, since the solution trajectories of the system under consideration are not known in general. Hence it is desirable to find techniques which allow to maintain the original partial differential inequality as far as possible, and this is where methods of nonsmooth analysis prove to be elegant and powerful tools in order to tackle this problem.

In the present paper we will use both inequalities along solutions and partial differential inequality, where we use the notion of viscosity (super–)solutions in order to overcome the lack of smoothness of our Lyapunov functions. After fixing the setup in Section 2, we first consider asymptotic stability in Section 3 and then asymptotic controllability in Section 4. Finally, in Section 5 we give an outline of the proofs providing suitable references for details.

2 Setup and preliminaries

We consider systems of the type

$$\dot{x}(t) = f(x(t), u(t), w(t)), \qquad (2.1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$ is supposed to be continuous in all variables and Lipschitz in x, uniformly in $(u, w) \in K$ for each compact subset $K \subset \mathbb{R}^m \times \mathbb{R}^l$. The functions u and w are supposed to lie in the spaces $\mathcal{U}$ and $\mathcal{W}$ given by

$$\begin{aligned} \mathcal{U} &:= \{u : \mathbb{R} \to U, \text{ measurable and loc. ess. bd.}\} \\ \mathcal{W} &:= \{w : \mathbb{R} \to W, \text{ measurable and loc. ess. bd.}\} \end{aligned}$$

where $U \subset \mathbb{R}^m$ is a compact set with $0 \in U$ and $W \subset \mathbb{R}^l$ is an arbitrary set with $0 \in W$. Here u can be interpreted either as perturbation or as control while w will always be considered as perturbation. If u is considered to be a control function, then the compactness assumption on U can be weakened, see Remark 2.3, below. The solution trajectories of (2.1) for initial value x_0 at initial time $t_0 = 0$ are denoted by $\varphi(t, x_0, u, w)$. We assume that $\varphi(t, x_0, u, w)$ exists for all times $t \geq 0$.

Our goal in this paper is to provide gain preserving Lyapunov function characterizations for input–to–state stability (ISS) like stability and controllability properties of (2.1). For the formulation of these stability concepts we need the following comparison functions.

A continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called of class $\mathcal{K}$ if it is strictly increasing with $\alpha(0) = 0$, and is called of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called of class $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first and strictly decreasing to 0 in the second argument. We define a continuous function $\mu : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ to be of class $\mathcal{KLD}$ if its restriction to $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ is of class $\mathcal{KL}$ and, in addition, it is a one dimensional dynamical system, i.e., it satisfies

$$\mu(r, t+s) = \mu(\mu(r, t), s) \text{ for all } t, s \in \mathbb{R}.$$

Observe that this condition implies $\mu(r, 0) = r$.

A system (2.1) is said to be ISS if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the inequality

$$\|\varphi(t, x_0, u, w)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|w\|_\infty)\}$$

holds for all $x_0 \in \mathbb{R}^n$, $t \geq 0$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$.

For a gain preserving Lyapunov function characterization it will be convenient to modify the ISS property as follows.

Definition 2.1 A system (2.1) is called *input-to-state dynamically stable* (ISDS), if there exists $\mu \in \mathcal{KLD}$ and σ, $\gamma \in \mathcal{K}_\infty$ such that the inequality

$$\|\varphi(t, x, u, w)\| \leq \max\{\mu(\sigma(\|x\|), t), \nu(w, t)\}.$$

holds for all $t \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathcal{U}$ and all $w \in \mathcal{W}$, where ν is defined by

$$\nu(w, t) := \operatorname{ess\,sup}_{\tau \in [0,t]} \mu(\gamma(\|w(\tau)\|), t - \tau) \qquad (2.2)$$

Here we call the function μ the *decay rate*, the function σ the *overshoot gain* and the function γ the *robustness gain.*

The ISDS property is equivalent to ISS, in the sense that ISDS implies ISS with $\beta(r, t) = \mu(\sigma(r), t)$ and same γ and that ISS with β and $\tilde{\gamma}$ implies ISDS for $\sigma(r) = \beta(r, 0)$ and any $\gamma > \tilde{\gamma}$, see [5, Proposition 3.4.4].

