Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization

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Chapter 1

Dynamics, Perturbation and Discretization

If anything can go wrong, it will go wrong.

Murphy's Law

Many people—and among them many scientists and mathematicians—will certainly agree to this well known saying. In fact, a constant scepticism towards the things that one expects to be true is probably one of the important driving forces in any kind of scientific development. In contrast to this slightly pessimistic attitude, when turning on a computer many people—and again among them many scientists and mathematicians—are willing to believe in whatever the machine tells them to be true.

It was only several years after the first observations of complicated dynamical behavior by means of numerical methods (like, e.g., the famous discovery of the Lorenz attractor [90]) that mathematicians started to ask whether the basic qualitative features of dynamical systems are correctly represented by numerical approximations. Fortunately, during the last two decades this question has been recognized as an important problem and many contributions have been made during this time. Dynamical objects for which the discretization and approximation behavior have been investigated are, for instance, invariant manifolds, (Beyn [10], Beyn and Lorenz [12], Lorenz [91], Zou and Beyn [128]), homoclinic orbits (Beyn [9], Fiedler and Scheurle [37]), attracting sets and attractors (Kloeden and Lorenz [77, 78], Lorenz [91], Garay and Kloeden [42]) and Morse–Smale systems (Garay [38, 39, 40, 41]). In addition, several survey articles (e.g., by Beyn [11] or by Stuart [111]) and monographs (like the one by Stuart and Humphries [113]) have been published and a number of specialized algorithms has been designed like, for instance, subdivision techniques for the computation of attractors, unstable manifolds and invariant measures by Dellnitz and Hohmann [29], Dellnitz and Junge [30] and Junge [68, 69] or methods for the computation of reachable sets and domains of attraction, see, e.g., Häckl [59, 60], Abu Hassan and Storey [1] or Genesio, Tartaglia and Vicino [43]. Of course, this list of references and topics is far from complete and can only give a short impression about which dynamical features have been addressed.

In this monograph we want to investigate several aspects of long time or asymptotic behavior under numerical discretization. More precisely, we want to consider asymptotically stable attracting sets, attractors and their respective domains of attraction. We will do this not only for classical dynamical systems (as induced, e.g., by the solutions of an autonomous ordinary differential equation), but also for systems with inputs, i.e., control systems or systems subject to some perturbation, for which these "asymptotic objects" can be generalized in a natural way. We will investigate several techniques which on the one hand allow us to conclude convergence (and related convergence rates) of the numerical approximations of these sets and on the other hand help identifying the cases in which convergence does not hold. Thus, in the context of Murphy's Law, the main intention of this book is to give a number of reasons why things do *not* go wrong, even if they could, and try to explain how we can tell the situations where things go wrong from those where things go well.

1.1 Starting Point

The result which can be considered as the starting point of our investigations was published in 1986 by Kloeden and Lorenz [77]. It states that if an ordinary differential equation has a compact attracting set A then any reasonable numerical one–step approximation (or, more precisely, the discrete time dynamical system induced by this discretization) with sufficiently small time step $h > 0$ has a nearby attracting set A_h which converges to A in the Hausdorff metric as the time step h tends to 0. One of the key contributions of this result is that it provides the right setting for obtaining such a general convergence statement. The crucial observation is that one has to formulate this result for the right definition of *attracting sets*, which here are chosen to be compact forward invariant sets A which uniformly attract a neighborhood $B \supset A$ under the respective dynamical system.

The following simple example (which is a slight modification of Example (0.12) in Garay and Kloeden [42]) illustrates this result and also shows what can go wrong even though we have convergence of attracting sets.

Example 1.1.1 Consider the two–dimensional ordinary differential equation given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x - \max\{\|x\| - 1, 0\}x$$

for $x = (x_1, x_2)^T \in \mathbb{R}^2$. □

Figure 1.1 shows, from left to right, two solutions of the original system, of its (explicit) Euler discretization (with time step $h = 1/2$) and of its implicit Euler discretization (with time step $h = 1/2$), respectively. The initial values for these solutions are $x_0' = (0, 2)$ and $x_0'' = (0, 1/2)$ and the solutions are computed for $t \in [0, 20]$. In addition, in the first two figures the shaded regions show the minimal attracting sets A and A_h.

