Quantitative Aspects of the Input–To–State–Stability Property

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1 Introduction

Since its introduction by Sontag [17] in 1989, the input–to–state stability (ISS) property has become one of the most influential concepts in nonlinear stability theory for perturbed systems. The property generalizes the well known asymptotic stability property by assuming that each trajectory φ of a perturbed nonlinear system with time–varying perturbation $u(t)$ satisfies the inequality

$$\|\varphi(t,x,u)\| \leq \{\beta(\|x\|,t),\, \rho(\|u\|_\infty)\}$$

for suitable functions β of class $\mathcal{KL}$ and ρ of class $\mathcal{K}_\infty$.

The fact that this concept was used by many authors (see, e.g., [1, 4, 9, 10, 12, 14, 16, 22]) is mainly due to the intuitive simplicity of the concept, which captures the qualitative essence of robust asymptotic stability in a truly nonlinear manner, for details and the relation to other nonlinear robust stability concepts see, e.g., [8, 18, 21] and the survey [20]. On the other hand, the use of the comparison functions β and ρ in its formulation immediately leads to the idea to explicitly use the *quantitative* information contained in the ISS inequality, i.e., the rate of convergence β and the robustness gain ρ, with one of the most prominent applications being the nonlinear small gain theorem [10], for which the quantitative information contained in ρ is crucial.

In this quantitative context, however, it turns out that the original ISS formulation has some drawbacks, which are caused by the fact that it does not yield explicit information about what happens for vanishing perturbations, i.e., for perturbations u with $u(t) \to 0$ as $t \to \infty$. Implicitly, ISS ensures that if $u(t)$ tends to 0 as t tends to infinity then also $\varphi(t,x,u)$ converges to 0 for t tending to infinity, but no explicit rate of convergence can be deduced. The main idea in order to overcome this difficulty is by introducing a certain "memory fading" effect into the u–term of the ISS formulation, an idea which was used before by Praly and Wang [14] in their notion of exp–ISS. There the perturbation is first fed into a one–dimensional control system whose output then enters the right hand side of the ISS estimate. Here, instead, we use the value of the perturbation at each time instant as an initial value of a one–dimensional dynamical system, which leads to the concept of *input–to–state dynamical stability* (ISDS). Proceeding this way, we are in particular able to "synchronize" the effects of past disturbances and large initial values by using the same dynamical system for both terms. It turns out that ISDS is qualitatively equivalent to ISS and, in addition, that we can pass from ISS to ISDS with only slightly larger robustness gains.

One of the most important features of the ISS property is that it can be characterized by a dissipation inequality using a so called ISS Lyapunov function, see [21]. One of the central properties of the ISDS estimate is that it admits an ISDS Lyapunov function, which not only characterizes ISDS as a qualitative property (the qualitative equivalence ISS $\Leftrightarrow$ ISDS immediately implies that the well known ISS Lyapunov function would be sufficient for this) but also represents the respective decay rate, the overshoot gain and the robustness gain. The respective results are given in Section 4.

Certainly, there are many applications where quantitative robust stability properties are of interest. A particular area of applications are numerical investigations, where one interprets a numerical approximation as a perturbation of the original system and vice versa. One example is given in Section 5, for a comprehensive treatment of this subject we refer to the monograph [5]. In Section 6 we present two control theoretic applications of the ISDS property, which also illustrate the difference to the ISS property.

2 Motivation

In order to explain and motivate our approach, in this section we briefly recall some classical results for systems without input, i.e., for nonlinear autonomous differential equations of the type

$$\dot{x}(t) = f(x(t)) \tag{1}$$

with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. The solutions of (1) for initial value $x \in \mathbb{R}^n$ at initial time $t = 0$ will be denoted by $\varphi(t, x)$. If we assume that the origin is globally asymptotically stable for (1), then it is well known that there exists a Lyapunov Function $V : \mathbb{R}^n \to \mathbb{R}$, i.e., a positive definite and proper function, which is C^∞ on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$DV(x) \cdot f(x) < 0 \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

By suitable rescaling of V we may assume that there exists a class $\mathcal{K}_\infty$–function σ (see Section 3 for a definition), such that the inequalities

$$\|x\| \le V(x) \le \sigma(\|x\|) \tag{2}$$

hold. Furthermore, it is easily seen that there exists a continuous function $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $g(r) > 0$ for $r > 0$ such that the inequality

$$DV(x) \cdot f(x) < -g(V(x)) \text{ for all } x \in \mathbb{R}^n \setminus \{0\} \tag{3}$$

holds.

Integrating inequality (3) and using (2) then yields the estimate

$$\|\varphi(t,x)\| \leq \mu(\sigma(\|x\|), t) \text{ for all } x \in \mathbb{R}^n,\ t \geq 0, \tag{4}$$

where μ is the solution of the 1d differential equation

$$\frac{d}{dt}\mu(r,t) = -g(\mu(r,t)), \quad \mu(r,0) = r.$$

This means that we get a special type of $\mathcal{KL}$–estimate for the norm of the solution trajectories $\varphi(t,x)$, which in turn implies global asymptotic stability.

