Lyapunov based continuous-time nonlinear controller redesign for sampled-data implementation*

revised version

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October 8, 2004

Abstract: Given a continuous-time controller and a Lyapunov function that shows global asymptotic stability for the closed loop system, we provide several results for modification of the controller for sampled-data implementation. The main idea behind this approach is to use a particular structure for the redesigned controller and the main technical result is to show that the Fliess series expansions (in the sampling period T) of the Lyapunov difference for the sampled-data system with the redesigned controller have a very special form that is useful for controller redesign. We present results on controller redesign that achieve two different goals. The first goal is making the lower order terms (in T) in the series expansion of the Lyapunov difference with the redesigned controller more negative. These control laws are very similar to those obtained from Lyapunov based redesign of continuous-time systems for robustification of control laws and they often lead to corrections of the well known "$-L_gV$" form. The second goal is making the lower order terms (in T) in the Fliess expansions of the Lyapunov difference for the sampled-data system with the redesigned controller behave as close as possible to the lower order terms of the Lyapunov difference along solutions of the "ideal" sampled response of the continuous-time system with the original controller. In this case, the controller correction is very different from the first case and it contains appropriate "prediction" terms. The method is very flexible and one may try to achieve other objectives not addressed in this paper or derive similar results under different conditions. Simulation studies verify that redesigned controllers perform better (in an appropriate sense) than the unmodified ones when they are digitally implemented with sufficiently small sampling period T.

Keywords: Controller design, asymptotic controllability, stabilization, sampled-data, nonlinear, robustness.

1 Introduction

Design of a controller based on the continuous-time plant model, followed by a discretization of the controller, is one of the most popular methods to design sampled-data controllers [4, 7, 14]. This method, which is often referred to as emulation, is very attractive since the controller design is carried out in two relatively simple steps. The first (design) step is done in continuous-time, completely ignoring sampling, which is easier than the design that takes sampling into account. The second step involves the discretization of the controller and there are many methods that can be used for this purpose. The classical discretization methods, such as the Euler, Tustin or matched pole-zero discretization are attractive for their simplicity but they may not perform well in practice since the required sampling rate may exceed the hardware limitations even for linear systems [11, 1]. This has lead to a range of advanced controller discretization techniques based on optimization ideas that

*The authors would like to thank the Alexander von Humboldt Foundation, Germany, for providing support for this work while the first author was using his Humboldt Research Fellowship.

compute "the best discretization" of the continuous-time controller in some sense. A nice account of these optimization based approaches for linear systems has been given in the Bode Lecture by Anderson in [1] and later in the book [4].

Emulation has been proved to preserve a range of important properties for nonlinear sampled-data systems in [14] if the discretized controller is consistent in some sense with the continuous-time controller and the sampling period is small enough. Hence, in [14] all the classical discretization techniques were shown to work for a large class of nonlinear systems under sufficiently fast sampling. While the optimization based approaches could probably be carried out for nonlinear systems, we are not aware of any results in this direction. This may be due to the fact that these approaches inevitably require solutions of partial differential equations of Hamilton-Jacobi type that are very hard to solve.

In this paper we present a Lyapunov based framework for redesign of continuous-time controllers for sampled-data implementation. We assume that an appropriate continuous-time controller $u_0(x)$ has been designed together with an appropriate Lyapunov function $V(\cdot)$ for the closed-loop continuous-time system. Then, we presuppose the following structure of the redesigned controller

$$u_{dt}(x) = u_0(x) + \sum_{i=1}^{N} T^i u_i(x) \ ,$$

where T is the sampling period and $u_i(x)$ are the extra terms that need to be determined through controller redesign. This controller structure yields a particularly useful structure of the Fliess series expansion (in the sampling period T) of the first difference for $V(\cdot)$ along solutions of the sampled-data system with the redesigned controller. The terms in the Fliess series depend explicitly on V, u_0, the continuous-time model and u_i and they can be used to systematically compute corrections u_i that achieve a particular objective of the redesign.

We were motivated to exploit this particular structure of the controller for several reasons. First, this structure was obtained in several different papers as an outcome of the design procedure. For instance, in [17] this controller structure was obtained as an outcome of a backstepping design based on the Euler approximate discrete-time model of the plant. In [2] this structure was obtained when approximately feedback linearizing a nonlinear system via sampled-feedback. Note that we *impose* this structure of the controller instead of obtaining it as an outcome of some design procedure. Furthermore, a robotic manipulator example was considered in [15] where the Euler model was used to redesign a continuous-time controller $u_{ct}(x)$ in the following way $u_{dt} = u_{ct}(x) + Tu_1$. Simulation studies in [15] showed that this redesign yielded better behaviour of the sampled-data system. We emphasize that [15] does not contain a systematic methodology for controller redesign, which is the purpose of this paper.

We present results that achieve two different objectives. We emphasize that the method is much more flexible and one can prove new results under different conditions or try to achieve other objectives not addressed in this paper. The first objective is to make the first terms in the Fliess series expansions more negative by choosing u_i. This often leads to correction terms of the form "$-L_gV$" that are known to be useful in robustification of continuous-time controllers by Lyapunov redesign (see, for instance, [5, 21]). Moreover, we show for a particular class of (optimal) control laws under appropriate conditions that we can always make the first two terms in the Fliess series expansions negative by choosing u_1. Note that in this case u_i always depend on the Lyapunov function $V(\cdot)$ and its derivatives with respect to x. The second objective is to make the first terms of the Fliess series expansions of the first difference for $V(\cdot)$ along solutions of the sampled-data system with the redesigned controller as close as possible to the first difference for $V(\cdot)$ along sampled solutions of the "ideal" response of the continuous-time system with the original controller. In this case, correction terms u_i take a completely different form and they do not explicitly depend on the Lyapunov function $V(\cdot)$ or its derivatives. Numerous simulations illustrate that our redesigned controllers work better (in an appropriate sense) than the original ones when they are implemented with sufficiently small sampling periods.

The paper is organized as follows. In Section 2 we present the notation, main assumptions and the problem formulation. Section 3 contains the main technical result on the Fliess series expansions of the Lyapunov difference for the sampled-data system with the redesigned controller. These results are used in Section 4 to show two distinct ways to redesign continuous-time controllers. Numerous simulations for different examples are given in Section 5. Conclusions are presented in the last section.

2 Preliminaries

The set of real numbers is denoted as $\mathbb{R}$, the set of natural numbers (excluding 0) as $\mathbb{N}$ and we use $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called class $\mathcal{KL}$ if it is continuous, of class $\mathcal{K}$ in the first and strictly decreasing to 0 in the second argument. The notation $|\cdot|$ always denotes the Euclidean norm. We will say that a function $G(T, x)$ is of order T^p and we write $G(T, x) = O(T^p)$ if, whenever G is defined, we can write $G(T, x) = T^p \tilde{G}(T, x)$ and there exists $\gamma \in \mathcal{K}_\infty$ such that for each $\Delta > 0$ there exists $T^* > 0$ such that $|x| \leq \Delta$ and $T \in (0, T^*)$ implies $\left|\tilde{G}(T, x)\right| \leq \gamma(|x|)$. We define multinomial coefficients

$$\begin{pmatrix} & n & \\ n_0 \; n_1 & \ldots & n_M \end{pmatrix} := \frac{n!}{n_0! n_1! \ldots n_M!} \; .$$

as well as multi-indices $\nu := (n_0, n_1, \ldots, n_M)$ and the notation $|\nu| := n_0 + n_1 + \ldots + n_M$. For these ,we need the following result, which can be found in [3, Theorem 4.5.1] or [12, Theorem 4.2]:

Proposition 2.1 We can write for any $a_i \in \mathbb{R}, i = 0, 1, 2, \ldots, M$ and $n \in \mathbb{N}$:

$$(a_0 + a_1 + \ldots + a_M)^n = \sum_{|\nu|=n} \begin{pmatrix} & n & \\ n_0 \; n_1 & \ldots & n_M \end{pmatrix} a_0^{n_0} \cdots a_M^{n_M} \; .$$

□

Consider the system

$$\dot{x} = g_0(x) + g_1(x)u \; , \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are respectively the state and the control input of the system. We will assume that all functions are sufficiently many times (r times) continuously differentiable. For simplicity, we concentrate on single input systems but the results can be extended to the multiple input case $u \in \mathbb{R}^m, m \in \mathbb{N}$.

For several classes of systems (2.1), there exist nowadays systematic methods to design a continuous-time control law of the form

$$u = u_0(x) \; , \tag{2.2}$$

and a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \; \leq \; V(x) \; \leq \; \alpha_2(|x|) \tag{2.3}$$

$$\frac{\partial V}{\partial x}[g_0(x) + g_1(x)u_0(x))] \; \leq \; -\alpha_3(|x|) \qquad \forall x \in \mathbb{R}^n \; . \tag{2.4}$$

Examples of such methods are backstepping [13, 8] and forwarding [21] or methods based on control Lyapunov functions, such as Sontag's formula [10].

However, in most cases the controller (2.2) is implemented digitally using a sampler and zero order hold. Since the controller (2.2) is static, it is often proposed in the literature to simply implement it digitally as follows (see [14]):

$$u(t) = u_0(x(k)) \qquad \forall t \in [kT, (k+1)T), \forall k \in \mathbb{N}_0 \; . \tag{2.5}$$

It was shown, for instance, in [14] that this digital controller will recover performance of the continuous-time system in a semiglobal practical sense (T is the parameter that needs to be chosen sufficiently small). However, this implementation typically requires very small sampling periods T to work well and, hence, it often does not produce a desired behaviour for a fixed given T. The purpose of this paper is to address the following problem:

Assuming that an appropriate continuous-time control law $u_0(\cdot)$ and a Lyapunov function $V(\cdot)$ have been found for the continuous-time system (2.1), redesign the controller $u_0(\cdot)$ so that the redesigned controller performs better than (2.5) in an appropriate sense when implemented digitally.

In our redesign technique we will aim at improving the quantitative behavior of the asymptotic stability property in terms of the transient behavior and overshoots and the attraction speed. However, as a side effect, we also expect that our procedure enlarges the domain of stability of the semiglobal practical stability property with respect to the emulated controller (2.5). These multiple objectives are the reason for the slightly vague phrase "appropriate sense" in the problem statement, above.