It is easily seen that for the original system each disc

$$D_a := \{x \in \mathbb{R}^2 \mid \|x\| \le a\}$$

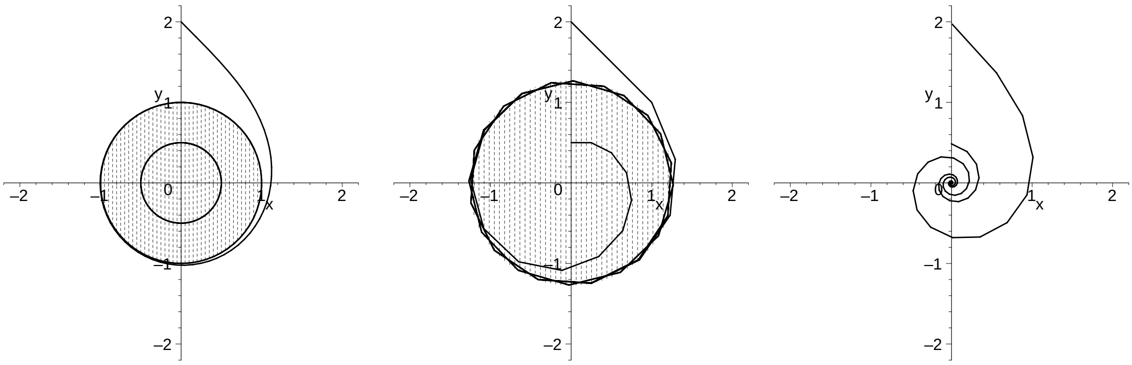

Figure 1.1: Exact, explicit and implicit Euler solutions of Example 1.1.1

with $a \geq 1$ is an attracting set, while for the Euler discretization each set D_a with $a \in [(1+h-\sqrt{1-h^2})/h, c(h)]$ is an attracting set, where $c(h)$ is a constant tending to infinity as h tends to 0. For the implicit Euler discretization it turns out that each set D_a with $a \in [0, c(h)]$ is an attracting set, with $c(h)$ as above. Hence, indeed, for both discretizations there exist attracting sets approaching D_a for each $a \geq 1$.

It is now tempting to try the converse implication: Given a family of attracting sets A_h for the numerical systems which for $h \to 0$ converge to some compact set $\tilde{A}$ in the Hausdorff metric, can we say that this set $\tilde{A}$ is an attracting set for the original system? For the explicit Euler discretization this seems to be true, because each sequence of sets $A_h = D_{a_h}$—if convergent—must converge to some D_a with $a \geq 1$ (in fact, this property *is* true, but a formal proof is more complicated since there exist attracting sets which are not discs). In contrast to this, for the implicit Euler scheme this implication is easily seen to be false, since for instance $A_h = \{0\}$ is a sequence of attracting sets which converges to $\tilde{A} = \{0\}$ which is not an attracting set for the original system.

Another way to look at this problem emerges if we consider *attractors* instead of *attracting sets*. Here we define an attractor to be a compact attracting set which in addition is invariant, i.e., which is mapped exactly onto itself under the respective solution map (this implies that the attractor is the minimal closed attracting set with the given attracted neighborhood B, for details see Section 6.2). Now one might ask whether the convergence result of Kloeden and Lorenz remains valid if we replace *attracting sets* by *attractors*. For the attracted neighborhood $B = D_2$ it is easily seen that the original system has the attractor $A = D_1$, the explicit Euler scheme for time step $h > 0$ sufficiently small has the attractor $A_h = D_{a_h}$ with $a_h = (1+h-\sqrt{1-h^2})/h$ and the implicit Euler scheme has the attractor $A_h = \{0\}$ for all time steps $h > 0$ which are sufficiently small. Hence, the stated convergence result does hold for the attractors of the explicit Euler discretization but it does not hold for the attractors of the implicit Euler discretization.

This example gives rise to two central questions which we want to investigate in this book, and which will be answered in Chapter 6:

(i) Given an attracting set for a numerical approximation, which conditions ensure the existence of a nearby attracting set for the original system?
(ii) Given a sequence of "numerical attractors" converging to some compact set, which conditions guarantee that the limiting set is an attractor for the original system?