Now the nice property of an inequality of type (4) is that it admits a converse Lyapunov theorem using a construction of Yoshisawa [23]. If we assume (4) and set

$$V(x) := \sup_{t\geq 0} \mu(\|\varphi(t,x)\|, -t),$$

then this function satisfies (2) and

$$V(\varphi(t,x)) \leq \mu(V(x), t) \text{ for all } x \in \mathbb{R}^n,\ t \geq 0,$$

from which we can in turn conclude (4). This function V, however, may be discontinuous, thus we cannot conclude (3). In order to obtain a smooth function we can fix an arbitrary $\varepsilon > 0$ and set

$$V(x) := \sup_{t\geq 0} \mu(\|\varphi(t,x)\|, -(1-\varepsilon)t).$$

This function is Lipschitz continuous and satisfies (2). It also satisfies (3) with $(1-\varepsilon)g$ instead of g in a nonsmooth sense, more precisely in the sense of viscosity supersolutions. Hence by an appropriate smoothing technique (see, e.g., [13]) we can obtain a smooth function (at least away from 0) satisfying

$$\|x\| \leq V(x) \leq (1+\varepsilon)\sigma(\|x\|)$$

and

$$DV(x) \cdot f(x) < -(1-2\varepsilon)g(V(x)) \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Thus, the particular form of the decay estimate (4) allows a converse Lyapunov theorem, which preserves the decay rate $\mu(\sigma(r), t)$ up to an arbitrarily small $\varepsilon > 0$.

Our aim in this paper is to generalize this approach to the ISS property, i.e.,

- formulate a suitable variant of ISS similar to (4), which leads to the ISDS property
- find a Lyapunov function which implies ISDS
- prove a converse Lyapunov theorem which preserves the rate and gains at least up to some arbitrarily small parameter $\varepsilon > 0$.

3 Input–to–state dynamical stability

We consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t), u(t)), \tag{5}$$

where we assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and that for each two compact subsets $K \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ there exists a constant $L = L(K, W)$ such that $\|f(x, u) - f(y, u)\| \le L\|x - y\|$ for all $x, y \in K$ and all $u \in W$. The perturbation functions u are supposed to lie in the space $\mathcal{U}$ of measurable and locally essentially bounded functions with values in U, where U is an arbitrary subset of $\mathbb{R}^m$. The trajectories of (5) with initial value x at time $t = 0$ are denoted by $\varphi(t, x, u)$.

We recall that a continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called of class $\mathcal{K}$ if it is strictly increasing with $\alpha(0) = 0$, and is called of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called of class $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first and strictly decreasing to 0 in the second argument. We define a continuous function $\mu : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ to be of class $\mathcal{KLD}$ if its restriction to $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ is of class $\mathcal{KL}$ and, in addition, it is a one dimensional dynamical system, i.e., it satisfies

$$\mu(r, t + s) = \mu(\mu(r, t), s) \text{ for all } t, s \in \mathbb{R}.$$

Observe that this condition implies $\mu(r, 0) = r$.

The expression $\| \cdot \|$ denotes the usual euclidean norm, $\|u\|_\infty$ is the L_∞ norm of $u \in \mathcal{U}$ and for $t > 0$ and any measurable function $g : \mathbb{R} \to \mathbb{R}_0^+$ the expression $\operatorname{ess\,sup}_{\tau \in [0,t]} g(\tau)$ denotes the essential supremum of g on $[0, t]$.

Using these notations we can now formulate the concept of input–to-state dynamical stability.

Definition 1. *System* (5) *is called* input-to-state dynamically stable *(ISDS), if there exists a function μ of class $\mathcal{KLD}$ and functions σ and γ of class $\mathcal{K}_\infty$ such that the inequality*

$$\|\varphi(t, x, u)\| \le \max\{\mu(\sigma(\|x\|), t),\ \nu(u, t)\}.$$

holds for all $t \ge 0$, $x \in \mathbb{R}^n$ and all $u \in \mathcal{U}$, where ν is defined by

$$\nu(u, t) := \operatorname{ess\,sup}_{\tau \in [0,t]}\ \mu(\gamma(\|u(\tau)\|), t - \tau) \tag{6}$$

Here we call the function μ the decay rate*, the function σ the* overshoot gain *and the function γ the* robustness gain.

Since $\mu(\sigma(r), t)$ is of class $\mathcal{KL}$ and $\nu(u, t) \le \gamma(\|u\|_\infty)$, ISDS implies ISS with $\beta(r, t) := \mu(\sigma(r), t)$ and robustness gain $\rho = \gamma$.

Conversely, a straightforward application of [19, Proposition 7] shows that any class $\mathcal{KL}$ function can be bounded from above by the composition of a class $\mathcal{KLD}$ and a class $\mathcal{K}_\infty$ function, see [5, Lemma B.1.4]. Hence the only real difference between ISS and ISDS is the decay property of the $\nu(u,t)$–term. The following theorem shows how one can pass from the ISS to the ISDS formulation. For the proof see [5, Proposition 3.4.4].

Theorem 1. *Assume that the system* (5) *is ISS for some* β *of class* $\mathcal{KL}$ *and* ρ *of class* $\mathcal{K}_\infty$*. Then for any class* $\mathcal{K}_\infty$ *function* γ *with* $\gamma(r) > \rho(r)$ *for all* $r > 0$ *there exists a class* $\mathcal{KLD}$ *function* μ *such that the system is ISDS with attraction rate* μ*, overshoot gain* $\sigma(r) = \beta(r,0)$ *and robustness gain* γ*.*

For some results in this paper we will need the following assumption.

Assumption 2 *The functions* μ*,* σ *and* γ *in Definition 1 are* C^∞ *on* $\mathbb{R}^+ \times \mathbb{R}$ *or* $\mathbb{R}^+$*, respectively, and the function* μ *solves the ordinary differential equation*

$$\frac{d}{dt}\mu(r,t) = -g(\mu(r,t))$$

for some locally Lipschitz continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$*, all* $r > 0$ *and all* $t \in \mathbb{R}$*.*

It was shown in [5, Appendix A] that for given nonsmooth rates and gains from Definition 1 one can find rates and gains arbitrarily close to the original ones, such that Assumption 2 holds and Definition 1 remains valid. Hence Assumption 2 is only a mild regularity condition.