In order to precisely state in which sense we can expect to improve the system's quantitative behavior with our approach we will below introduce our main Assumption 2.2. Before doing this, we need to recall some standard facts about Lyapunov functions. It is a well known fact (see [16]) if (2.3) and (2.4) hold, then there exists a function $\beta \in \mathcal{KL}$ such that solutions of the closed loop system (2.1), (2.2) satisfy:

$$|x(t,x_0)| \le \beta(|x_0|,t) \qquad \forall x_0 \in \mathbb{R}^n,\ t \ge 0\ . \tag{2.6}$$

Moreover, the function β is completely determined by $\alpha_1, \alpha_2, \alpha_3$ in the following manner. Consider the solution of the following scalar differential equation[1]:

$$\dot{y} = -\alpha_3 \circ \alpha_2^{-1}(y) \qquad y(0) = y_0\ . \tag{2.7}$$

Proposition 4.4 in [16] states that there exists $\sigma \in \mathcal{KL}$ such that the solution $y(\cdot)$ of equation (2.7) is defined for all $t \ge 0$ and can be written as $y(t) = \sigma(y_0, t)$. Finally, using a standard proof technique and comparison principle we can write that:

$$\beta(s,t) := \alpha_1^{-1}\left(\sigma(\alpha_2(s),t)\right)\ . \tag{2.8}$$

Based on these considerations we can now state our main assumption.

Assumption 2.2 Suppose that a continuous static state feedback controller (2.2) has been designed for the system (2.1) so that the following holds:

(i) There exists a Lyapunov function $V(\cdot)$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ satisfying (2.3) and (2.4).

(ii) The function $\beta \in \mathcal{KL}$ defined in (2.8) satisfies all performance specifications in terms of overshoot and speed of convergence.

(iii) The controller (2.2) is to be implemented digitally using a sampler and zero order hold, that is for a given sampling period $T > 0$ we measure $x(k) := x(kT), k \in \mathbb{N}_0$ and $u(t) = u(k) = const., t \in [kT, (k+1)T), k \in \mathbb{N}_0$.

□

Remark 2.3 It may seem strange that we use both items (i) and (ii) in Assumption 2.2, since either (i) or (ii) may seem enough. Since our approach is Lyapunov based it is obvious that we need to have a Lyapunov function satisfying item (i) of Assumption 2.2. There are two reasons for using item (ii) of Assumption 2.2:

(i) In Subsection 4.1 we provide a controller redesign with the objective of achieving more Lyapunov decrease with the redesigned controller. We note that while such redesign improves the speed of response, it may in general lead to arbitrarily worse overshoots. This situation is illustrated by Example 4.1. Item (ii) guards against such undesirable situations[2].

[1] Without loss of generality we need to assume here that $\alpha_3 \circ \alpha_2^{-1}(\cdot)$ is a locally Lipschitz function (see footnote in [16, pg. 153]).

[2] Actually, in this case it is sufficient to assume that overshoots obtained from the β function are satisfactory – speed of convergence can be improved by the redesign.

(ii) In Subsection 4.2, the redesign objective is to minimize the mismatch between the values of the Lyapunov function along solutions of the sampled continuous-time and the sampled-data systems. In this case, it is assumed that the $\mathcal{KL}$ estimate for the continuous-time system is "ideal" and it is used as a reference for controller redesign. Hence, using item (ii) of Assumption 2.2 is natural in this context. We will discuss the implications of this redesign technique in terms of the $\mathcal{KL}$ estimate in more detail in Remark 4.15, after the presentation of the necessary technical results.

In general, finding a Lyapunov function that satisfies both items (i) and (ii) of Assumption 2.2 is hard but in some cases it is possible, cf. the examples in Section 5. □

The exact discrete-time model of the system with the zero order hold assumption is obtained (whenever it exists) by integrating the equation (2.1) starting from $x(k)$ with the control $u(t) = u(k), t \in [kT, (k+1)T)$:

$$x(k+1) = x(k) + \int_{kT}^{(k+1)T} [g_0(x(s)) + g_1(x(s))u(k)]ds \ ,$$

where $x(s)$ denotes the solution of (2.1) with constant control $u(k)$ and initial value $x(k)$ at time kT. We abbreviate this to

$$x(k+1) = F_T^e(x(k), u(k))$$

with

$$F_T^e(x,u) := x + \int_0^T [g_0(x(s)) + g_1(x(s))u]ds \tag{2.9}$$

where $x(s)$ now denotes the solution of (2.1) with constant control u and initial value x at time 0. We use this notation in the sequel and for given $x \in \mathbb{R}^n$, $u \in u$ and $T > 0$ we say that $F_T^e(x,u)$ is *well defined* if the solution of (2.1) with initial value x and control u exists on the interval $[0,T]$.

3 Fliess expansion of the Lyapunov difference

In this section we propose a particular structure for the redesigned controller. This structure of the controller yields an interesting structure of the series expansion of the Lyapunov difference along the solutions of closed loop system with the redesigned controller and will allow us to redesign the controller in a systematic manner. We propose to modify the continuous-time controller as follows:

$$u_{dt}(x) := \sum_{j=0}^{M} u_j(x)T^j \ , \tag{3.1}$$

where $u_0(x)$ comes from Assumption 2.2 and $u_j = u_j(x), j = 1,2,\ldots,M$ are corrections that we want to determine.

The idea is to use the Lyapunov function V as a control Lyapunov function for the discrete-time model (2.9) of the sampled-data system with the modified controller (3.1) where we treat $u_i, i = 1,2,\ldots,M$ as new controls, and then from the Lyapunov difference:

$$\frac{V(F_T^e(x, u_{dt}(x))) - V(x)}{T} \tag{3.2}$$

determine $u_i, i = 1,2,\ldots,M$.

Since in general it is not possible to compute $F_T^e(x,u)$ in (3.2) exactly we will have to use an approximation technique for the controller redesign. Results in [18, 20] show that if we use (3.1) and we can show that it stabilizes any reasonable (more precisely consistent[3]) approximate model of (2.9), then the exact model (2.9) will be stabilized by the same controller for sufficiently small sampling

[3]The notion of consistency is borrowed from the numerical analysis literature and can be checked easily for a given approximate model.

periods T. In our approach in this paper we do not explicitly use such consistent discrete time approximations. Instead, below we present a series expansion of the Lyapunov difference (3.2) in T that is particularly useful for controller redesign. The expansion is based on truncated Fliess series and the special structure of the modified controller (3.1). In the context of discrete time approximations, the truncated Fliess series can be interpreted as a consistent approximation of the Lyapunov difference which in our approach replaces the discrete approximation of the system itself. It should, however, be noted that Fliess series approximations applied to the system itself can also be used to construct consistent discrete time approximations, see [9] for details.

Theorem 3.1 Consider system (2.1) and controller (3.1) and suppose that Assumption 2.2 holds. Then, for sufficiently small T, there exist functions $p_i(x, u_0, \ldots, u_{i-1})$ such that we can write:

$$\begin{aligned} \frac{V(F^e_T(x, u_{dt})) - V(x)}{T} &= L_{g_0}V + L_{g_1}V \cdot u_0 + \sum_{s=1}^{M} T^s[L_{g_1}V \cdot u_s + p_s(x, u_0, \ldots, u_{s-1})] \\ &\quad + G(T, x, u_0, u_1, \ldots, u_M) \ , \end{aligned} \tag{3.3}$$

where $G(T, x, u_0, u_1, \ldots, u_M) = O(T^{M+1})$. □

Proof of Theorem 3.1: Consider, the solutions of (2.1) initialized at $x(0) = x$ with some input $u(\cdot)$ and with the "output"

$$y(t) = V(x(t)) \ . \tag{3.4}$$

Then, for sufficiently small t, using the Fliess series expansions (see [6] or formula (3.7) in [10, Section 3.1]) we can write:

$$V(x(t)) - V(x) = \sum_{k=0}^{\infty} \sum_{i_0,\ldots,i_k=0}^{m=1} L_{g_{i_0}} \cdots L_{g_{i_k}} V(x) \int_0^t d\xi_{i_k} \cdots d\xi_{i_0} \ , \tag{3.5}$$

where $\int_0^t d\xi_{i_k} \cdots d\xi_{i_0}$ are the so called iterated integrals (see [10, pg. 106]). Note that since we consider single input systems we obtain $m = 1$ in [10, formula (3.7)] and the indices i_k take values on the set $\{0, 1\}$. The iterated integrals are defined as follows:

$$\begin{aligned} \xi_0(t) &= t \\ \xi_1(t) &= \int_0^t u(\tau) d\tau \\ \int_0^t d\xi_{i_k} \cdots d\xi_{i_0} &= \int_0^t d\xi_{i_k}(\tau) \int_0^\tau d\xi_{i_{k-1}} \cdots d\xi_{i_0} \ . \end{aligned}$$

Several integrals for the single input case are given below:

$$\int_0^t d\xi_0 d\xi_0 = \frac{t^2}{2}, \qquad \int_0^t d\xi_0 d\xi_1 = \int_0^t \int_0^\tau u(\theta) d\theta d\tau$$

$$\int_0^t d\xi_1 d\xi_0 = \int_0^t u(\tau)\tau d\tau, \qquad \int_0^t d\xi_1 d\xi_1 = \int_0^t u(\tau) \int_0^\tau u(\theta) d\theta d\tau.$$

If we write (3.5) for the case when $t = T$ is sufficiently small and $u(\cdot) = u = const.$, then we have that

$$\begin{aligned} x(T) &= F^e_T(x, u) \\ \int_0^T d\xi_{i_k} \cdots d\xi_{i_0} &= \frac{T^{(k+1)}}{(k+1)!} u^{|I_k|} \ , \end{aligned}$$

where $|I_k| := i_0 + i_1 + \ldots + i_k$. We can write:

$$\frac{V(F^e_T(x, u)) - V(x)}{T} = \sum_{k=0}^{\infty} \sum_{i_0=0,\ldots,i_k=0}^{m=1} L_{g_{i_0}} \cdots L_{g_{i_k}} V(x) \frac{T^k}{(k+1)!} u^{|I_k|} \ . \tag{3.6}$$