1.2 Different Approaches

There are basically three ways to obtain statements that tell us about the validity of numerical findings of long time behavior; all of them are used in this monograph. The first approach is to impose suitable conditions on the approximated system (or on the asymptotic object we are interested in), which ensure a faithful numerical approximation and exclude the appearance of numerical artifacts. This approach is closely related to the concept of *structural stability* in the theory of dynamical systems, which, roughly speaking, describes properties of dynamical systems which are robust against small perturbations. Typical examples of this approach are, for instance, the results on the numerical approximation of Morse–Smale systems by Garay [38, 39, 40, 41] and the investigation of gradient systems under discretization as presented in Section 7.7 of the monograph by Stuart and Humphries [113]. We will utilize a condition of this type for the approximation of domains of attraction in Chapter 7.

The second approach is to design algorithms which can be shown to converge to the right objects under no or under very mild conditions on the approximated system. An example for this approach is the subdivision algorithm for the computation of attractors based on a *rigorous discretization* as proposed by Junge [68, 69]. We will investigate this algorithm in Chapter 6 and present a related technique for the computation of domains of attraction in Chapter 7.

Most of the results we will develop here, however, follow a third approach. Instead of imposing conditions on the approximated system or designing clever—but expensive—algorithms we will consider standard methods (like one–step approximations of ordinary differential equations) and formulate *conditions on the behavior of the numerical systems* under which we can ensure convergence of the respective sets or the existence of respective nearby sets for the approximated system. A typical example for this approach in the literature is the study of the behavior of attracting sets in the Galerkin approximation to Navier–Stokes equations by Kloeden [74]. Here we will be able to give a number of conditions on the dynamical behavior of the numerical systems for the existence and convergence of attracting sets, attractors and domains of attraction. A typical statement of this type for one–step approximations is a robustness condition for numerical attracting sets which ensures the existence of a nearby attracting set for the approximated system, cf. Theorem 6.1.3. An example for a convergence result is Theorem 6.2.8, which—among other criteria—shows that a sequence of numerical attractors for vanishing time step converges to some real attractor if and only if we find nearby attracting sets for the numerical system which attract with a rate which is independent of the time step. Due to the fact that convergence occurs for $h \to 0$ we believe that in a general setting this is the strongest result one can obtain, i.e., we do not expect that there is a condition which can be verified using a finite number of time steps only. Of course, we are aware of the fact that a condition for an infinite sequence of vanishing time steps is impossible to check rigorously in practice. Nevertheless, apart from the fact that these results precisely show what we consider to be the principles of convergence of numerically approximated asymptotic objects, there are indeed ways to derive justified heuristic criteria for numerical approximations, cf. Remark 6.1.5.

While all of these approaches have their own advantages and disadvantages, there are a number of reasons why we believe this last approach to be particularly useful. For example, it applies to standard schemes which are implemented in most scientific software packages. Even though sophisticated algorithms are now available for many problems in numerical dynamics, it is a common practice to use standard tools for numerical simulations when one wants to obtain a first impression about what is going on in a system, and clearly it is important to have criteria at hand which facilitate the interpretation of these simulation results. Another reason for which we consider these results to be helpful is the fact that often structural stability conditions are difficult to verify (like, e.g., hyperbolicity) or do not hold for systems coming from real applications. Nevertheless, one might expect that—like in the explicit Euler discretization in Example 1.1.1 and in contrast to Murphy's law—numerical simulations yield reasonable results even for "fragile" objects, and our results allow a precise description of the cases where this is true. Finally, the use of numerical criteria does not exclude the use of structural stability conditions, on the contrary, sometimes these concepts can be efficiently combined as indicated in Remark 6.2.10.

Many of the results which are formulated in this book do not only give qualitative existence or convergence results but also quantitative information about the discretization error. The question behind this is the following: Given a numerical approximation (obtained from a time and/or space discretization) with some local discretization error, what can we say about the global discretizations error, i.e., the distance between the "real" and the "numerical" attracting sets, attractors or domains of attraction? While for finite time approximations the local error (plus some stability condition) directly implies a corresponding global error, the situation is more complicated for objects which are defined via the asymptotic behavior. Nevertheless, it turns out that the local error still determines the global error, however, not directly but in connection with a suitably defined *robustness gain* for the set which we want to approximate. Although in general these gains are not available explicitly from the systems equation, we can show that they always exist and that for an attracting set they are strongly related to the rate of attraction to this set, cf. Theorems 3.4.6 and 4.4.5.

1.3 Basic Idea

The basic idea we will use for the development of our results is to interpret the numerical approximation as a perturbation of the approximated system, and, vice versa, to interpret the approximated system as a perturbation of the numerical approximation. This classical technique from numerical analysis allows us to use abstract results about perturbed dynamical systems which we will develop for this purpose.