It should be noted that the ISDS formulation and many of its properties can be generalized to arbitrary compact attracting sets, local attraction and systems with additional perturbation input; some results also carry over to systems with control input, leading to the definition of weak or controlled ISDS (wISDS/cISDS), see [5] for an extensive discussion and [6] for a shorter overview. Here, in order to make this short presentation concise, we stay with the basic ISDS formulation as given in Definition 1.

4 Lyapunov function characterization

One of the main tools for working with ISS systems is the ISS Lyapunov function whose existence is a necessary and sufficient condition for the ISS property, see [21]. In this section we provide two theorems on a Lyapunov function characterization of the ISDS property. We start with a version for discontinuous Lyapunov functions, which can exactly represent the rate and gains in the ISDS formulation. The proof of the following theorem is given in Section 7.

Theorem 3. *A system* (5) *is ISDS with rate* μ *of class* $\mathcal{KLD}$ *and gains* σ *and* γ *of class* $\mathcal{K}_\infty$ *if and only if there exists a (possibly discontinuous)* ISDS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ *satisfying*

$$\|x\| \le V(x) \le \sigma(\|x\|) \tag{7}$$

and

$$V(\varphi(t,x,u)) \le \max\{\mu(V(x),t),\, \nu(u,t)\} \tag{8}$$

for all $x \in \mathbb{R}^n$*,* $t \ge 0$ *and all* $u \in \mathcal{U}$*, where* ν *is given by* (6)*.*

For many applications it might be desirable to have ISDS Lyapunov functions with some more regularity. The next theorem, which is also proved in Section 7, shows that if we slightly relax the sharp representation of the gains, then we can always find smooth (i.e., C^∞) Lyapunov functions, at least away from the origin.

Theorem 4. *A system* (5) *is ISDS with rate* μ *of class* $\mathcal{KLD}$ *and gains* σ *and* γ *of class* $\mathcal{K}_\infty$ *satisfying Assumption 2 if and only if for each* $\varepsilon > 0$ *there exists a continuous function* $V : \mathbb{R}^n \to \mathbb{R}_0^+$ *which is smooth on* $\mathbb{R}^n \setminus \{0\}$ *and satisfies*

$$\|x\|/(1+\varepsilon) \le V(x) \le \sigma(\|x\|) \tag{9}$$

and

$$\gamma(\|u\|) < V(x) \quad \Rightarrow \quad DV(x) \cdot f(x,u) \le -(1-\varepsilon) g(V(x)) \tag{10}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ *and all* $u \in U$*.*

It should be noted that there exists an intermediate object between the discontinuous and the smooth ISDS Lyapunov function, namely a Lipschitz Lyapunov function which satisfies (10) in a suitable generalized sense using the theory of viscosity solutions, see [5] for details. While both smooth and Lipschitz Lyapunov functions characterize the optimal gains "in the limit", we conjecture that there are examples in which gains can be exactly characterized by Lipschitz but not by smooth ISDS Lyapunov functions, similar to what was shown recently for H_∞ Lyapunov functions in [15].

Theorem 4 gives rise to a constructive procedure of computing ISDS robustness gains from Lyapunov functions for the unperturbed system $f(x,0)$. We illustrate this procedure by three examples.

Example 1. Consider a linear system $\dot{x} = f(x,u) = Ax + Bu$. If we assume ISDS then the matrix A needs to be Hurwitz and we can find a quadratic Lyapunov function $W(x) = x^T P x$ for some positive definite matrix P satisfying $c_1\|x\|^2 \le W(x) \le c_2\|x\|^2$ and $DW(x)Ax \le -c_3\|x\|^2$. Setting $V(x) = \sqrt{W(x)/c_1}$ we obtain $\|x\| \le V(x) \le c_4\|x\|$, $DV(x)Ax \le -c_5 V(x)$ and $\|DV(x)\| \le c_4$ for $c_4 = \sqrt{c_2/c_1}$ and $c_5 = c_3/(2c_2)$. Fixing some $\lambda \in (0,1)$ we set $\gamma(r) = c_4\|B\|r/(\lambda c_5)$. Then we obtain

$$\gamma(\|u\|) \le V(x) \quad \Rightarrow \quad DV(x) \cdot f(x,u) \le -(1-\lambda)c_5 V(x) =: -g(V(x)).$$

Hence V is an ISDS Lyapunov function in the sense of Theorem 4 (for each $\varepsilon > 0$) and we obtain ISDS with $\mu(r,t) = e^{-(1-\lambda)c_5 t} r$, $\sigma(r) = c_4 r$ and $\gamma(r) = c_4\|B\|r/(\lambda c_5)$, i.e., exponential convergence and linear overshoot and robustness gains.

This example nicely illustrates the (typical) tradeoff between the attraction rate μ and the robustness gain γ, which is represented here by the choice of λ: the smaller γ becomes the slower convergence can be guaranteed. In the next two examples, showing ISDS estimates for two simple nonlinear systems, we set $\lambda = 3/4$.

Example 2. Consider the system $\dot{x} = f(x,u) = -x + u^3/2$ with $x \in \mathbb{R}$, $u \in \mathbb{R}$. Using the Lyapunov function $V(x) = |x|$ one obtains $DV(x)f(x,0) = -|x| = -V(x)$. We choose γ such that $\gamma(|u|) \le V(x) = |x|$ implies $|u^3/2| \le 3|x|/4$, i.e., $\gamma(r) = 2r^3/3$. Then we obtain

$$\gamma(\|u\|) \le V(x) \quad \Rightarrow \quad DV(x) \cdot f(x,u) \le -\frac{1}{4}V(x) =: -g(V(x)),$$

and consequently ISDS with $\mu(r,t) = e^{-t/4}r$, $\sigma(r) = r$ and $\gamma(r) = 2r^3/3$.