Using Proposition 2.1, the following holds:

$$\left(\sum_{j=0}^{M} u_j T^j\right)^{|I_k|} = \sum_{|\nu|=|I_k|} \binom{|I_k|}{n_0\ n_1\ \dots\ n_M} u_0^{n_0}\cdots u_M^{n_M}\cdot T^{\|\nu\|}\ , \tag{3.7}$$

where we used the multi-index $\nu := (n_0, n_1, \dots, n_M)$ and the notation:

$$\|\nu\| := \sum_{j=0}^{M} j\cdot n_j\ .$$

Substituting (3.1) into (3.6) and using (3.7), we can write:

$$\frac{V(F_T^e(x,u)) - V(x)}{T} = H(T, x, u_0, \dots, u_M) + H_1(T, x, u_0, \dots, u_M)\ , \tag{3.8}$$

where $H_1(T, x, u_0, \dots, u_M) = O(T^{M+1})$ and $H(T, x, u_0, \dots, u_M)$ is equal to:

$$\sum_{k=0}^{M}\sum_{i_0,\dots,i_k=0}^{m=1} L_{g_{i_0}}\cdots L_{g_{i_k}} V(x)\frac{T^k}{(k+1)!}\left(\sum_{|\nu|=|I_k|}\binom{|I_k|}{n_0\ n_1\ \dots\ n_M}\prod_{j=0}^{M} u_j^{n_j}\cdot T^{\|\nu\|}\right)\ .$$

The proof is completed by introducing a new index $s := k + \sum_{j=0}^{M} jn_j$ and then collecting first terms that multiply $T^s, s = 0, 1, 2, \dots, M$ in the expression for H. Indeed, H in (3.8) can be written as follows:

$$\sum_{s=0}^{M} T^s \sum_{k=0}^{s}\sum_{i_0,\dots,i_k=0}^{m=1}\frac{L_{g_{i_0}}\cdots L_{g_{i_k}} V(x)}{(k+1)!}\left(\sum_{\substack{|\nu|=|I_k|\\ \|\nu\|=s-k}}\binom{|I_k|}{n_0\ n_1\ \dots\ n_M}\prod_{j=0}^{M} u_j^{n_j}\right) + O(T^{M+1})\ .$$

Direct calculations show that the term for $s = 0$ is

$$L_{g_0}V\binom{0}{0\ 0\ \dots\ 0}u_0^0u_1^0\cdots u_M^0 + L_{g_1}V\binom{1}{1\ 0\ \dots\ 0}u_0^1u_1^0\cdots u_M^0 = L_{g_0}V + L_{g_1}V\cdot u_0$$

and the terms for arbitrary $s = 1, ..., M$ and $k = 0$ are

$$L_{g_1}V\underbrace{\binom{1}{0\ 0\ \dots 1 \dots\ 0}}_{1 \text{ is in } s\text{th place}}u_0^0u_1^0\cdots u_s^1\cdots u_M^0 = L_{g_1}V\cdot u_s\ .$$

Hence, we can write H as follows:

$$H = L_{g_0}V + L_{g_1}V\cdot u_{ct} + \sum_{s=1}^{M} T^s[L_{g_1}V\cdot u_s + p_s(x, u_0, \dots, u_{s-1})] + O(T^{M+1})\ ,$$

where

$$p_s := \sum_{k=1}^{s}\sum_{i_0=0,\dots,i_k=0}^{m=1}\frac{L_{g_{i_0}}\cdots L_{g_{i_k}} V(x)}{(k+1)!}\left(\sum_{\substack{|\nu|=|I_k|\\ \|\nu\|=s-k}}\binom{|I_k|}{n_0\ n_1\ \dots\ n_M}\prod_{j=0}^{M} u_j^{n_j}\right)\ ,$$

which completes the proof by noting that p_s are functions of x and $u_0, ..., u_{s-1}$. □

It is instructive to write down the expressions for the first couple of p_s and we do this below for p_1 and p_2. Direct calculations show that

$$\begin{aligned} p_1 &= \frac{L_{g_0}L_{g_0}V}{2!}\begin{pmatrix} 0 \\ 0\ 0\ \ldots\ 0 \end{pmatrix} u_0^0 u_1^0 \cdots u_M^0 + \\ &\quad \frac{(L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V)}{2!}\begin{pmatrix} 1 \\ 1\ 0\ \ldots\ 0 \end{pmatrix} u_0^1 u_1^0 \cdots u_M^0 + \\ &\quad \frac{L_{g_1}L_{g_1}V}{2!}\begin{pmatrix} 2 \\ 2\ 0\ \ldots\ 0 \end{pmatrix} u_0^2 u_1^0 \cdots u_M^0 \\ &= \frac{L_{g_0}L_{g_0}V + (L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V)u_0 + L_{g_1}L_{g_1}V u_0^2}{2!}\,. \end{aligned} \tag{3.9}$$

$$\begin{aligned} p_2 &= \frac{L_{g_0}L_{g_1}V + L_{g_1}L_{g_0}V}{2!}\begin{pmatrix} 1 \\ 0\ 1\ \ldots\ 0 \end{pmatrix} u_0^0 u_1^1 u_2^0 \cdots u_M^0 + \\ &\quad \frac{L_{g_1}L_{g_1}V}{2!}\begin{pmatrix} 2 \\ 1\ 1\ \ldots\ 0 \end{pmatrix} u_0^1 u_1^1 u_1^0 \cdots u_M^0 + \\ &\quad \frac{L_{g_0}L_{g_0}L_{g_0}V}{3!}\begin{pmatrix} 0 \\ 0\ 0\ \ldots\ 0 \end{pmatrix} u_0^0 u_1^0 u_2^0 \cdots u_M^0 + \\ &\quad \frac{L_{g_0}L_{g_0}L_{g_1}V + L_{g_0}L_{g_1}L_{g_0}V + L_{g_1}L_{g_0}L_{g_0}V}{3!}\begin{pmatrix} 1 \\ 1\ 0\ \ldots\ 0 \end{pmatrix} u_0^1 u_1^0 u_2^0 \cdots u_M^0 + \\ &\quad \frac{L_{g_0}L_{g_1}L_{g_1}V + L_{g_1}L_{g_1}L_{g_0}V + L_{g_1}L_{g_0}L_{g_1}V}{3!}\begin{pmatrix} 2 \\ 2\ 0\ \ldots\ 0 \end{pmatrix} u_0^2 u_1^0 u_2^0 \cdots u_M^0 + \\ &\quad \frac{L_{g_1}L_{g_1}L_{g_1}V}{3!}\begin{pmatrix} 3 \\ 3\ 0\ \ldots\ 0 \end{pmatrix} u_0^3 u_1^0 u_2^0 \cdots u_M^0 \\ &= \frac{u_1(L_{g_0}L_{g_1}V + L_{g_1}L_{g_0}V + (2!)\cdot L_{g_1}L_{g_1}V u_0)}{2!} + \\ &\quad \frac{L_{g_0}L_{g_0}L_{g_0}V + (L_{g_0}L_{g_0}L_{g_1}V + L_{g_0}L_{g_1}L_{g_0}V + L_{g_1}L_{g_0}L_{g_0}V)u_0}{3!} + \\ &\quad \frac{(L_{g_0}L_{g_1}L_{g_1}V + L_{g_1}L_{g_0}L_{g_1}V + L_{g_1}L_{g_1}L_{g_0}V)u_0^2 + L_{g_1}L_{g_1}L_{g_1}V u_0^3}{3!}\,. \end{aligned} \tag{3.10}$$

Other functions p_s can be obtained in a similar manner.

Remark 3.2 Computer algebra systems, such as Maple, can be used to compute expansions of the Lyapunov difference for particular examples. We note that this is the approach we took when solving the examples in Section 5. While these formulas can be in general very complex, we illustrate in the next section how Theorem 3.1 can be used for controller redesign under relatively weak conditions. □

4 Lyapunov based controller redesign

In this section we propose controller redesign procedures that are based on the structure of (3.3) in Theorem 3.1. The main idea behind the redesign is to use the Lyapunov function of the continuous-time closed loop system (2.1), (2.2) as a control Lyapunov function for the discrete-time model of the sampled-data closed loop system with the redesigned controller $u_{dt}(x)$ of the form (3.1). Moreover, since the exact discrete-time model of the system is not available, we will use the Fliess series expansions from the previous section for this purpose.

There is a lot of flexibility in this procedure and in general one needs to deal with systems on a case-by-case basis. Hence, we concentrate below on two different goals for controller redesign and the issues involved that are respectively presented in Subsections 4.1 and 4.2. The first case is reminiscent of the Lyapunov controller redesign of continuous-time systems for robustification of the system (see [5, 16]). In this case, the redesigned controller $u_{dt}(x)$ is providing more negativity to the Lyapunov difference than the original controller $u_0(x)$. This typically yields high gain controllers that may have the well known "$-L_gV$" structure which was used, for example, in [21]. In the second subsection, the goal is to redesign the controller so that the Lyapunov difference along the solutions of the discrete-time model with the redesigned controller $u_{dt}(x)$ is as close as possible to the Lyapunov difference of the sampled solutions of the continuous-time closed loop system with the original controller $u_0(x)$, which can be thought of as providing the "ideal" reference response.

Examples in the next section are serving to further illustrate how to use this method to systematically improve performance of the redesigned controller.

4.1 High gain controller redesign

The goal in this section is to provide a controller redesign satisfying the following objective. Suppose that we have:

$$V(F_T^e(x, u_{ct})) - V(x) \leq -\alpha(T, x) + G(T, x)$$

where $G(T, x) = O(T^M)$ for some $M \in \mathbb{N}$, α is positive definite in x for all $T > 0$. Then we want to design u_{dt} so that

$$V(F_T^e(x, u_{dt})) - V(x) \leq -\alpha_1(T, x) + G_1(T, x)$$

where α_1 is positive definite, $G_1(T, x) = O(T^N)$, $N \in \mathbb{N}$ and the following two conditions hold:

$$\alpha_1(T, x) > \alpha(T, x) \;\; \forall T > 0, x \neq 0 \qquad \text{and} \qquad N \geq M \; .$$

Hence, we want to achieve more Lyapunov decrease with the redesigned controller (i.e., $\alpha_1(T, x) > \alpha(T, x)$) while not decreasing the order in T of the (possibly) positive terms (i.e., $N \geq M$). Hence, it is expected that the redesigned controller will provide faster decrease of the Lyapunov function for sufficiently small sampling periods. This results in a faster response and typically enlarges the domain of attraction for sufficiently small sampling periods. Before we present the details we give an example that motivates the use of item (ii) in Assumption 2.2 in this context:

Example 4.1 Note that redesigning the controller to provide more decrease of the Lyapunov function along solutions of a given system is aimed at improving convergence but, in general, it may worsen the overshoots. Item (ii) of Assumption 2.2 guards from having extremely bad situation illustrated by the following example. Indeed, consider the two input linear system:

$$\dot{x}_1 = u_1; \qquad \dot{x}_2 = u_2 \; . \tag{4.1}$$

and let

$$u_1(x) = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} \; . \tag{4.2}$$