The main principle we are going to use for the treatment of perturbed system is adopted from mathematical control theory, namely we will work with a variant of the *input–to–state stability* property. The concept of input–to–state stability was introduced by Sontag [102] and provides a way to characterize the asymptotic behavior of nonlinear systems in the presence of perturbations. It can be considered as a nonlinear generalization of the "finite energy gain" property for linear control systems, where, actually, the term "nonlinear generalization" can be made mathematically precise using suitable nonlinear coordinate transformations, see [57]. Several variants and modifications of this property have been introduced in order to describe various aspects of the asymptotical behavior of nonlinear systems. Here we are going to introduce yet another variant, which is qualitatively equivalent to input–to–state stability (i.e., it describes the same dynamical behavior) but turns out to be more convenient when we want to deduce quantitative estimates, cf. Section 3.2 and Proposition 3.4.4.

In order to use this abstract concept for analyzing the behavior of numerical systems we will then investigate ways to embed numerical systems into suitably perturbed systems. While for internally perturbed systems the internal and the "numerical" perturbation act in the same way, for control systems the control and the perturbation can be considered as opponents. For instance, the control might want to achieve attraction to some set while the perturbation wants to keep the system away from it. This leads to the adoption of ideas from dynamical game theory, namely the use of nonanticipating strategies for modeling the "numerical" perturbations. It turns out that one has to be careful with the definition of "nonanticipation" in order to cover all possible numerical errors, cf. Examples 4.2.4 and 5.3.9.

In all cases we will use a very rich set of perturbation values, which allows the perturbation to act in any possible direction, with the only restriction being on its amplitude. This concept of an *inflated system* enables us to capture all possible numerical errors of a given magnitude without using any further information about their structure. Since most of the abstract perturbation results are formulated for more specific perturbations (i.e., not only for inflated systems) one could well include additional information about the numerical error in its modeling via perturbations. Here, however, we do not follow this idea because we want to consider general purpose numerical schemes without any additional structure. A typical objection against this type of "worst case analysis" is that it usually leads to very conservative results. While this criticism is justified in our case as long as the *quantitative* results are concerned (certainly, a numerical system can by chance or by good reasons perform much better than an inflated system), this does not apply to our *qualitative* results. The reason for this is the basic idea indicated above: Embedding the numerical system into the inflated original system *and* the original system into the inflated numerical system, we are able to use results for inflated systems in order to obtain necessary and sufficient conditions (i.e., equivalence statements), e.g. for the convergence of numerical attractors.

1.4 Outline of the Results

The approach we have just sketched is reflected in the arrangement of the material in the following chapters. After fixing notation and defining the types of systems we are going to consider, we start with the development of a perturbation theory for attracting sets. This part is split into the Chapters 3 and 4, where Chapter 3 is devoted to internally perturbed systems (in which case we speak of *strongly attracting sets*) while Chapter 4 contains the results for control systems (where we speak of *weakly attracting sets*). These two chapters have identical structure, at least as far as the differences between weak and strong attraction permit. We first define the respective concepts of attraction (along with other dynamical properties which will be needed) and then introduce a number of robustness concepts for these sets, i.e., methods to measure how much external perturbations affect the respective attraction properties. For the strongest of these concepts, which we call *input–to–state dynamical stability* we will then give alternative characterizations by means of a geometric criterion and using Lyapunov functions. On the one hand, these characterization are important tools for the application of this abstract concept, on the other hand they allow an exact description of the relation between the different robustness concepts we have introduced. In addition, we can use these characterizations to show that input–to–state dynamical stability is in fact an inherent property of asymptotically stable attracting sets, at least for sufficiently small compact perturbation ranges. We will further provide a stability analysis of these robustness properties, which includes the definition of the important concept of *embedding* systems into each other, and then state a number of results which are valid for inflated systems and go beyond what we could prove for general perturbations. Finally, we investigate the relation between these robustness concepts for continuous and discrete time systems, which will be needed for the interpretation of numerical results, since, in practice, a continuous time system can only be approximated by a discrete time system.