If u is interpreted as a control function, then we can also consider the following controlled version of the ISDS property. Here $\mathcal{P}$ is the space of nonanticipating strategies, i.e., the space of mappings $p : \mathcal{U} \to \mathcal{W}$ which for all $t > 0$ satisfy the implication $u_1(s) = u_2(s)$ for all $s \leq t \Rightarrow p(u_1, t) = p(u_2, t)$ for all $s \leq t$ (see [2, Chapter VIII] for details). Here we only require the respective inequality to hold for discrete times $t \geq 0$, since this allows a Lyapunov function characterization without assuming continuity properties with respect to u or using relaxed controls, cf. also Remark 4.3, below.

Definition 2.2 A system (2.1) is called *controlled input-to-state dynamically stable* (cISDS), if there exists a function $\mu \in \mathcal{KLD}$ and $\sigma, \gamma \in \mathcal{K}_\infty$ such that for all $p \in \mathcal{P}$, all $x \in \mathbb{R}^n$ and all sequences $0 \leq t_1 < t_2 < \ldots$ there exists $u \in \mathcal{U}$ such that

$$\|\varphi(t, x, u, p[u])\| \leq \max\{\mu(\sigma(\|x\|), t), \tilde{\nu}(p, t)\}$$

holds for all $t = t_1, t_2, \ldots$, where

$$\tilde{\nu}(p, t) := \sup_{u \in \mathcal{U}} \nu(p[u], t) \qquad (2.3)$$

for ν from (2.2).

Note that for $W = \{0\}$ Definition 2.1 includes the asymptotic stability property for perturbed systems and Definition 2.2 includes asymptotic controllability.

Remark 2.3 For the cISDS property the condition that U is compact can be weakened, if desired. For instance, for $U = \mathbb{R}^m$ one could modify the cISDS property by requiring that the control functions u satisfy $\|u(t)\| \le \eta(\|\varphi(t, x, u, p[u])\|)$ for some continuous function $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, similar to what is often assumed for asymptotic controllability, see, e.g., [16]. Then the modified system given by $\tilde{f}(x, \tilde{u}, w) = f(x, \eta(\|x\|)\tilde{u}, w)$ with $\tilde{u} \in \widetilde{U} = \{\tilde{u} \in \mathbb{R}^m \mid \|\tilde{u}\| \le 1\}$ fits into our class of systems and has the cISDS property.

In the remainder of this paper we will sometimes need the following Assumption.

Assumption 2.4 The function μ in Definition 2.1 and 2.2 is C^∞ on $\mathbb{R}^+ \times \mathbb{R}$ and solves the ordinary differential equation

$$\frac{d}{dt}\mu(r, t) = -g(\mu(r, t))$$

for some Lipschitz continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$, all $r > 0$ and all $t \in \mathbb{R}$.

It was shown in [5, Appendix B] that for given nonsmooth rates and gains from Definition 2.1 or 2.2 one can find rates and gains arbitrarily close to the original ones, such that Assumption 2.4 holds and Definition 2.1 or 2.2, respectively, remain valid. Hence Assumption 2.4 is only a mild regularity condition.

It should be noted that ISDS and cISDS are easily generalized to arbitrary compact sets (instead of the origin) and to local settings, where the ISDS inequality only holds for x_0 from a given neighborhood of the origin (or the respective set), see [5, Section 3 and 4] for details.

For completeness, we give the definition of viscosity supersolutions, a concept from nonsmooth analysis which will be important in what follows. For details we refer to [2].

Definition 2.5 Consider an open set $O \subset \mathbb{R}^n$ and a continuous function $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. Then a lower semicontinuous function $V : O \to \mathbb{R}$ is called a *viscosity supersolution* of the partial differential inequality $H(x, V, DV) \ge 0$ if for all $x \in O$ and all $p \in D^-V(x)$, where $D^-V(x) :=$

$$\left\{ p \in \mathbb{R}^n \;\middle|\; \begin{array}{l} V(y) \ge V(x) + p(y - x) - o(\|x - y\|) \\ \text{for all } y \text{ in a neighborhood of } x \end{array} \right\}$$

we have

$$H(x, V(x), p) \ge 0.$$

3 Lyapunov functions for ISDS

In this section we give two characterizations of the ISDS property by means of Lyapunov functions. The first theorem states that we can exactly represent the gains and rate of the ISDS estimate using a possibly discontinuous Lyapunov function.