Using the Lyapunov function $V_1(x) = \frac{1}{2}(x_1^2 + x_2^2)$ we have that solutions of the system (4.1), (4.2) satisfy:

$$|x(t)| \leq \exp(-t)\,|x(0)| \qquad \forall t \geq 0 \; . \tag{4.3}$$

Hence, there is no overshoot in the response. Suppose now that we use $V_2(x) := \frac{1}{2}(x_1^2 + \frac{1}{c}x_2^2)$ for some $c < 1$. Using V_2 we obtain that the solutions of the system (4.1), (4.2) satisfy:

$$|x(t)| \leq \frac{1}{\sqrt{c}} \exp(-t)\,|x(0)| \qquad \forall t \geq 0 \; . \tag{4.4}$$

Note that in this case the estimate (4.3) is very tight and if the convergence is satisfactory, then we can say that the Lyapunov function V_1 satisfies item (ii) of Assumption 2.2. However, the overshoot in the estimate (4.4) obtained via V_2 can be arbitrarily bad and for sufficiently small c we would have that the estimate of the overshoot $\frac{1}{\sqrt{c}}$ is too large, i.e. for sufficiently small c we would have that item (ii) of Assumption 2.2 does not hold. Suppose now that we use V_2 to design a new controller which would provide more decrease of V_2 along trajectories of the closed loop system. Let the new (redesigned) control law be

$$u_2(x) = \begin{pmatrix} -kx_1 - \frac{1}{c}x_2 \\ -kx_2 + x_1 \end{pmatrix} . \tag{4.5}$$

Along solutions of the system (4.1), (4.5) we have:

$$\dot{V} \leq -2kV , \tag{4.6}$$

and hence we have that the new estimate on trajectories holds:

$$|x(t)| \leq \frac{1}{\sqrt{c}} \exp(-kt) \, |x(0)| \qquad \forall t \geq 0 , \tag{4.7}$$

and using V_2 we could conclude that the controller (4.2) is better than (4.5) since the estimate of overshoots in (4.4) and (4.7) are the same while (4.7) decays faster. However, the actual overshoot with (4.5) can be arbitrarily worse than the overshoot with (4.2). For instance, if we let $k = \frac{2}{\sqrt{c}}$ and consider $t^* = \frac{\pi}{2}$, then we have that $|x(t^*)| = \frac{e^{-\pi}}{\sqrt{c}} \, |x(0)|$ and as $c \to 0$, we have that the real overshoot grows unbounded.

Note that the actual overshoot with the redesigned controller can not be worse than the estimated overshoot via the Lyapunov function. Hence, item (ii) in Assumption 2.2 guarantees that the actual overshoot will be satisfactory after the controller redesign. Without this assumption, we can not avoid having the situations like in the above example. $\square$

Note that the special structure of (3.3) is due only to the controller structure (3.1) that we proposed to use and this is crucial in our controller redesign approach. Indeed, the first $M+1$ terms in the series expansion have the following form:

$$\begin{array}{llcl} O\left(T^0\right) \text{ term}: & L_{g_1}V \cdot u_0 & + & L_{g_0}V \qquad (4.8) \\ O\left(T^1\right) \text{ term}: & L_{g_1}V \cdot u_1 & + & p_1(x,u_0) \qquad (4.9) \\ O\left(T^2\right) \text{ term}: & L_{g_1}V \cdot u_2 & + & p_2(x,u_0,u_1) \qquad (4.10) \\ O\left(T^3\right) \text{ term}: & L_{g_1}V \cdot u_3 & + & p_3(x,u_0,u_1,u_2) \qquad (4.11) \\ & \vdots & & \vdots \\ O\left(T^M\right) \text{ term}: & L_{g_1}V \cdot u_M & + & p_M(x,u_0,u_1,u_2,\ldots,u_{M-1}) . \qquad (4.12) \end{array}$$

This special triangular structure allows us to use a recursive redesign. We already assumed that u_0 is designed based on the continuous-time plant model (2.1). At the next step we design u_1 from (4.9) since $p_1(x,u_0)$ and u_0 are known by assumption. We will choose u_1 so that $O(T)$ terms in the expansion (3.3) are more negative than when $u_1 = 0$. At step $s \in \{2,\ldots,M\}$ we design u_s to make $O(T^s)$ more negative and for this purpose we can use $p_s(x,u_0,\ldots,u_{s-1})$ since all previous $u_i, i = 0,1,2,\ldots,s-1$ have already been designed. The question is how to design u_s at each step of the above described procedure. We present some choices below and point out some issues that have to be taken into account. It is obvious from (3.3) that any function u_j with

$$u_j = u_j(x) \;\text{ such that }\; \begin{cases} u_j \leq 0 & \text{if } L_{g_1}V \geq 0 \\ u_j \geq 0 & \text{if } L_{g_1}V \leq 0 \end{cases}$$

will achieve more decrease of $V(\cdot)$ if we neglect the terms of order $\geq j+1$. For example, one such choice is

$$u_j(x) = -\gamma_j(V(x)) \cdot (L_{g_1}V(x)) , \tag{4.13}$$

where $\gamma_j \in \mathcal{K}$ is a design parameter that can be determined using the $p_s(x,u_0,\ldots,u_{s-1})$ functions from (3.3). In particular, one would like to dominate the sign indefinite function $p_s(x,u_0,\ldots,u_{s-1})$ as much as possible with the available control via the negative term $u_s(x)L_{g_1}V(x)$. Hence, we can state formally the following:

Theorem 4.2 Consider the system (2.1) and suppose that Assumption 2.2 holds. For any $j \in \{0,1,2,\dots,M\}$ denote $u^j(x) := \sum_{i=0}^{j} T^i u_i(x)$. Then, suppose that for some $x \in \mathbb{R}^n$ and $j \in \{0,1,2,\dots,M\}$ the function $F_T^e(x,u^j(x))$ is well defined and the following holds:

$$\frac{V(F_T^e(x,u^j(x))) - V(x)}{T} \leq -\alpha_3(|x|) + G_1(T,x) \ , \tag{4.14}$$

and $G_1(T,x) = O(T^p)$ for some $p \in \mathbb{N}$. Suppose now that the controller $u^{j+1}(x)$ is implemented, where $u_{j+1}(x) := -\gamma_{j+1}(V(x)) \cdot L_{g_1}V(x)$. Then, whenever $F_T^e(x,u^{j+1}(x))$ is well defined, we have that:

$$\frac{V(F_T^e(x,u^{j+1}(x))) - V(x)}{T} \leq -\alpha_3(|x|) - T^{j+1}\gamma_{j+1}(V(x))\left(\frac{\partial V}{\partial x} g_1(x)\right)^2 + G_1(T,x) + G_2(T,x) \ , \tag{4.15}$$

where $G_1(T,x)$ is the same as in (4.14) and $G_2(T,x) = O(T^{j+2})$. □

The proof of the above result follows directly from Theorem 3.1. If the function p_s has the special form

$$p_s(x,u_0,\dots,u_{s-1}) = L_{g_1}V \cdot \bar{p}_s(x,u_0,\dots,u_{s-1}) \ ,$$

then it is possible to make the $O(T^s)$ term in (3.3) negative for all $x \in \mathbb{R}^n$. Unfortunately, this condition is too strong in general. On the other hand, it is often useful to use corrections of a more general form than (4.13). This situation is illustrated in the following theorem that is derived under stronger assumptions than Theorem 4.2. The conditions we use allow us to use a construction very similar to the well known Sontag's formula [22]. Indeed, we can state:

Theorem 4.3 Consider the system (2.1) and suppose that the following conditions hold:

(i) Assumption 2.2 holds;

(ii) $u_0(x) = -(L_{g_1}V(x))R(x)$, where $R(x) > 0, \forall x \in \mathbb{R}^n$;

(iii) for all $x \neq 0$ we have that $L_{g_1}V(x) = 0$ implies $L_{g_0}L_{g_0}V(x) < 0$;

(iv) for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x| \leq \delta, x \neq 0$ there exists some u, with $|u| \leq \epsilon$, such that
$$\frac{L_{g_0}L_{g_0}V(x)}{2} + L_{g_1}V(x)u < 0 \ .$$

Then, the controller $u_{dt}(x) = u_0(x) + Tu_1(x)$ with

$$u_1(x) = \tilde{u}_1(x) - \frac{-(L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V)R(x) + (L_{g_1}L_{g_1}V)\cdot(L_{g_1}V)\cdot R(x)^2}{2!} \tag{4.16}$$

and

$$\tilde{u}_1(x) = \begin{cases} 0 & \text{if } L_{g_1}V(x) = 0 \\ -\dfrac{\frac{L_{g_0}L_{g_0}V}{2} + \sqrt{\frac{(L_{g_0}L_{g_0}V)^2}{4} + (L_{g_1}V)^4}}{L_{g_1}V} & \text{if } L_{g_1}V(x) \neq 0 \end{cases} \tag{4.17}$$

yields

$$\frac{V(F_T^e(x,u_{dt}(x))) - V(x)}{T} \leq -\alpha_3(|x|) + TG_1(x) + G_2(T,x) \ , \tag{4.18}$$

with α_3 from (2.4),

$$G_1(x) := -\sqrt{\frac{(L_{g_0}L_{g_0}V(x))^2}{4} + (L_{g_1}V(x))^4}$$

being negative definite and $G_2(T,x) = O(T^2)$. □

Proof of Theorem 4.3: From item (i) of Theorem 4.3 and Theorem 3.1 we have that

$$\begin{aligned}\frac{V(F_T^e(x,u_{dt}(x)))-V(x)}{T} &= L_{g_0}V+L_{g_1}Vu_0+T[L_{g_1}Vu_1+p_1]+O(T^2)\\ &\le -\alpha_3(|x|)+T[L_{g_1}Vu_1+p_1]+O(T^2) \qquad (4.19)\end{aligned}$$

where p_1 comes from (3.9) and has the following form:

$$p_1=\frac{L_{g_0}L_{g_0}V+(L_{g_1}L_{g_0}V+L_{g_0}L_{g_1}V)u_0+L_{g_1}L_{g_1}Vu_0^2}{2!} \; .$$

From item (ii) of Theorem 4.3 the $O(T)$ terms in (4.19) can be written as

$$L_{g_1}V\cdot\left(u_1+\frac{-(L_{g_1}L_{g_0}V+L_{g_0}L_{g_1}V)R(x)+(L_{g_1}L_{g_1}V)\cdot(L_{g_1}V)\cdot R(x)^2}{2!}\right)+\frac{L_{g_0}L_{g_0}V}{2!}\;, \qquad (4.20)$$

which by using (4.16) can be simplified to

$$L_{g_1}V\cdot\tilde{u}_1+\frac{L_{g_0}L_{g_0}V}{2}$$

Now the proof is completed by using (4.17), items (iii) and (iv) of the theorem and arguments identical to the ones used to prove Sontag's formula (see [22]).