Example 3. Consider the system $\dot{x} = f(x,u) = -x^3 + u$ with $x \in \mathbb{R}$, $u \in \mathbb{R}$. Again using the Lyapunov function $V(x) = |x|$ one obtains $DV(x)f(x,0) = -|x|^3 = -V(x)^3$. Here we choose γ such that $\gamma(|u|) \le V(x) = |x|$ implies $|u| \le 3|x|^3/4$, i.e., $\gamma(r) = \sqrt[3]{4r/3}$. Then we obtain

$$\gamma(\|u\|) \le V(x) \quad \Rightarrow \quad DV(x) \cdot f(x,u) \le -\frac{1}{4}V(x)^3 =: -g(V(x)),$$

and consequently ISDS with $\mu(r,t) = \sqrt{2t + 4/r^2}/(t + 2/r^2)$ (the solution of $\dot{\mu} = -\mu^3/4$), $\sigma(r) = r$ and $\gamma(r) = \sqrt[3]{4r/3}$.

5 Applications in Numerical Analysis

In order to illustrate the way in which ISDS–like properties can be used in numerical analysis, we consider a problem from numerical dynamics. We briefly describe an algorithm for the computation of attractors developed by Dellnitz and Hohmann [2]; here we describe a version due to Junge [11].

Consider the differential equation (1) and its time–1 map $\Phi(x) := \varphi(1,x)$. Consider a rectangular domain $\Omega \subset \mathbb{R}^n$ and a partition $\mathcal{C}^0$ of Ω into N^0 rectangular cells $\mathcal{C}^0 = \{C_1^0, C_2^0, \ldots, C_{N^0}^0\}$.

Setting $k = 0$ we compute a collection of cells $\widetilde{\mathcal{C}}^k \subset \mathcal{C}^k$ by defining

$$\widetilde{\mathcal{C}}^k := \{C_i^k \in \mathcal{C}^k : \text{there exists } C \in \mathcal{C}^k with \Phi(C) \cap C_i^k \neq \emptyset\}. \tag{11}$$

For simplicity we assume here that $\Phi(C)$ can be computed, which will not be the case in general, cf. Remark 1 (ii), below. In the next step each cell contained in $\widetilde{\mathcal{C}}^k$ is refined (e.g., by subdividing it into a number of finer rectangles) and the resulting collection of cells is denoted by $\mathcal{C}^{k+1}$. Now we set $k = k + 1$ and restart this procedure by going to step (11).

This generates a sequence of collections $\mathcal{C}^k$, $k = 0, 1, \ldots$, satisfying $\bigcup_i C_i^{k+1} \subseteq \bigcup_i C_i^k$. Now let $A \subset \Omega$ be an attractor, i.e., a minimal asymptotically stable set which attracts $\Omega \setminus \{A\}$. Then it is known that the convergence

$$d_H(C^k, A) \to 0$$

holds in the Hausdorff metric d_H for compact sets, however, estimates for the corresponding rate of convergence are dificult to obtain.

Such estimates can be derived from the ISDS property. Consider the perturbed system

$$\dot{x}(t) = f(x(t)) + u(t) \tag{12}$$

with solution trajectories $\varphi(t, x, u)$ and assume that the attractor A has the ISDS property, i.e.,

$$\|\varphi(t, x, u)\|_A \leq \max\{\mu(\sigma(\|x\|_A), t),\, \nu(u, t)\},$$

where $\|x\|_A := \inf_{y \in A} \|x - y\|$. Let

$$\operatorname{diam}(\mathcal{C}^k) := \max_{i=1,\ldots,N^k} \max_{x,y \in C_i^k} \|x - y\|$$

be the maximal diameter of the cells in $\mathcal{C}^k$. Then we obtain the estimate

$$d_H(C^k, A) \leq \max\left\{\mu(\sigma(d_H(\Omega, A), k),\, \max_{j=0,\ldots,k-1} \mu(\gamma(\operatorname{diam}(\mathcal{C}^{k-j-1})), j)\right\}. \tag{13}$$

For a proof of this estimate see [5, Theorem 6.3.3].

Remark 1. (i) In fact, for this estimate to hold we only need that the ISDS estimate is valid for $x \in \Omega$. It can be shown that any asymptotically stable set for the unperturbed system (1) for which Ω lies in its domain of attraction has this "local" ISDS property for the perturbed system (12) for suitable μ, σ and γ and suitable perturbation range U, see [5, Theorem 3.4.6]. Hence estimate (13) holds for all attractors without any additional asumptions for suitable functions μ, σ and γ.
(ii) It is possible to incorporate numerical errors in the computation of the image $\Phi(C)$ in (11) in the analysis of the algoroithm. We refer to [5, Section 6.3] for details.

6 Applications in Control Theory

As a first application, we derive an estimate on a nonlinear stability margin. In [21] it was shown that ISS implies the existence of a stability margin for a perturbed system, however, for ISS it is difficult to derive an estimate for this margin. In contrast to this, the ISDS property easily allows to give an estimate based on the ISDS robustness gain.

Theorem 5. *Consider a system* (5) *and assume ISDS with* μ, σ *and* γ *and* $U = \mathbb{R}^m$, *satisfying Assumption 2. Consider a Lipschitz map* $k : \mathbb{R}^n \to \mathbb{R}_0^+$ *satisfying* $k(x) \le \max\{\gamma^{-1}(\|x\|),\, k_0\}$ *for some value* $k_0 \ge 0$. *Then for each* $x \in \mathbb{R}^n$ *and all* $u \in \mathcal{U}$ *with* $\|u\|_\infty \le 1$ *the trajectories* $\varphi_k(t,x,u)$ *of the system* $\dot{x} = f_k(x,u) := f(x, k(x)u)$ *satisfy*

$$\|\varphi_k(t,x,u)\| \le \max\{\mu(\sigma(\|x\|),t), \gamma(k_0)\}$$

for all $t \ge 0$.