Theorem 3.1 A system (2.1) is ISDS with rate $\mu \in \mathcal{KLD}$ and gains $\sigma, \gamma \in \mathcal{K}_\infty$ if and only if there exists a (possibly discontinuous) function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ satisfying

$$\|x\| \le V(x) \le \sigma(\|x\|)$$

and

$$V(\varphi(t, x, u)) \le \max\{\mu(V(x), t), \nu(u, t)\}$$

for all $x \in \mathbb{R}^n$, $t \ge 0$ and all $u \in \mathcal{U}$, where ν is given by (2.2).

For many applications it might be desirable to have ISDS Lyapunov functions with some more regularity, such that we can in particular avoid the use of the solution trajectories φ in the formulation. The next theorem shows that if we slightly relax the sharp representation of the gains, then we can always find Lipschitz Lyapunov functions, at least away from the origin.

Theorem 3.2 A system (2.1) is ISDS with rate $\mu \in \mathcal{KLD}$ satisfying Assumption 2.4 and gains $\sigma, \gamma \in \mathcal{K}_\infty$ if and only if for each $\varepsilon > 0$ there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ which is Lipschitz on $\mathbb{R}^n \setminus \{0\}$, satisfies

$$\|x\|/(1 + \varepsilon) \le V(x) \le \sigma(\|x\|)$$

and is a viscosity supersolution of

$$\inf_{\substack{u \in U \\ \gamma(\|w\|) < V(x)}} \{-DV(x) f(x, u, w) - (1 - \varepsilon) g(V(x))\} \ge 0$$

for all $x \in \mathbb{R}^n$.

Here one could even go one step further and apply the smoothing technique [11, Theorem B.1] in order to obtain a characterization by smooth Lyapunov functions, see [6] for details. This causes a slight enlargement of the gain γ, however, in the limit (i.e., for $\varepsilon \to 0$) the same gains can be characterized, hence we obtain a similar behavior as for H_∞ control problems, see [17].

On the other hand, a great advantage of the concept of viscosity solutions is that if V is smooth then the viscosity supersolution condition in this theorem "automatically" reduces to the classical partial differential inequality

$$\sup_{u \in U,\, \gamma(\|w\|) < V(x)} \{DV(x) f(x, u, w)\} \le -(1 - \varepsilon) g(V(x)),$$

hence no separate statement for smooth V is necessary.

In this context it should be noted that Theorem 3.2 gives rise to a constructive procedure for estimating ISDS robustness gains γ from "ordinary" Lyapunov functions. We illustrate this procedure by two examples.

Example 3.3 Consider a linear system $\dot{x} = f(x, w) = Ax + Bw$. If we assume ISDS then the matrix A needs to be Hurwitz and we can find a quadratic Lyapunov function $W(x) = x^T P x$ for some positive definite matrix P satisfying $c_1\|x\|^2 \le W(x) \le c_2\|x\|^2$ and $DW(x)Ax \le -c_3\|x\|^2$. Setting $V(x) = \sqrt{W(x)/c_1}$ we obtain $\|x\| \le V(x) \le c_4\|x\|$, $DV(x)Ax \le -c_5 V(x)$ and $\|DV(x)\| \le c_4$ for $c_4 = \sqrt{c_2/c_1}$ and $c_5 = c_3/(2c_2)$. Fixing some $\lambda \in (0, 1)$ we set $\gamma(r) = c_4\|B\|r/(\lambda c_5)$. Then for $\alpha = 1 - \lambda$ we obtain

$$\inf_{\gamma(\|w\|)\le V(x)} \{-DV(x)f(x, w)\} \ge \alpha c_5 V(x) = g(V(x)).$$

Hence V is an ISDS Lyapunov function in the sense of Theorem 3.2 (for each $\varepsilon > 0$) and we obtain ISDS with $\mu(r, t) = e^{-(1-\lambda)c_5 t} r$, $\sigma(r) = c_4 r$ and $\gamma(r) = c_4\|B\|r/(\lambda c_5)$, i.e., exponential convergence and linear overshoot and robustness gains.