Remark 4.4 Note that a large class of optimal and inverse optimal control laws satisfy the item (ii) of Theorem 4.3 (see [21, Sections 3.3, 3.4 and 3.5]). □

Remark 4.5 It is obvious from the proof of Theorem 4.3 that if one has $u_s=-L_{g_1}V\cdot R(x)$, then we can make the $O(T^{s+1})$ term in the Lyapunov difference expansion negative definite. The main obstruction to propagating this construction to terms $O(T^j)$, $j\ge s+2$ is that the constructed u_{s+1} will not have the same dependence on $L_{g_1}V$ that is crucial. □

Remark 4.6 An important point is that whenever $L_{g_1}V(x)\neq 0$ then in principle we can dominate the terms $p_s(x,u_0,\ldots,u_{s-1})$ by increasing the gain of u_s. However, due to saturation in actuators that is always present in the system, arbitrary increase in gain is not feasible. If we know an explicit bound on the control signals, such as $|u_j|\le\gamma(|x|)$, then the control that produces most decrease of $V(\cdot)$ under this constraint is

$$u_j(x)=\begin{cases}-\gamma(|x|) & \text{if } L_{g_1}V(x)\ge 0\\ \gamma(|x|) & \text{if } L_{g_1}V(x)\le 0\end{cases}\;.$$

We will use such a controller in the jet engine example presented in Section 5.2, below. □

Remark 4.7 We emphasize that one should exercise caution when applying the above reasoning. Indeed, the approach indicated above can work well only if the sampling period T is sufficiently small so that terms of order $O(T^{M+1})$ are negligible. However, $O(T^{M+1})$ terms depend in general on $u_0,u_1,\ldots,u_M$ and larger magnitudes of u_i will in general increase $O(T^{M+1})$ terms. Hence, making $O(T^i), i=1,2,\ldots,M$ more negative will in general mean that we are making $O(T^{M+1})$ less negligible. See, for example, the dependence of p_1 and p_2 (see equations (3.9) and (3.10)) on u_1. If we want to achieve more decrease in $O(T)$ in (3.3) by increasing the gain in u_1, then this will in general increase the magnitude of p_2 and, hence, of the $O(T^2)$ term in (3.3). Nevertheless, we will show in examples that a judicious choice of u_i and of the sampling period T does produce controllers that perform better than the original non-redesigned controller (2.5). □

Remark 4.8 We again emphasize that the procedure we described above is very flexible and we only outlined some of the main guiding principles and issues in controller redesign. However, even the simplest choice of redesigned controller of the form $u_{dt}(x)=u_{ct}(x)-TL_{g_1}V(x)$ will in general improve the transients of the sampled-data system. It is well known (see [21]) that control laws of this form robustify the controller to several classes of uncertainties and lead to improved stability margins. This theory has connections with inverse optimality and passivity and is relatively well understood.

Our results show that adding the $-L_{g_1}V$ terms of the form (4.13) robustifies the controller also with respect to sampling (i.e. small time varying time delays). In particular, the effect of $-L_{g_1}V$ on the sampled data system is explicitly stated in Theorem 4.2. Note that while one could design $-L_{g_1}V$ controllers using other techniques, its effect on the sampled data system is not clear without analysis similar to our main results. Moreover, the intricate interplay between the controller gain and the sampling period is revealed in our Theorem 4.2: adding too much negativity for too large sampling rates may lead to undesirable behaviour, and it is this situation where more sophisticated techniques exploiting the structure of p_s terms become important and show better performance, see Section 5.1.2 for an example. □

Remark 4.9 Note that the controller correction $u_1(\cdot)$ defined by (4.16) and (4.17) in Theorem 4.3 does not have the form (4.13). Hence, by exploiting the structure of the terms p_s, as well as the properties of the control law u_0 it is possible to obtain control laws that provide better Lyapunov function decrease than the general corrections (4.13).

Another approach to take into account higher order terms is obtained using the expansion (3.3) setting $u_i = 0$ for $i = 2, 3, \ldots$. This leads to the expansion

$$\frac{V(F^e_T(x, u_{dt})) - V(x)}{T} = L_{g_0}V + L_{g_1}V \cdot (u_0 + Tu_1) + \sum_{s=1}^{M} T^s p_s(x, u_0, u_1) + O(T^{M+1}). \tag{4.21}$$

Neglecting the $O(T^{M+1})$ term, for moderate values of M one may end up with an expression in u_1 which is easy to minimize, e.g., a quadratic form in u_1. Choosing the term u_1 as the minimizer of this expression we can *simultaneously* take into account several terms in (3.3) instead of looking at them separately as in Theorem 4.2. Clearly, this approach is less systematic than the recursive design in Theorem 4.2 and its feasability crucially depends on the system structure. If applicable, however, it may result in a redesign with higher accuracy and lower gain than the recursive design, see the example in Section 5.1.2. □

Remark 4.10 For nonlinear systems whose linearization is stabilizable, one can use linear techniques to guarantee stability and performance of the nonlinear system locally around an equilibrium using linear design techniques. Furthermore, close to the origin the simple emulated controller (2.5) often performs satisfactorily. Hence, in many cases our redesign is more important for states away from the origin, an observation which may facilitate the search for a suitable Lyapunov function, as it may happen that we can find a Lyapunov function satisfying Assumption 2.2 only on a subset of the state space. Then, we can use that Lyapunov function to redesign the controller only on this region of a state space. This situation is presented in the jet engine example that we consider in Section 5.2, below. □

4.2 Model reference based controller redesign

In this subsection, the goal of the controller redesign is to make the sampled data Lyapunov difference $V(F^e_T(x, u_{dt}(x))) - V(x)$ as close as possible to the continuous time Lyapunov difference $V(\phi(T, x)) - V(x)$, where $\phi(T, x)$ is the solution of the continuous time closed loop system (2.1), (2.2) at time $t = T$ and initialized at $x(0) = x$. This makes sense in situations when we want the bound on our sampled-data response with the redesigned controller to be as close as possible to the "ideal" bound on the response generated by sampling the solution of the continuous-time closed-loop system (2.1), (2.2). Note that this is a plausible goal when Assumption 2.2 holds. We will see that in this case the redesigned controller has a completely different form from the ones obtained in the previous subsection. We present an explicit construction for the case $u_{dt}(x) = u_0(x) + Tu_1(x)$ and comment on more general controller structures. We use the following notation:

$$\Delta V_{dt}(T, x, u) := V(F^e_T(x, u)) - V(x); \qquad \Delta V_{ct}(T, x) := V(\phi(T, x)) - V(x)\ .$$

The main result of this subsection is presented below:

Theorem 4.11 Suppose that Assumption 2.2 holds. Then we have

$$\Delta V_{ct}(T,x) - \Delta V_{dt}(T,x,u_0(x)) = O(T^2) \ . \tag{4.22}$$

Defining the redesigned controller by $u_{dt}(x) = u_0(x) + Tu_1(x)$, with

$$u_1(x) = \frac{1}{2}\frac{\partial u_0(x)}{\partial x}[g_0(x) + g_1(x)u_0(x)] \tag{4.23}$$

then we have

$$\Delta V_{ct}(T,x) - \Delta V_{dt}(T,x,u_{dt}(x)) = O(T^3) \ . \tag{4.24}$$

□

Proof: Using Theorem 3.1 we have that

$$\Delta V_{dt}(T,x,u_0+Tu_1) = T[L_{g_0}V + L_{g_1}V\cdot u_0] + T^2[L_{g_1}V\cdot u_1 + p_1(x,u_0)] + O(T^3) \ , \tag{4.25}$$

where p_1 is given by (3.9). Using Taylor series expansions of the solution $V(\phi(t,x))$ in t and evaluating them at $t=T$, we have:

$$V(\phi(T,x)) = V(x) + \sum_{i=1}^{\infty}\frac{T^i}{i!}\left.\frac{d^iV(\phi(t,x))}{dt^i}\right|_{t=0} \ .$$

Note that

$$\left.\frac{d^iV(\phi(t,x))}{dt^i}\right|_{t=0} = L^i_{g_0+g_1u_0}V(x) \ .$$

By direct calculations, we can compute:

$$\left.\frac{dV(\phi(t,x))}{dt}\right|_{t=0} = L_{g_0}V + L_{g_1}V\cdot u_0 \ , \tag{4.26}$$

which together with (4.25) shows that (4.22) holds. Computing further:

$$\begin{aligned}\left.\frac{d^2V(\phi(t,x))}{dt^2}\right|_{t=0} &= L^2_{g_0+g_1u_0}V(x) \\ &= \frac{\partial(L_{g_0}V + L_{g_1}V\cdot u_0)}{\partial x}[g_0+g_1u_0] \\ &= L_{g_0}L_{g_0}V + [L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V]u_0 + L_{g_1}L_{g_1}Vu_0^2 \\ &\quad + L_{g_1}V\cdot\frac{\partial u_0}{\partial x}[g_0+g_1u_0] \ . \end{aligned} \tag{4.27}$$

Using now (3.9), (4.23), (4.25), (4.26) and (4.27) the proof follows by comparing the T^0, T^1 and T^2 terms in the expansions of $\Delta V_{ct}(T,x)$ and $\Delta V_{dt}(T,x,u_{dt}(x))$.