The proof can be found in [7].

As a second application we consider the stability of coupled systems. The following theorem is a version of the generalized small gain theorem [10, Theorem 2.1] (in a simplified setting). As for Theorem 5, the qualitative result (i.e., asymptotic stability of the coupled system) can be proved using the original ISS property. The advantage of ISDS lies in the estimates for the overshoot and the decay rates of the coupled system.

Theorem 6. *Consider two systems* $\dot{x}_i = f(x_i, u_i)$, $i = 1, 2$, *of type* (5) *where the* f_i *are Lipschitz in both* x_i *and* u_i. *Let* $x_i \in \mathbb{R}^{n_i}$, $U_1 = \mathbb{R}^{n_2}$ *and* $U_2 = \mathbb{R}^{n_1}$. *Assume that the systems are ISDS with rates* μ_i *and gains* σ_i *and* γ_i *and assume that the inequalities* $\gamma_1(\gamma_2(r)) \le r$ *and* $\gamma_2(\gamma_1(r)) \le r$ *hold for all* $r > 0$. *Then the coupled system*

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_2, x_1) \tag{14}$$

is globally asymptotically stable and the trajectories $(x_1(t), x_2(t))$ *of* (14) *satisfy*

$$\|x_i(t)\| \le \delta_i\Big(\max\{\sigma_i(\|x_i(0)\|),\, \gamma_i(\sigma_j(\|x_j(0)\|))\},\, t\Big) \tag{15}$$

for $i = 1, 2$, $j = 3 - i$ *and functions* δ_i *given by*

$$\delta_i(r,t) := \sup\left\{ \theta_i^{t_1,s_1} \circ \ldots \circ \theta_i^{t_k,s_k}(r) \;\middle|\; k \ge 1,\, t_j, s_j \ge 0,\, \sum_{j=1}^{k} t_j + s_j = t \right\}$$

with $\theta_1^{t,s}(r) := \mu_1(\gamma_1(\mu_2(\gamma_1^{-1}(r), s)), t)$ *and* $\theta_2^{t,s}(r) := \mu_2(\gamma_2(\mu_1(\gamma_2^{-1}(r), s)), t)$. *In particular, for all* $t \ge 0$ *from* (15) *we obtain the overshoot estimates*

$$\|x_i(t)\| \le \max\{\sigma_i(\|x_i(0)\|), \gamma_i(\sigma_j(\|x_j(0)\|))\}.$$

Again, the proof can be found in [7].

Remark 2. A different characterization of the decay rates δ_i in Theorem 6 can be obtained if we assume that the gains γ_i and the class $\mathcal{KLD}$ functions μ_i satisfy Assumption 2 for functions g_i. In this case, derivating the expressions in the definition of $\delta_i(r,t)$, $i = 1, 2$, with respect to t, one sees that the δ_i are bounded from above by the solutions of the one–dimensional differential equations $\dot{r}_i = \max\{-g_i(r_i),\, -\gamma_i'(\gamma_i^{-1}(r_i))g_j(\gamma_i^{-1}(r_i))\}$, $r_i(r,0) = r$, where γ_i' denotes the derivative of γ_i and $j = 3 - i$.

In the following example we illustrate the quantitative information one can obtain from Theorem 6 and Remark 2.

Example 4. Consider the two systems from Examples 2 and 3 with robustness gains $\gamma_1(r) = 2r^3/3$ and $\gamma_2(r) = \sqrt[3]{4r/3}$. Then the coupled system reads $\dot{x}_1(t) = -x_1(t) + x_2(t)^3/2$, $\dot{x}_2(t) = -x_2(t)^3 + x_1(t)$. One verifies that the gain condition of Theorem 6 is satisfied, hence we can conclude asymptotic stability with overshoot estimates

$$\|x_1(t)\| \le \max\{\|x_1(0)\|, 2\|x_2(0)\|^3/3\},$$
$$\|x_2(t)\| \le \max\{\|x_2(0)\|, \sqrt[3]{4\|x_1(0)\|/3}\}.$$

Using the formula from Remark 2 we obtain

$$\dot{r}_1 = \max\{-c_1 r_1, -c_2 r_1^{\frac{5}{3}}\}, \quad \dot{r}_2 = \max\{-c_3 r_2^3, -c_4 r_2\}$$

for suitable constants $c_1, \ldots, c_4 > 0$. This shows that far away from the equilibrium exponential convergence can be expected, while in a neighborhood of 0 the rates of convergence in both components will slow down.

7 Proofs of the main results from Section 4

The following Lemma will be crucial for all our proofs.

Lemma 1. *Consider a (possibly discontinuous) function $V : \mathbb{R}^n \to \mathbb{R}_0^+$. Then the following two statements are equivalent*

(i) $V(\varphi(t,x,u)) \le \max\{\mu(V(x),t), \nu(u,\tau)\}$ for all $t \ge 0$ and all $u \in \mathcal{U}$.
(ii)$V(\varphi(t,x,u)) \le \mu(a,t)$ for all times $t \ge 0$, all values $a \in \mathbb{R}$ with $a \ge V(x)$ and all $u \in \mathcal{U}$ satisfying $\gamma(\|u(\tau))\|) \le \mu(a,\tau)$ for almost all $\tau \in [0,t]$.