This example also illustrates the (typical) tradeoff between the attraction rate μ and the robustness gain γ, which is represented here by the choice of λ. In the next example, showing an ISDS estimate for a simple nonlinear system, we set $\lambda = 3/4$.

Example 3.4 Consider the system $\dot{x} = f(x, w) = -x + w^3/2$ with $x \in \mathbb{R}$, $w \in \mathbb{R}$. Using the Lyapunov function $V(x) = |x|$ one obtains $DV(x)f(x, 0) = -|x| = -V(x)$. We choose γ such that $\gamma(|w|) \le V(x) = |x|$ implies $|w^3/2| \le 3|x|/4$, i.e., $\gamma(r) = 2r^3/3$. Then we obtain

$$\sup_{\gamma(\|w\|)\le V(x)} \{-DV(x)f(x, w)\} \ge \frac{1}{4}V(x) = g(V(x)),$$

and consequently ISDS with $\mu(r, t) = e^{-t/4} r$, $\sigma(r) = r$ and $\gamma(r) = 2r^3/3$.

4 Lyapunov functions for cISDS

We will now present similar theorems as in the previous section for the cISDS property. Compared to the ISDS case, two major difficulties arise here. One originates from the fact that our system's trajectories will not in general depend continuously on u. Even if the structure of the system allows such a continuous dependence (e.g., for control affine systems) the strategies $p \in \mathcal{P}$ will in general destroy this continuity property. This difficulty results in additional approximation parameters ε in the following theorem, cf. also Remark 4.3, below.

Theorem 4.1 Consider the system (2.1) and functions $\gamma, \sigma \in \mathcal{K}_\infty$ and $\mu \in \mathcal{KLD}$. Define $\mu_\varepsilon(r, t) = \mu(r, (1-\varepsilon)\, t)$, $\gamma_\varepsilon = (1+\varepsilon)\gamma$ and $\tilde{\nu}_\varepsilon$ by (2.3) with $\gamma = \gamma_\varepsilon$ and $\mu = \mu_\varepsilon$. Then the following properties are equivalent:

(i) For each $\varepsilon > 0$ system (2.1) is wISDS with robustness gain γ_ε, overshoot gain σ and attraction rate μ_ε.

(ii) For each $\varepsilon > 0$ there exists a (possibly discontinuous) function $V_\varepsilon : \mathbb{R}^n \to \mathbb{R}_0^+$ which satisfies

$$\|x\| \le V_\varepsilon(x) \le \sigma(\|x\|)$$

and

$$\inf_{u\in\mathcal{U}} V_\varepsilon(\varphi(t, x, u, p[u])) \le \max\{\mu_\varepsilon(V_\varepsilon(x), t),\, \tilde{\nu}_\varepsilon(p, t)\}$$

for all $x \in \mathbb{R}^n$, all $p \in \mathcal{P}$ and all $t \ge 0$.

The second difficulty arising in the cISDS case concerns the controlled analogue of Theorem 3.2, because here the construction of the continuous Lyapunov function from the ISDS case does not carry over to the cISDS case, cf. Remark 5.3, below. Hence we only obtain a sufficient condition for the cISDS property.

Theorem 4.2 Consider system (2.1) and assume there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ which satisfies

$$\|x\| \le V(x) \le \sigma(\|x\|)$$

and is a viscosity supersolution of the equation

$$\sup_{u\in U} \inf_{\gamma(\|w\|)\le V(x)} \{-DV(x)f(x, u, w) - g(V(x))\} \ge 0.$$

Then for each $\varepsilon > 0$ the system (2.1) is cISDS with gains σ and γ and rate $\mu_\varepsilon(r, t) = \mu(r, (1 - \varepsilon)t)$ with μ from Assumption 2.4.

It is an open question at the moment whether the existence of a continuous Lyapunov function meeting the assumptions of Theorem 4.2 is indeed a strictly stronger property than cISDS. Nevertheless, Theorem 4.2 is useful as a verification theorem for cISDS robustness gains in the same way as Theorem 3.2 is for ISDS, cf. the Examples 3.3 and 3.4.

Remark 4.3 If we assume that for all $p \in \mathcal{P}$, all $x \in \mathbb{R}^n$ and all sequences $u_n \in \mathcal{U}$ there exists a subsequence $n_k \to \infty$ such that $\varphi(t, x, u_{n_k}, p[u_{n_k}]) \to \varphi(t, x, u, p[u])$ for each $t \ge 0$, then both Theorem 4.1 and Theorem 4.2 also hold for $\varepsilon = 0$, see [5, Section 4].