Remark 4.12 Note that the correction (4.23) satisfies:

$$u_1(x) = \frac{1}{2}\left.\frac{du(\phi(t,x))}{dt}\right|_{t=0} \ . \tag{4.28}$$

Hence, the modification term is in some sense trying to extrapolate (predict) what the continuous-time control law would be like at time $T/2$. Note also that this controller does not depend on the Lyapunov function as opposed to control laws derived in Subsection 4.1. □

Remark 4.13 It may be tempting to conjecture that the control law of the form:

$$u_{dt}(x) = u_0(x) + \sum_{i=1}^{N}\frac{T^i}{(i+1)!}\left.\frac{d^iu(\phi(t,x))}{dt^i}\right|_{t=0} \tag{4.29}$$

for some fixed $N \in \mathbb{N}$ will yield:

$$\Delta V_{ct}(T,x) - \Delta V_{dt}(T,x,u_{dt}(x)) = O(T^{N+2}) \ .$$

However, this is not true even for $N = 2$, as we show next. By taking another derivative of (4.27) along solutions of (2.1), (2.2) we obtain

$$\begin{aligned}
\left.\frac{d^3V(\phi(t,x))}{dt^3}\right|_{t=0} &= \frac{\partial}{\partial x}\left\{L_{g_0}L_{g_0}V + [L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V]u_0 + L_{g_1}L_{g_1}Vu_0^2\right\}[g_0 + g_1u_0] \\
&\quad + \frac{\partial}{\partial x}\left\{L_{g_1}V \cdot \frac{\partial u_0}{\partial x}[g_0 + g_1u_0]\right\}[g_0 + g_1u_0] \\
&= L_{g_0}L_{g_0}L_{g_0}V + [L_{g_1}L_{g_0}L_{g_0}V + L_{g_0}L_{g_1}L_{g_0}V + L_{g_0}L_{g_0}L_{g_1}V]u_0 \\
&\quad + [L_{g_1}L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}L_{g_1}V + L_{g_1}L_{g_0}L_{g_1}V]u_0^2 + L_{g_1}L_{g_1}L_{g_1}V \cdot u_0^3 \\
&\quad + [L_{g_1}L_{g_0}V + L_{g_0}L_{g_1}V]\frac{\partial u_0}{\partial x}[g_0 + g_1u_0] \\
&\quad + 2L_{g_1}L_{g_1}V\frac{\partial u_0}{\partial x}[g_0 + g_1u_0]u_0 \\
&\quad + L_{g_0}L_{g_1}V \cdot \frac{\partial u_0}{\partial x}[g_0 + g_1u_0] + L_{g_1}L_{g_1}V \cdot \frac{\partial u_0}{\partial x}[g_0 + g_1u_0] \cdot u_0 \\
&\quad + L_{g_1}V \cdot \frac{\partial}{\partial x}\left[\frac{\partial u_0}{\partial x}[g_0 + g_1u_0]\right] \cdot [g_0 + g_1u_0] \ .
\end{aligned} \tag{4.30}$$

Let the control law be

$$u_{dt}(x) = u_0(x) + Tu_1 + T^2u_2 \tag{4.31}$$

where u_1 is given by (4.23) and u_2 is

$$u_2(x) = \frac{1}{3!}\frac{\partial}{\partial x}\left[\frac{\partial u_0}{\partial x}[g_0 + g_1u_0]\right] \cdot [g_0 + g_1u_0] \ .$$

Using (3.10), (4.30) and expressions for u_1 and u_2, direct computations show that

$$\begin{aligned}
\frac{1}{3!}\left.\frac{d^3V(\phi(t,x))}{dt^3}\right|_{t=0} - [L_{g_1}V \cdot u_2 + p_2(x,u_0,u_1)] &= \left(\frac{1}{3!} - \frac{1}{2!}\right)L_{g_1}L_{g_0}V \cdot u_1 + \left(\frac{4}{3!} - \frac{1}{2!}\right)L_{g_0}L_{g_1}V \cdot u_1 \\
&\neq 0 \ .
\end{aligned}$$

Hence, it is impossible in general to satisfy the above hypothesis. However, note that u_2 did cancel the term

$$\frac{T^2}{3!}L_{g_1}V \cdot \left.\frac{d^2u_0(\phi(t,x))}{dt^2}\right|_{t=0} = \frac{T^2}{3!}L_{g_1}V \cdot \frac{\partial}{\partial x}\left[\frac{\partial u_0}{\partial x}[g_0 + g_1u_0]\right] \cdot [g_0 + g_1u_0]$$

that is due to (4.30). This is true in general, if we use the controller structure (4.29), we will cancel some terms in $\Delta V_{ct}(T,x) - \Delta V_{dt}(T,x,u_{dt})$ but as we have shown above we can not in general make this difference of order higher than $O(T^3)$. Note that while we can not exactly cancel them, we can still use our technique to "minimize" the effect of higher order terms and reduce the mismatch between responses of sampled data and continuous time systems. □

Remark 4.14 It may seem too restrictive to use in our main results only the corrections u_1 in the redesigned controller. However, we observed in simulations that adding corrections u_k for $k \geq 2$ often does not improve the response considerably with respect to the redesigned controller with only the first correction u_1. The reason for this behaviour lies in the fact that the higher order corrections often introduce additional high gain which implies that the sampling rates have to be reduced in order to ensure satisfactory performance, cf. Remark 4.7 and the discussion in Section 5.1.1, below. For small sampling rates, however, the sampled continuous time trajectories usually show satisfactory results, hence the need for redesign is not given. This does not mean that the higher order terms cannot give valuable information, but this has to be handled with care, preferably using additional structure of the system, cf. Remark 4.9 and Section 5.1.2, below. □

Remark 4.15 The function $\beta \in \mathcal{KL}$ appearing in our Assumption 2.2(ii) does not enter explicitly in our feedback design methods, however, it is necessary for our controller redesing technique to be plausible. While for the high gain approach Example 4.1 illustrates this fact, for the model reference redesing technique this can be seen as follows: recall that the continuous time system satisfies

$$|x(t,x_0)| \le \beta(|x_0|,t) = \alpha_1^{-1}(\sigma(\alpha_2(s),t))$$

which is what we want to recover in our model reference redesign technique.

Assuming for simplicity of exposition that α_1^{-1} from Assumption 2.2(i) is Lipschitz and denoting the solutions of the sampled data system with emulated controller $u_0 = u$ by $x_s(k, x_0, u_0)$, by induction over the inequality for $\Delta V_{dt}(T, x, u_0(x))$ from Theorem 4.11 we obtain

$$\begin{aligned} |x_s(\lfloor \tau/T \rfloor, x_0, u_0)| &\le \alpha_1^{-1}(\sigma(\alpha_2(|x_0|), \lfloor \tau/T \rfloor) + O(T)) \\ &\le \alpha_1^{-1}(\sigma(\alpha_2(|x_0|), \lfloor \tau/T \rfloor)) + O(T) \quad = \quad \beta(|x_0|, \lfloor \tau/T \rfloor) + O(T). \end{aligned}$$

Here $\tau > 0$ is a fixed time and $\lfloor \tau/T \rfloor$ denotes the largest integer $k \le \tau/T$.

In contrast to this, for the redesigned controller u_{dt} from Theorem 4.11 we obtain

$$\begin{aligned} |x_s(\lfloor \tau/T \rfloor, x_0, u_{dt})| &\le \alpha_1^{-1}(\sigma(\alpha_2(|x_0|), \lfloor \tau/T \rfloor) + O(T^2)) \\ &\le \alpha_1^{-1}(\sigma(\alpha_2(|x_0|), \lfloor \tau/T \rfloor)) + O(T^2) \quad = \quad \beta(|x_0|, \lfloor \tau/T \rfloor) + O(T^2). \end{aligned}$$

i.e., for small sampling rates T the bound on the norm is closer to that of the continuous time system. □

5 Examples

In this section we illustrate our proposed techniques with two examples. For both examples we use several redesign techniques in order to demonstrate the flexibility of our approach and the different behaviour of the resulting discrete time controllers.

5.1 A first order example

Our first example is a simple first order nonlinear system given by

$$\dot{x} = x^3 + u. \tag{5.1}$$

For this system we use the stabilizing continuous time controller

$$u_0(x) = -x^3 - x\sqrt{x^4+1}$$

and the Lyapunov function

$$V(x) = \frac{x^2}{2}.$$

5.1.1 Lyapunov based redesign

Using the controller structure $u_{dt} = u_0 + Tu_1 + T^2 u_2$ we obtain the following expansion from Theorem 3.1.

$$\begin{aligned} \frac{V(F_T^e(x,u_{dt})) - V(x)}{T} &= x^4 + xu_0 \\ &\quad + T\Big(xu_1 + 2x^6 + \frac{5}{2}x^3u_0 + \frac{1}{2}u_0^2\Big) \\ &\quad + T^2\Big(xu_2 + \frac{5}{2}x^3u_1 + 4x^8 + \frac{13}{2}x^5u_0 + u_0u_1 + \frac{5}{2}x^2u_0^2\Big) \quad + \quad O(T^3) \end{aligned}$$

With this example we illustrate the redesign technique from Theorem 4.2, where we will design γ_j in such a way that the inequality

$$\frac{V(F_T^e(x, u_{dt})) - V(x)}{T} \leq -10x^2 + G_2(T, x)$$

holds, where the term "$-10x^2$" is a design parameter we chose arbitrarily and which could be replaced by other terms if desired. In order to achieve this, knowing u_0 we choose u_1 such that

$$x^4 + xu_0 + T\Big(xu_1 \; + \; 2x^6 + \frac{5}{2}x^3 u_0 + \frac{1}{2}u_0^2\Big) = -10x^2$$

holds and knowing u_0 and u_1 we choose u_2 such that

$$\begin{aligned} x^4 + xu_0 \quad &+ \quad T\Big(xu_1 \; + \; 2x^6 + \frac{5}{2}x^3 u_0 + \frac{1}{2}u_0^2\Big) \\ &+ \quad T^2\Big(xu_2 \; + \; \frac{5}{2}x^3 u_1 + 4x^8 + \frac{13}{2}x^5 u_0 + u_0 u_1 + \frac{5}{2}x^2 u_0^2\Big) \quad = \quad -10x^2 \end{aligned}$$

holds. Note that in both cases these equations are linear in u_1 and u_2, respectively, hence they can be solved explicitly.

Figure 5.1 shows the corresponding solution trajectories (left) and sampled data control values (right) for sampling rate $T = 0.2$ and innitial value $x_0 = 1$. The left figure shows the continuous time trajectory (no markers), and the sampled trajectory with $u_{dt} = u_0$ (marked with circles), with $u_{dt} = u_0 + Tu_1$ (crosses) and with $u_{dt} = u_0 + Tu_1 + T^2 u_2$ (squares). Note that since the Lyapunov function here is simply $V(x) = x^2/2$, its value is easily deduced from this figure. In particular, it is clearly visible that the Lyapunov difference (i.e., the difference between two consecutive values in this figure) is much smaller for the redesigned controllers.