In the next Chapter 5 we will study the relation between numerical discretization and the perturbation concepts from the previous chapters. We present abstract frameworks first for time and then for space discretizations and show how the resulting numerical systems can be embedded into the perturbed systems considered in Chapter 3 and 4. In addition, we discuss a number of numerical schemes for systems without inputs, for internally perturbed systems and for control systems. A great part of this chapter is devoted to the presentation of a systematic development of high–order one–step schemes for systems affine in the input, which were recently proposed by Kloeden and the author in [55], and show how they fit into the abstract framework. Similarly, we discuss space discretization techniques, where particular attention is payed to *rigorous discretization* techniques as developed by Junge [68, Section 2.2] and [69]. In both presentations we will not go into too much implementational details, but restrict ourselves to a description of those main ideas, which we believe to be necessary in order to understand how these schemes work and to show that they are indeed implementable schemes

After all these preparatory investigations, in Chapter 6 we finally come to the presentation of the results on the discretization of attracting sets and attractors. In terms of the robustness properties from Chapter 3 and 4 we give conditions under which the existence of such a set in the approximated system implies the existence of a nearby set in the numerical scheme, and vice versa, which also include quantitative estimates for the distances between these sets. Furthermore, we give several conditions, which ensure that the limit of a sequence of numerical attracting sets (or attractors) is an attracting set (or an attractor) for the approximated system. For attractors we also formulate a sufficient condition on the behavior of the numerical scheme which not only implies convergence but also allows an estimate for the convergence rate. In addition to these results, which apply to general time and space discretizations, we also provide a convergence analysis for the rigorous subdivision algorithm for the computation of attractors from [68, 69], which turns out to be very straightforward using the “right” robustness concept for attractors. Preliminary versions of some of the results in this chapter for systems without inputs (i.e., without control or internal perturbation) have appeared in the papers [50, 51, 54], which were written during the research for this monograph. However, thanks to the systematic development of the abstract perturbation theory the results given here considerably improve these preliminary versions, even for systems without inputs.

The final Chapter 7 then focuses on domains of attraction and reachable sets. It turns out that essentially the same concepts which are used for attracting sets can be used here, because the complement of a domain of attraction is nothing but an attracting set for the time reversed system, provided that the system is reversible in time. Since for discrete time systems this is not necessarily the case we will not directly use this observation but formulate a *dynamical robustness* property for domains of attraction in forward time which is equivalent to the input–to–state dynamical stability for their complements under time reversal. Since we do not want to rephrase all the statements from Chapter 3 and 4 we use a "shortcut" and define this property directly in terms of Lyapunov functions. In order to prove that—similar to attracting sets—this dynamical robustness is an inherent property of domains of attraction we then have to show the existence of a suitable Lyapunov function. For this purpose we use generalizations of what is called Zubov's method for perturbed and for controlled systems, which were recently obtained by Camilli, Wirth and the author [16, 17] and Wirth and the author [58]. After summarizing (and slightly extending) the results from these references we show that the Lyapunov functions obtained by Zubov's method can be used to construct Lyapunov functions characterizing the dynamical robustness property. Having established this result we turn to the analysis of domains of attractions under discretization. Just as limits of numerical attracting sets do not need to be "real" attracting sets, limits of numerical domains of attraction do not need to be "real" domains of attraction. Hence we end up with similar results as those for attracting sets and attractors in Chapter 6 based on conditions for the behavior of the numerical system. In addition, we introduce a structural stability condition for domains of attractions (via the forward invariance of the complement of the domain of attraction) which allows to conclude convergence without imposing conditions on the numerical systems. After these results for general schemes we formulate a subdivision algorithm for the computation of domains of attraction, and show its convergence both without the structural stability condition (provided that the underlying space discretization is rigorous) and with this condition (in this case we can obtain an estimate also for non–rigorous discretizations). Finally, we discuss reachable sets, and show how the results for domains of attraction can be transferred to these sets. In this context we re–investigate the structural stability condition introduced before and show that for reachable sets this condition can be reformulated via chain reachable sets. Hence this condition turns out to be equivalent to a robustness condition well known in the geometric analysis of nonlinear control systems as presented, e.g., in the monograph by Colonius and Kliemann [22].