Note however, that for this condition to hold we will in general have to restrict our class of perturbation strategies $\mathcal{P}$.

5 Outline of the proofs

In this section we present the main ideas needed for the proofs of the results in this paper. Detailed proofs can be found in [5, Sections 3 and 4]. The proofs in [5], however, use an indirect construction of the function V via its level sets. Here we outline a direct approach following ideas from [6], which in turn extend a construction from [20] to perturbed systems.

The main technical step in the ISDS case is the observation formulated in the following Lemma. The proof is straightforward, see [6, Lemma 5.1] and also the proof of Lemma 5.2, below.

Lemma 5.1 Consider a (possibly discontinuous) function $V : \mathbb{R}^n \to \mathbb{R}_0^+$. Then the following two statements are equivalent

(i) $V(\varphi(t,x,u,w)) \le \max\{\mu(V(x),t),\, \nu(w,t)\}$ for all $t \ge 0$, all $u \in \mathcal{U}$ and all $w \in \mathcal{W}$.

(ii) $V(\varphi(t,x,u,w)) \le \mu(a,t)$ for all times $t \ge 0$, all values $a \in \mathbb{R}$ with $a \ge V(x)$, all $u \in \mathcal{U}$ and all $w \in \mathcal{W}$ satisfying $\gamma(\|w(\tau))\|) \le \mu(a,\tau)$ for almost all $\tau \in [0,t]$.

Proof of Theorem 3.1: The existence of V immediately implies the ISDS property using the bounds on V. Conversely, if a system has the ISDS property, then we define

$$V(x) := \inf\left\{ b \ge 0 \;\middle|\; \begin{array}{l} \|\varphi(t,x,u,w)\| \le \max\{\mu(b,t),\, \nu(w,t)\} \\ \text{for all } u \in \mathcal{U},\ w \in \mathcal{W} \text{ and all } t \ge 0 \end{array} \right\}$$

For this function one can verify the stated bounds as well as Lemma 5.1(ii), which completes the proof. □

Proof of Theorem 3.2: Let $\varepsilon > 0$ and assume that V is given. The results in [19, Section 4] imply that then V satisfies

$$V(\varphi(t,x,u,w)) \le \mu(V(x),t)$$

for all $u \in \mathcal{U}$ and $w \in \mathcal{W}$ with $\gamma(\|w(\tau)\|) \le \mu(V(x),t)$ for almost all $\tau \in [0,t]$. where μ solves $\dot\mu = -(1-\varepsilon)g(\mu)$, $\mu(r,0) = r$.

Applying this inequality inductively for small $t > 0$ one proves Lemma 5.1(ii) for $\mu(r,(1-\varepsilon)t)$. Using the bounds on V we obtain

$$\|\varphi(t,x,u,w)\| \le \frac{\max\{\mu(\sigma(\|x\|),(1-\varepsilon)t),\nu_\varepsilon(w,t)\}}{1-\varepsilon}$$

with ν_ε given by (2.2) using $\mu(r,(1-\varepsilon)t)$. By continuity, this implies the desired ISDS estimate for $\varepsilon \to 0$.

Conversely, assume ISDS. Then for $\varepsilon > 0$ we define $\rho_\varepsilon(r) := \varepsilon(1-e^{-r}) + 1$ and

$$V(x) := \inf\left\{ b \ge 0 \;\middle|\; \begin{array}{c} \|\varphi(t,x,u,w)\| \le \rho_\varepsilon(\mu(b,t)) \\ \max\{\mu(b,(1-\varepsilon)t),\, \nu(w,t)\} \\ \text{for all } u \in \mathcal{U},\ w \in \mathcal{W} \text{ and all } t \ge 0 \end{array} \right\}$$

For this function one verifies Lemma (5.1)(ii), which immediately implies the desired viscosity supersolution property by Definition 2.5, provided V is continuous.