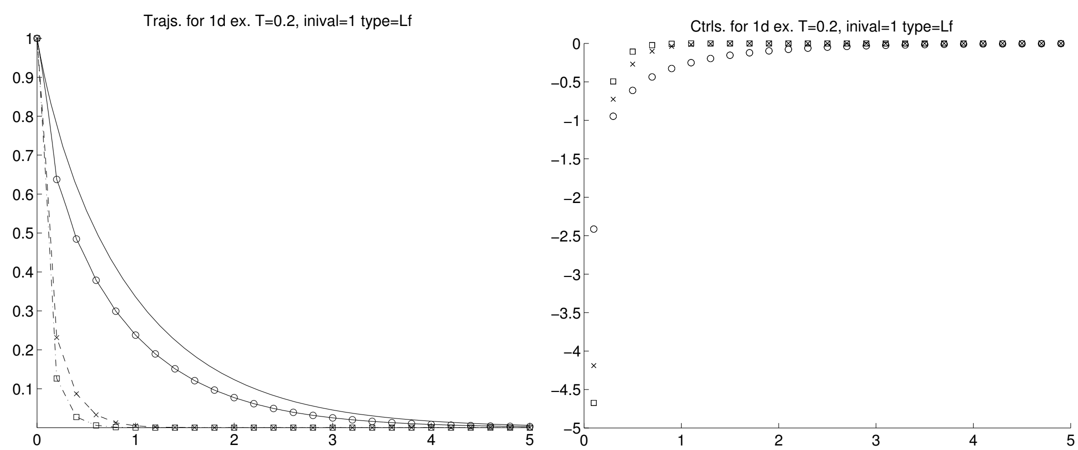

Figure 5.1: Solutions for controllers from Theorem 4.2

The right part of Figure 5.1 shows the corresponding control values for the sampled data controllers. Here one sees that the redesigned controllers introduce a higher gain, which also means that the higher order terms in (3.3) become larger and consequently for larger sampling rates the respective trajectory behave worse. Recall that all our results are asymptotic, i.e., they hold for sufficiently small sampling rate, where "sufficiently small" is substantially affected by the size of $|u_j|$, cf. Remark 4.7. Indeed, in the example above for the larger sampling rate $T = 0.3$ and $x_0 = 1$ the above redesign strategy turns out to yield oscillatory behaviour, cf. Figure 5.2, below. There are several ways to avoid this undesirable response. Introducing suitable gains for the correction terms u_1 and u_2 is one way, which does, however, affect the performance of the redesigned controller also in regions where it shows good behaviour. Using higher order terms in (3.3) is another way, however, the recursive design approach chosen here will typically result in even higher gain for u_3, u_4, ... and thus in u_{dt}, which is why our simulation experience suggests that this recursive approach is best applied for a moderate number of terms in the expansion, cf. Remark 4.14.

5.1.2 Lyapunov based minimizing redesign

For example 5.1 the minimizing redesign technique sketched in Remark 4.9 provides an alternative approach to take into account higher order terms in (3.3). For this example it turns out that the expansion (4.21) for $M = 5$ is a quadratic expression in u_1, hence it is easily minimized. For sampling rate $T = 0.3$ Figure 5.2 shows the corresponding trajectory together with the results for the controllers from the last section. The oscillatory behaviour (due to large remainder terms in the expansion) of the latter is clearly observable and in fact the trajectory with $u_{dt} = u_0 + Tu_1 + T^2u_2$ (marked with squares) is the least satisfactory — due to its high gain. In contrast to this, the minimizing strategy from Remark 4.9 with $M = 5$ (marked with diamonds) shows much better performance. In particular, this example shows that a more sophisticated redesign taking into account the higher order p_i terms may indeed outperform simpler redesign ideas which just add negativity to the Lyapunov difference, cf. Remark 4.8.

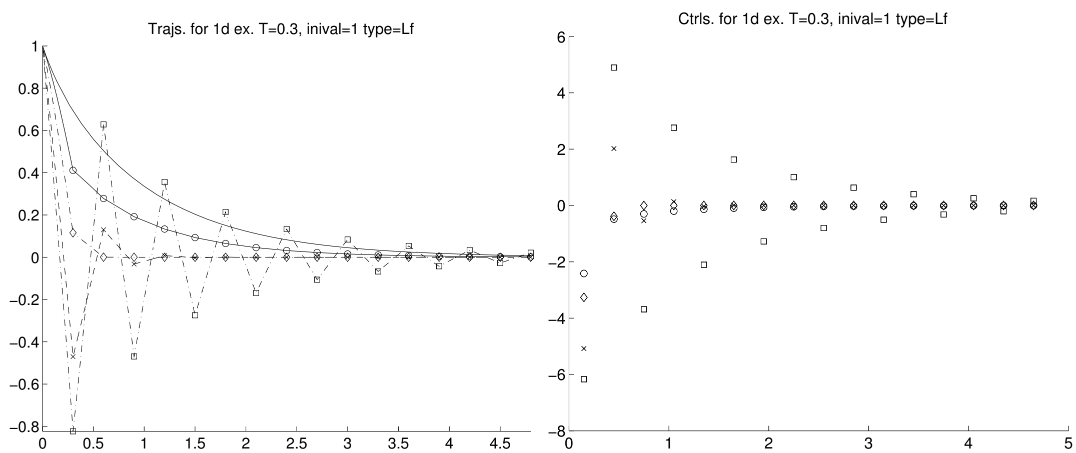

Figure 5.2: Solutions for controllers from Theorem 4.2 and Remark 4.9

5.1.3 Model reference based redesign

Let us finally illustrate the model reference controller correction u_1 from Theorem 4.11 for example 5.1. For this example, this formula yields

$$u_1(x) = \frac{1}{2}(3x^3\sqrt{x^4+1} + 3x^5 + x).$$

The Figures 5.3 and 5.4 compare this controller (marked with crosses) with the continuous time trajectory (unmarked) and the sampled continuous time controllers (marked with circles). As expected, this controller manages to keep the sampled data trajectory closer to the continuous time trajectory. In addition, it yields lower gain and, as Figure 5.4 shows, it can help avoiding oscillatory phenomena even for rather large sampling rates.

5.2 A second order example

As a second order example we consider the following model that is taken from [13, Section 2.4.3], a simplified Moore-Greitzer model of a jet engine with the assumption of no stall given by

$$\begin{aligned} \dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 &= -u \,, \end{aligned}$$

where x_1 and x_2 are respectively related to the mass flow and the pressure rise though the engine after an appropriate change of coordinates (see [13] for more details). The control law $u_0(x) = -k_1x_1 + k_2x_2$ and the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{c_0}{8}x_1^4 + \frac{1}{2}(x_2 - c_0x_1)^2$, have been derived in [13, pg. 72],

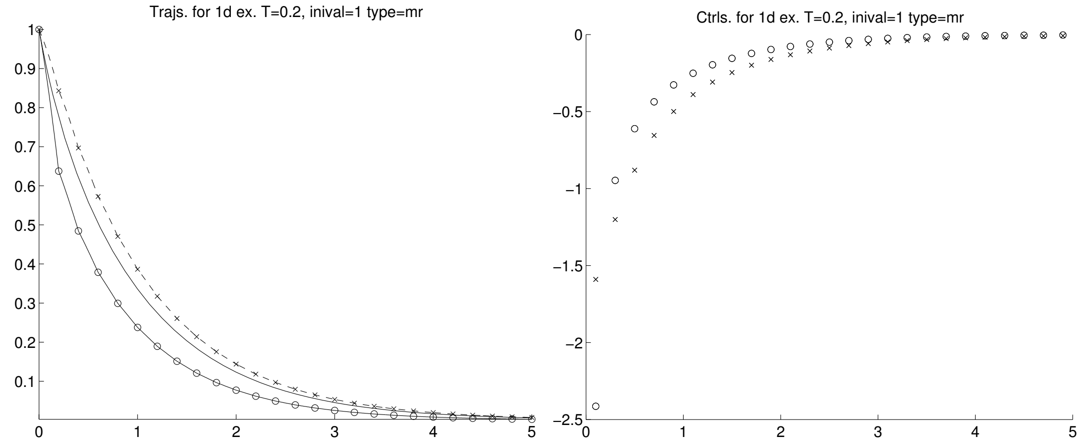

Figure 5.3: Solutions for controller from Theorem 4.11

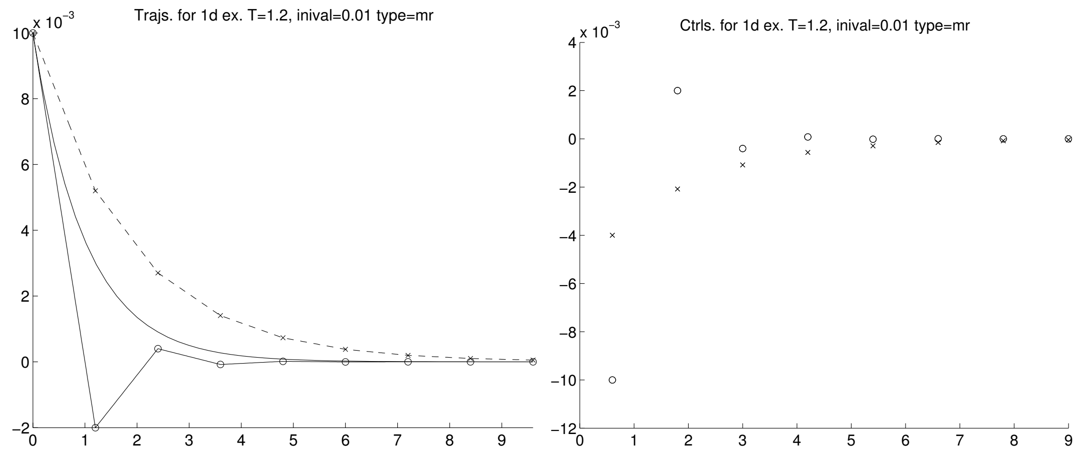

Figure 5.4: Solutions for controller from Theorem 4.11

where $k_1 = 1 + c_0 c_2 + \frac{9c_0^2}{8c_1}$, $k_2 = c_2 + c_0 + \frac{9c_0}{8c_1}$, $c_0 = c_1 + \frac{9}{8}$ and $c_1, c_2 > 0$ are design parameters. We use the choice $c_1 = \frac{7}{8}$, $c_2 = \frac{3}{7}$, which yield $c_0 = 2$, $k_1 = 7$, $k_2 = 5$. With these particular choices of parameters, we obtain

$$\begin{aligned} u_0(x) &= -7x_1 + 5x_2 && (5.2)\\ V(x) &= \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}(x_2 - 2x_1)^2 \ , && (5.3) \end{aligned}$$

and the closed loop system becomes

$$\begin{aligned} \dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 && (5.4)\\ \dot{x}_2 &= 7x_1 - 5x_2 \ , && (5.5) \end{aligned}$$

This continuous-time system has a very nice response and we will now proceed to redesign the controller (5.2) for digital implementation.