Two concepts which are used extensively throughout this book are *viscosity solutions* and *comparison functions*. Viscosity solutions are a generalized notion of solutions to partial differential equations and play a vital role in the Lyapunov function characterization of robustness properties in Chapter 3 and 4, as well as for the generalization of Zubov's method for controlled and perturbed systems in Chapter 7. Comparison functions provide an elegant way to formulate robustness, attraction and asymptotic stability properties without using ε–δ formalisms and in addition lead to a natural definition of robustness gains and rates of attraction, for which reason we use them throughout all the Chapters in this monograph. Since these notions might not be well known to all readers we have compiled some elementary background information in the two Appendices A and B. In addition, we use these appendices to formulate and prove several statements about viscosity solutions and comparison functions which did not fit into the other chapters, but are nevertheless needed for the formulation or proofs of some results.

1.5 Open Questions and Future Research

Although this monograph tries to give a self contained treatment of the mentioned problems regarding the asymptotic behavior of systems under perturbation and discretization, it is clear that not all questions arising in this context can be ultimately answered here. Before starting with the development of our results in the next chapter, we therefore want to summarize some open questions and some ideas for future research.

First of all, we believe that the characterization of the robustness of attracting sets by means of Lyapunov functions has not yet reached its final form. Both for internally perturbed and for control systems we are able to prove the existence of *discontinuous* Lyapunov functions, which exactly represent the attraction rate and the robustness gain. While these functions are sufficient for our applications in this book, from a theoretical point of view it is nevertheless interesting to know whether one can find *continuous* Lyapunov functions with the same properties. In the perturbed case we were at least able to show the existence of such functions which *approximately* represent these rates and gains, while in the case of control systems we could not even achieve this result.

Concerning the results on numerical approximations, the probably most important case which is not covered here is the analysis of schemes with adaptive timestepping. Our results only apply to one–step discretizations with fixed time step $h > 0$, but we conjecture that the principles used for these schemes can also be used for the analysis of schemes with step–size control. The reason why we did not include results for this case is that adaptive schemes in general do not induce a standard discrete time dynamical (or control) system. Recently, Kloeden and Schmalfuß [80] and Lamba [85] have proposed different techniques to overcome this difficulty, and we believe that based on these ideas results can be obtained, cf. the Discussion after Definition 5.1.5 in Chapter 5.

Another important issue is the development of an efficient implementation of the subdivision algorithm for the computation of domains of attraction. An first straightforward implementation of this algorithm shows very promising results, cf. Appendix C. We plan to develop such an efficient implementation and hope to be able to present results for more complex systems in the near future.

We have intentionally formulated the perturbation theory in Chapter 3 and 4 in much more generality than needed for our applications to numerical error analysis. Due to this fact we believe that these results are of independent interest and can be used in various different contexts. Certainly, since these results emerge from mathematical control theory, there should be a number of control theoretic applications, in particular for those problems where quantitative results are of interest. An example is the analysis of coupled systems by Jiang, Teel and Praly [67] and Teel [119], where the particular form of the robustness gain decides about stability or instability of the coupled system. In fact, our perturbation analysis is based on conceptionally similar ideas as used in this reference and we believe that many of the results in these references can be recovered and even refined using our approach, see [53, Section 4] for first steps in this direction. Another example is the relation between dynamical and control systems as investigated by Colonius and Kliemann

[22]. One result in this area states that under suitable conditions on a control system one can conclude the existence of control sets (i.e., regions of complete controllability) around chain recurrent attractors of the corresponding uncontrolled system. Our results immediately lead to an estimate about the distance between the attractor and the control set, cf. [51, Section 8.3] for a first result in this direction.

A special type of perturbation occurs if we consider the effect of sampling on the performance of a control system. This effect can be investigated by similar techniques as used for numerical approximations, see, e.g., Nešić, Teel and Kokotović [93]. A particular question in this context is whether asymptotic stability can be achieved by sampled controls with some positive lower bound on the sampling rate, a property which by now could only be verified for homogeneous systems [49, 52]. It seems reasonable to expect that the perturbation theory used in this book can provide some new insights into this class of problems.

Apart from control theoretic applications, also for the analysis of numerical errors one could used more sophisticated perturbation models than simple inflated systems. For instance, it might be possible to figure out more restrictive classes of perturbations which still suffice to capture the error caused by specialized numerical schemes like, e.g., energy preserving schemes or symplectic Runge–Kutta methods.

Finally, discretization effects do not only occur in the approximation of finite dimensional systems. It would be a challenging project to investigate which parts of the perturbation theory can be carried over to infinite dimensional systems in order to analyze the behavior of partial differential equations under finite difference, finite element or other kinds of discretizations.

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