Proof: "(i) ⇒ (ii)": The definition of ν immediately implies $\nu(u,t) \le \mu(a,t)$ for t, a and u satisfying the assumptions from (ii), hence (i) implies (ii).
"(ii) ⇒ (i)": Consider an arbitrary $u \in \mathcal{U}$ and $t > 0$. We set $a = \max\{V(x), \mu(\nu(u,t),-t)\}$ which implies $\gamma(\|u(\tau))\|) \le \mu(a,\tau)$ for almost all $\tau \in [0,t]$. Now either $a = V(x)$ or $\mu(a,t) = \nu(u,t)$ holds. In the first case we obtain $V(\varphi(t,x,u)) \le \mu(a,t) = \mu(V(x),t)$ while in the second case we have $V(\varphi(t,x,u)) \le \mu(a,t) = \nu(u,t)$. Thus we can conclude (i).

Now we can turn to the **Proof of Theorem 3:**
"(i) ⇒ (ii)" We construct a function for which Lemma 1(ii) can be verified. We define

$$V(x) := \inf\{b \ge 0 \mid \|\varphi(t,x,u)\| \le \max\{\mu(b,t), \nu(u,t)\} \text{ for all } u \in \mathcal{U},\, t \ge 0\}.$$

Clearly, the ISDS assumption implies $\|x\| \le V(x) \le \sigma(\|x\|)$. It remains to show Lemma 1(ii). To this end, fix $x \in \mathbb{R}^n$, $a \ge V(x)$, $t > 0$ and $u \in \mathcal{U}$

with $\gamma(\|u(\tau)\|) \le \mu(a,\tau)$ for almost all $\tau \in [0,t]$. This implies $\nu(u, t+s) \le \max\{\mu(\mu(a,t),s),\, \nu(u(t+\cdot),s)\}$ for each $s > 0$, thus by the definition of V for any $b > a$ we obtain

$$\|\varphi(t{+}s,x,u)\| \le \max\{\mu(b,t{+}s),\, \nu(u,t{+}s)\} \le \max\{\mu(\mu(b,t),s),\, \nu(u(t{+}\cdot),s)\}$$

which implies $V(\varphi(t,x,u)) \le \mu(a,t)$ and thus Lemma 1(ii).
"(ii) $\Rightarrow$ (i)" This implication follows immediately using the assumed bounds on V.

Throughout the rest of this section we assume Assumption 2. For the proof of Theorem 4 we need four preliminary lemmata.

Lemma 2. *Let μ be a class $\mathcal{KLD}$ function, let γ be a class $\mathcal{K}_\infty$ function and let $x \in \mathbb{R}^n$. If a continuous function $V : \mathbb{R}^n \to \mathbb{R}_0^+$, which is differentiable in x, satisfies the inequality*

$$V(\varphi(t,x,u)) \le \max\{\mu(V(x),t),\, \nu(u,t)\}$$

for all $t \ge 0$, all $u \in \mathcal{U}$ and ν from (6), then for all $u \in U$ it satisfies

$$\gamma(\|u\|) < V(x) \quad \Rightarrow \quad DV(x)\cdot f(x,u) \le -g(V(x)). \tag{16}$$

Proof: Fix $u_0 \in U$ with $\gamma(\|u_0\|) < V(x)$ and consider the constant function $u(t) \equiv u_0$. By continuity, for all $\tau > 0$ small enough we obtain $V(\varphi(\tau,x,u)) \le \mu(V(x),\tau)$, which implies

$$\begin{aligned} DV(x)\cdot f(x,u_0) &\le \limsup_{\tau\to 0} \frac{V(\varphi(\tau,x,u)) - V(x)}{\tau} \\ &\le \limsup_{\tau\to 0} \frac{\mu(V(x),\tau) - V(x)}{\tau} \quad = \quad -g(V(x)), \end{aligned}$$

and thus the claim.

We cannot in general conclude the result for $\gamma(\|u\|) = V(x)$ using continuity in u because U is an arbitrary set which might in particular be discrete. The following Lemma shows that we can nevertheless obtain (16) for $\gamma(\|u\|) = V(x)$ if V is continuously differentiable. Furthermore, if V is smooth, then also the converse implication holds.

Lemma 3. *Let μ be a class $\mathcal{KLD}$ function satisfying Assumption 2 and let γ be a class $\mathcal{K}_\infty$ function. Then a continuous function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ which is smooth on $\mathbb{R}^n \setminus \{0\}$ satisfies the inequality*

$$V(\varphi(t,x,u)) \le \max\{\mu(V(x),t),\, \nu(u,t)\} \tag{17}$$

for all $x \in \mathbb{R}^n$, $t \ge 0$ and all $u \in \mathcal{U}$, where ν is given by (6), if and only if it satisfies

$$\gamma(\|u\|) \le V(x) \quad \Rightarrow \quad DV(x)\cdot f(x,u) \le -g(V(x)) \tag{18}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and all $u \in U$.

Proof: "(17) $\Rightarrow$ (18)": From (16) we already know the desired inequality for $\gamma(\|u\|) < V(x)$. Hence fix $u \in \mathcal{U}$ and $x \in \mathbb{R}^n \setminus \{0\}$ with $\gamma(\|u\|) = V(x)$. Since by (16) we know $DV(x) \neq 0$ the point x cannot be a local maximum. Hence there exists a sequence of points $x_i \to x$ with $V(x_i) > V(x) = \gamma(\|u\|)$. From (16) we obtain $DV(x_i) \cdot f(x_i, u) \leq -g(V(x_i))$ for all $i \in \mathbb{N}$, which implies (18) by continuity.