By simulation studies one observes that in this example we are in the situation of Remark 4.10: the simple emulated sampled–data controller (2.5) shows good results near the origin but exhibits rather poor performance, in particular large overshoots, for initial values farther away from the origin, which is in contrast to the nice response of the continuous–time system, whose trajectories converge very quickly with no overshoot. This nice response, however, is not captured by the Lyapunov function V from (5.3), which is due to the fact that for large values $c > 0$ the Lyapunov function (5.3) has level sets $V^{-1}(c)$ that are elongated very much along the x_2 axis. This yields very large functions

α_1^{-1} and α_2 in (2.8) and consequently the resulting function $\beta \in \mathcal{KL}$ does not satisfy performance requirements, because overshoots are just too big. In summary, the function V from (5.3) does not satisfy our Assumption 2.2(ii) and hence, following Remark 4.10, we try to find a better Lyapunov function outside a neighbourhood of the origin.

To this end, since simulations reveal that any sufficiently large ball around the origin is a forward invariant set for the trajectories, we try to use the Lyapunov function

$$V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \ . \tag{5.6}$$

Direct calculations show that

$$\begin{aligned} \dot{V}_1 &= \frac{\partial V_1}{\partial x_1}(-x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3) + \frac{\partial V_1}{\partial x_2}(7x_1 - 5x_2) \\ &= -\frac{3}{2}x_1^3 - \frac{1}{2}x_1^4 + 6x_1x_2 - 5x_2^2 \\ &= \underbrace{2\,(x_1^2 - \frac{3}{2}x_1^3 - \frac{1}{2}x_1^4)}_{Term\ 1} - \underbrace{(2x_1^2 - 6x_1x_2 + 5x_2^2)}_{Term\ 2} \ , \end{aligned} \tag{5.7}$$

where in the last step we just added and subtracted the term $2x_1^2$. Note that Term 1 in (5.7) is negative on the set $S_1 := \{x \in \mathbb{R}^2 \,:\, x_1 \notin [-4, +1],\ x_2 \in \mathbb{R}\}$ achieving the maximum value of about 18.1 on its complement. On the other hand, Term 2 is a positive definite quadratic form that is positive everywhere and radially unbounded. In particular, we have that the value of Term 2 is larger than 18.1 on the set $S_2 := \{x \in \mathbb{R}^2 \,:\, 2x_1^2 - 6x_1x_2 + 5x_2^2 > 18.1\}$. Hence, $\dot{V}_1$ in (5.7) is strictly negative on the set:

$$S := S_1 \cup (S_1^C \cap S_2) \ ,$$

where S_1^C denotes the complement of the set S_1. Hence, V_1 is a Lyapunov function on the above set and, moreover, it satisfies our Assumption 2.2 since it shows that trajectories are converging without any overshoot.

Using V_1 one sees that the complement S^c is a forward invariant neighbourhood of the origin, on which we can use the original Lyapunov function V to conclude asymptotic stability for the continuous time system and thus, by the results in [14], also for the emulated controller (2.5) for sufficiently small sampling rate. It turns out that for a large interval of sampling rates the emulated controller shows satisfactory results on S^c, thus we use (2.5) on S^c and perform our redesign on S. Overall asymptotic stability then follows from the asymptotic stability on S^c and the fact that our redesigned controller steers any trajectory to S^c in finite time.

For our redesign on the set S we now use V_1 as a control Lyapunov function. Based on Theorem 4.2 and Remark 4.6 and noting that $L_{g_1}V_1 = -x_2$, we implemented the controller

$$u_{dt}^{Lf}(x) = \begin{cases} u_0(x) + Tu_1^{Lf}(x) & \text{if } x \in S \\ u_0(x) & \text{otherwise} \end{cases}$$

with

$$u_1^{Lf}(x) = \begin{cases} x_1^2 + x_2^2 & \text{if } L_{g_1}V_1 = -x_2 < 0 \\ -(x_1^2 + x_2^2) & \text{otherwise} \end{cases} \ .$$

The chosen gain $\gamma(|x|) = |x|^2$ here was selected using the following guidelines: first we identified parameter domains (i.e., combinations of initial value x_0 and sampling rate T) for which the sampled continuous time controller did not yield satisfactory response. Particularly, we chose a region where the corresponding trajectories exhibit overshoots; for sampling rate $T = 0.1$ the domain $[-25, 25]^2$ (and specifically initial values close to the boundary of this set) turns out to be such a region. In the second step we tuned the gain $\gamma(|x|)$ such that the redesigned controller yields a significant improvement in the response in this region.

As an alternative to the Lyapunov function based controller we also used the model reference controller from Theorem 4.11, which here reads

$$u_1^{mr}(x) = \frac{35}{2}x_1 + \frac{21}{4}x_1^2 + \frac{7}{4}x_1^3 - 9x_2.$$

For the parameter region of interest it turned out that this controller yields a gain which induces too large remainder terms, hence we used a saturation for u_1^{mr} with $\pm|x|^2$. This choice also allows a "fair" comparison between the two controllers u_{dt}^{Lf} and u_{dt}^{mr} because this way their first order correction terms u_1^{Lf} and u_1^{mr} satisfy the same constraints.

Figure 5.5 shows the trajectories, (sampled) control values and the Lyapunov function $V_1(x)$ along the trajectories for the different controllers for initial value $x_0 = [22,\ 21]$[4] and sampling rate $T = 0.1$. The unmarked curves show the continuous time system, the curves marked with circles show the sampled continuous time controller $u_{dt} = u_0$. The Lyapunov based redesigned controller u_{dt}^{Lf} is marked with squares while the model reference type controller u_{dt}^{mr} is marked with crosses.

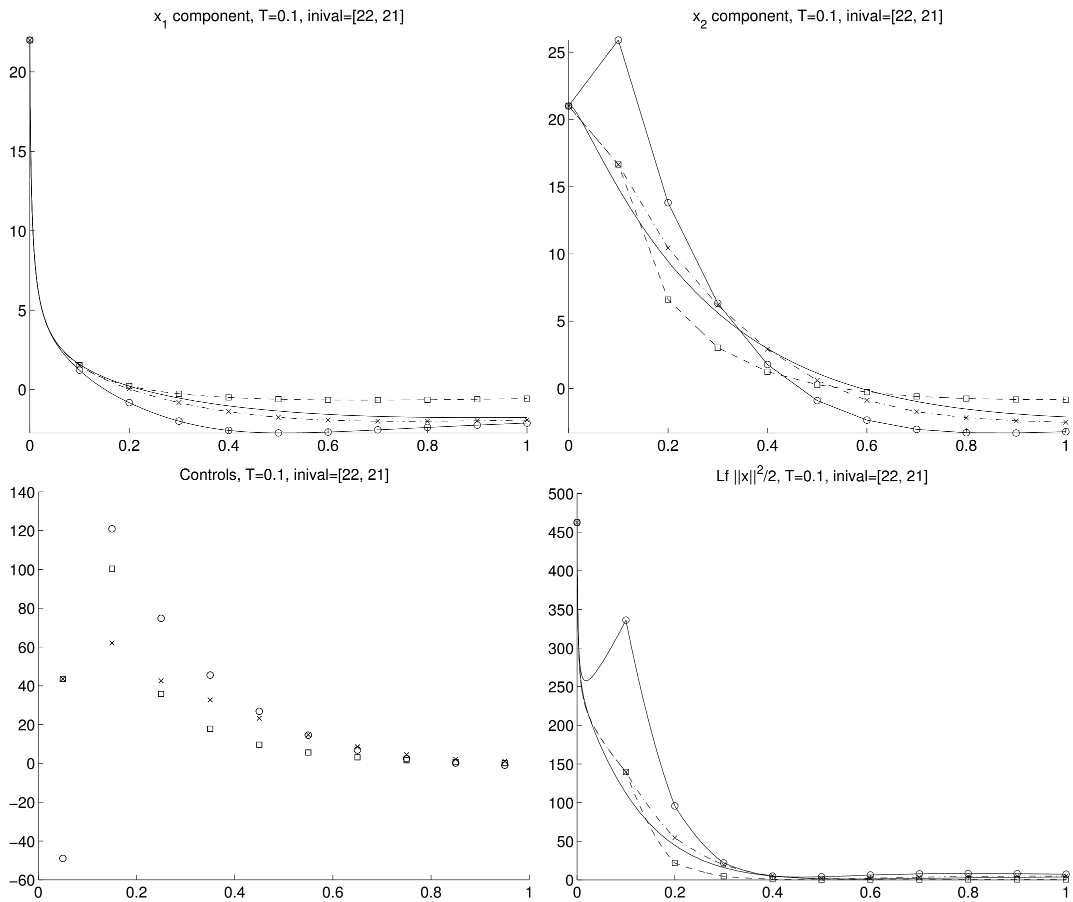

Figure 5.5: Solutions for different controllers

Note that on the first sampling interval the controllers u_{dt}^{Lf} and u_{dt}^{mr} coincide (u_1^{mr} saturates) and cause a substantial decrease of the Lyapunov difference compared to the emnulated controller $u_{dt} = u_0$. Afterwards the trajectory corresponding to u_{dt}^{Lf} tends to 0 faster, while u_{dt}^{mr} keeps the trajectory closer to the continuous time one. Both redesigned controllers avoid the overshoot in the x_2–component clearly visible in the sampled continuous time controller.

6 Conclusions

We have presented a method for a systematic redesign of continuous-time controllers for digital implementation. This method is very flexible and we illustrated its usefulness through several examples. Many variations of this method are possible and the main directions for further improvement are including dynamical and observer based controllers and relaxing some of the assumptions that we use

[4]This initial value has been chosen in order to illustrate the performance of our method and has no further physical meaning.

at the moment.

Acknowledgements

We thank the anonymous reviewers for their constructive comments and suggestions that improved the paper considerably.

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