In order to see continuity, one first uses compactness arguments to obtain that for any compact set $K \subset \mathbb{R}^n$ with $0 \ne K$ there exist constants $R > 0$ and a compact interval $I \subset \mathbb{R}^+$ such that in the definition of V for $x \in K$ we can restrict ourselves to $b \in I$ and $\|w\| \le R$. Then the fact that the the attraction rate is slowed down by $(1-\varepsilon)$ implies that there exists $T > 0$ such that the inequality in the definition of V is always satisfied for all $t \ge T$. Hence the inequality needs to be satisfied on the compact interval $[0,T]$ only, which eventually implies continuity. The stated Lipschitz property then follows by a careful estimation of all the functions involved in the definition of V. □

For the cISDS case the analogue to Lemma 5.1 is as follows.

Lemma 5.2 Consider a (possibly discontinuous) function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ and a point $x \in \mathbb{R}^n$. Then the following two statements are equivalent

(i) For all $t > 0$ and all $p \in \mathcal{P}$ there exists $u \in \mathcal{U}$ such that $V(\varphi(t,x,u,p[u])) \le \max\{\mu(V(x),t),\, \tilde\nu(p,t)\}$.

(ii) For all $t > 0$, all $a \in \mathbb{R}$ with $a \ge V(x)$ and all $p \in \mathcal{P}$ satisfying $\gamma(\|p[u](\tau)\|) \le \mu(a,\tau)$ for almost all $\tau \in [0,t]$ and all $u \in \mathcal{U}$, there exists $u \in \mathcal{U}$ with $V(\varphi(t,x,u,w)) \le \mu(a,t)$.

Proof: "(i)⇒(ii)": Consider a perturbation strategy p meeting the assumptions of (ii). Then one immediately obtains $\tilde\nu(p,t) \le \mu(a,t)$ and consequently the control function u from (i) also satisfies (ii).

"(ii)⇒(i)": Consider an arbitrary $p \in \mathcal{P}$ and set $a = \max\{V(x), \mu(\tilde\nu(p,t),-t)\}$, which implies $\gamma(\|p[u](\tau)\|) \le \mu(a,\tau)$ for almost all $\tau \in [0,t]$ and all $u \in \mathcal{U}$. Then either $a = V(x)$ or $\mu(a,t) = \tilde\nu(p,t)$ holds. From (ii) we find $u \in \mathcal{U}$ with $V(\varphi(t,x,u,p[u])) \le \mu(a,t)$, which in the first case implies $V(\varphi(t,x,u,p[u])) = \mu(V(x),t)$ and in the second case yields $V(\varphi(t,x,u,p[u])) = \tilde{\nu}(p,t)$. This shows the assertion. □

Proof of Theorem 4.1: The existence of V_ε implies the existence of u such that

$$V(\varphi(t,x,u,p[u])) \leq \max\{\mu_{2\varepsilon}(V(x),u),\, \tilde{\nu}_{2\varepsilon}(p,t)\}$$

From this for any sequence $0 \leq t_1 < t_2 < \ldots$ by induction we obtain the stated cISDS estimate for 2ε, and since $\varepsilon > 0$ was arbitrary for all $\varepsilon > 0$.

Conversely, if a system has the stated cISDS property, then for $\varepsilon > 0$ we define

$$V_\varepsilon(x) := \inf\left\{ b \geq 0 \;\middle|\; \begin{array}{l} \text{for all sequences } 0 \leq t_1 < t_2 < \ldots \\ \text{and all } p \in \mathcal{P} \text{ there exists } u \in \mathcal{U} \text{ with} \\ \|\varphi(t,x,u,w)\| \leq \max\{\mu_\varepsilon(b,t),\, \nu_\varepsilon(w,t)\} \\ \text{for } t = t_1, t_2, \ldots \end{array} \right\}$$

For this function one verifies the desired bounds as well as Lemma 5.2(ii) for $\mu = \mu_{2\varepsilon}$ and $\tilde{\nu} = \tilde{\nu}_{2\varepsilon}$. □

Proof of Theorem 4.2: Analogous to the first part of the proof of Theorem 3.2, again using the results from [19, Section 4]. □

Remark 5.3 The construction of a continuous V using the trick from the second part of the proof of Theorem 3.2 does not work here, because in the cISDS case it cannot be guaranteed that the inequality in the definition of V is satisfied for all $t > 0$ sufficiently large.

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