"(18) $\Rightarrow$ (17)": Fix $x \in \mathbb{R}^n$ and $t > 0$. Integrating (18) we obtain

$$V(\varphi(t,x,u)) \leq \mu(V(x),t) \tag{19}$$

for all $u \in \mathcal{U}$ with $\gamma(\|u(\tau)\|) \leq \mu(V(x),t)$ f.a.a. $\tau \in [0,t]$, where μ solves $\dot\mu = -g(\mu)$, $\mu(r,0) = r$. We claim that (19) implies Lemma 1(ii).

In order to prove the assertion fix $x \in \mathbb{R}^n$, $a \geq V(x)$ and $t > 0$, let $u \in \mathcal{U}$ satisfy $\gamma(\|u(\tau))\|) \leq \mu(a,\tau)$ for almost all $\tau \in [0,t]$ and assume $V(\varphi(t,x,u)) > \mu(a,t)$. Then there exists $\delta > 0$ such that $V(\varphi(t,x,u)) > \mu(a,t) + \delta$. Now pick an arbitrary $\varepsilon < \delta$ and choose $t^* > 0$ such that $V(\varphi(t^*,x,u)) = \mu(a,t^*) + \varepsilon$ and $V(\varphi(\tau,x,u)) > \mu(a,\tau) + \varepsilon$ for all $\tau \in [t^*,t]$. From the assumption on u we obtain $\gamma(\|u(\tau)\|) \leq V(\varphi(\tau,x,u)) - \varepsilon$ for almost all $\tau \in [t^*,t]$. Using the continuity of $V(\varphi(\tau,x,u))$ in τ and the Lipschitz property of g we can now conclude the existence of times t_i, $i = 0,\ldots,k$ such that $t_0 = t^*$, $t_k = t$ and $\mu(V(\varphi(t_i,x,u), t_{i+1} - t_i) \geq V(\varphi(t_i,x,u)) - \varepsilon$, which implies $\|u(\tau)\| \leq \mu(V(\varphi(t_i,x,u))$ for almost all $\tau \in [t_i,t_{i+1}]$. Using (19) inductively and applying Gronwall's Lemma we obtain

$$V(\varphi(t,x,u)) \leq \mu(V(\varphi(t^*,x,u)), t-t^*) \leq \mu(\mu(a,t^*)+\varepsilon, t-t^*) \leq \mu(a,t) + C\varepsilon$$

for some suitable $C > 0$ which contradicts $V(\varphi(t,x,u)) > \mu(a,t) + \delta$ as $\varepsilon \to 0$ and hence shows Lemma 1(ii) and thus the assertion.

The next lemma shows the existence of a Lipschitz ISDS Lyapunov function.

Lemma 4. *If a system* (5) *is ISDS with rate μ of class $\mathcal{KLD}$ satisfying Assumption 2 and gains σ and γ of class $\mathcal{K}_\infty$ then for each $\varepsilon > 0$ there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}_0^+$, which is Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies*

$$\|x\|/(1+\varepsilon) \leq V(x) \leq \sigma(\|x\|) \tag{20}$$

for all $x \in \mathbb{R}^n$ and

$$\gamma(\|u\|) < V(x) \quad \Rightarrow \quad DV(x) \cdot f(x,u) \leq -(1-\varepsilon)g(V(x)) \tag{21}$$

for almost all $x \in \mathbb{R}^n$ and all $u \in U$.

Proof: Fix some $\varepsilon > 0$ and set $\rho_\varepsilon(r) := \varepsilon(1 - e^{-r}) + 1$. Then ρ_ε is strictly increasing for $r > 0$, $\rho_\varepsilon(0) = 1$ and $\rho_\varepsilon(r) \nearrow 1 + \varepsilon$ as $r \to \infty$. Using this function we define

$$V(x) := \inf\left\{ b \geq 0 \;\middle|\; \begin{array}{l} \|\varphi(t,x,u)\| \leq \rho_\varepsilon(\mu(b,t)) \max\{\mu(b,(1-\varepsilon)t),\, \nu(u,t)\} \\ \text{for all } u \in \mathcal{U} \text{ and all } t \geq 0 \end{array} \right\}. \tag{22}$$

Similar to the proof of Theorem 3 one verifies (20) and (21).

We now show the Lipschitz property of V. In order to do this pick a compact set $N \subset \mathbb{R}^n$ not containing the origin. From the bounds on V we can conclude that there exists a compact interval $I = [c_1, c_2] \subset \mathbb{R}^+$ such that for $x \in N$ the infimum over $b \geq 0$ in (22) can be replaced by the infimum over $b \in I$. Now the ISDS property implies the existence of a constant $R > 0$ such that $\|\varphi(t, x, u)\| \leq \max\{\mu(R, t),\, \nu(u, t)\}$ holds for all $x \in N$, all $u \in \mathcal{U}$ and all $t \geq 0$, which implies that we can restrict ourselves to those $u \in \mathcal{U}$ with $\|u\|_\infty \leq R$. Furthermore, there exists $T > 0$ such that $\mu(R, t) < \mu(c_1, (1-\varepsilon)t)$ holds for all $t \geq T$, which implies that we only have to check the inequality for $\|\varphi(t, x, u)\|$ in (22) for $t \in [0, T]$. Thus the definition of V eventually reduces to

$$V(x) := \inf \left\{ b \in I \;\middle|\; \begin{array}{l} \|\varphi(t,x,u)\| \leq \rho_\varepsilon(\mu(b,t)) \max\{\mu(b,(1-\varepsilon)t),\, \nu(u,t)\} \\ \text{for all } u \in \mathcal{U} \text{ with } \|u\|_\infty \leq R \text{ and all } t \in [0,T] \end{array} \right\}. \tag{23}$$

Now we find constants $L_1 > 0$ and $C_1 > 0$ such that the inequalities $\|\varphi(t, x_1, u) - \varphi(t, x_2, u)\| \leq L_1\|x_1 - x_2\|$ and $|\rho_\varepsilon(\mu(a_1, t)) - \rho_\varepsilon(\mu(a_2, t))| \geq C_1|a_1 - a_2|$ hold for all $u \in \mathcal{U}$ with $\|u\|_\infty \leq R$, all $t \in [0, T]$, all a_1, $a_2 \in I$ and all x_1, $x_2 \in N$.

We set $L_N = L_1/(C_1\mu(c_1, T))$, pick x_1, $x_2 \in N$ and fix $\delta > 0$. From (23) we can conclude the existence of $b^* \in I$, $t^* \in [0, T]$ and $u^* \in \mathcal{U}$ with $\|u\|_\infty \leq R$ such that $b^* \geq V(x_1) - \delta$ and $\|\varphi(t^*, x_1, u^*)\| > \rho_\varepsilon(\mu(b^*, t^*)) \max\{\mu(b^*, (1 - \varepsilon)t^*),\, \nu(u^*, t^*)\}$. Then $\|\varphi(t^*, x_2, u^*)\| \geq \rho_\varepsilon(\mu(b^{**}, t^*)) \max\{\mu(b^{**}, (1 - \varepsilon)t^*),\, \nu(u^*, t^*)\}$ holds for all $b^{**} < b^*$ with $|b^{**} - b^*| \geq L_N\|x_1 - x_2\|$, implying $V(x_2) \geq b^{**}$ and thus $V(x_1) - V(x_2) \leq L_N\|x_1 - x_2\| + \delta$. Since $\delta > 0$ was arbitrary and this estimate is symmetric in x_1 and x_2 we obtain the desired Lipschitz estimate with constant L_N.

Finally, since by Rademacher's Theorem (see, e.g., [3, page 216]) a Lipschitz function is differentiable almost everywhere, inequality (21) follows from Lemma 2.

The following lemma gives a smoothing result for Lipschitz Lyapunov functions.

Lemma 5. *Consider a continuous function $V : \mathbb{R}^n \to \mathbb{R}_0^+$, which is Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies*

$$\gamma(\|u\|) < V(x) \quad \Rightarrow \quad DV(x) \cdot f(x, u) \leq -g(V(x))$$

for almost all $x \in \mathbb{R}^n$. Then for each two continuous functions $\alpha_1, \alpha_2 : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ there exists a continuous function $\widetilde{V} : \mathbb{R}^n \to \mathbb{R}_0^+$, which is smooth on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\|V(x) - \widetilde{V}(x)\| \leq \alpha_1(x)$$

and

$$\gamma(\|u\|) \leq V(x) \;\;\Rightarrow\;\; D\widetilde{V}(x) \cdot f(x, u) \leq -g(\widetilde{V}(x)) + \alpha_2(x)$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

Proof: This follows from Theorem B.1 in [13], observing that the proof in [13] (which requires compact U) remains valid if for any compact subset $K \subset \mathbb{R}^n$ we can restrict ourselves to a compact subset of U, which is the case here since we only need to consider $\|u\| \le \gamma^{-1}(\max_{x\in K} V(x))$.

Finally, we can turn to the **Proof of Theorem 4:**
Assume ISDS, fix $\varepsilon > 0$ and let $\varepsilon_1 > 0$ be such that $1/(1+\varepsilon_1)^2 \ge (1-\varepsilon)$, $(1+\varepsilon_1)^2 \le (1+\varepsilon)$ and $(1-\varepsilon_1)^2 \ge (1-\varepsilon)$. Applying Lemma 4 with $\varepsilon = \varepsilon_1$ we can conclude the existence of a locally Lipschitz (away from 0) Lyapunov function V satisfying $\|x\|/(1+\varepsilon_1) \le V(x) \le \sigma(\|x\|)$ for all $x \in \mathbb{R}^n$ and $\gamma(\|u\|) < V(x) \Rightarrow DV(x) \cdot f(x,u) \le -(1-\varepsilon_1)g(V(x))$ for almost all $x \in \mathbb{R}^n$. Applying Lemma 5 with $\alpha_1(x) = \min\{\gamma((1+\varepsilon)\gamma^{-1}(V(x))) - V(x),\, \varepsilon_1 V(x)\}$ and $\alpha_2(x) = \varepsilon_1 g(V(x))$ we obtain a smooth (away from 0) Lyapunov function $\widetilde{V}$ satisfying the desired bounds and, since the choice of α_1 implies $\gamma((1+\varepsilon)\|u\|) \le \widetilde{V}(x) \Rightarrow \gamma(\|u\|) \le V(x)$ we obtain

$$\begin{aligned} &\gamma((1+\varepsilon)\|u\|) \le \widetilde{V}(x) \\ \Rightarrow \quad & D\widetilde{V}(x) \cdot f(x,u) \le -(1-\varepsilon_1)^2 g(\widetilde{V}(x)) \le -(1-\varepsilon) g(\widetilde{V}(x)) \end{aligned}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Hence $\widetilde{V}$ is the desired Lyapunov function.

Conversely, assume the existence of V for any $\varepsilon > 0$ and fix $t > 0$. By Lemma 3 we obtain $(1-\varepsilon)\|\varphi(t,x,u)\| \le \{\mu((1+\varepsilon)\sigma(\|x\|), (1-\varepsilon)t),\, \nu_\varepsilon(u,t)\}$ where

$$\nu_\varepsilon(u,t) := \operatorname{ess\,sup}_{\tau\in[0,t]} \mu(\gamma(\|(1+\varepsilon)u(\tau)\|),\, (1-\varepsilon)(t-\tau)).$$

Since all these expressions are continuous in ε we obtain the desired inequality.

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