Constructing Robust Feedback Laws by Set Oriented Numerical Methods

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In [8, 6] a numerical method for the construction of optimally stabilizing feedback laws was proposed. The method is based on a set oriented discretization of phase space in combination with graph theoretic algorithms for the computation of shortest paths in directed weighted graphs. The resulting approximate optimal value function is piecewise constant, yielding an approximate optimal feedback which might not be robust with respect to perturbations of the system. In this contribution we extend the approach to the case of perturbed control systems. Based on the concept of a multivalued game we show how to derive a directed weighted hypergraph from the original system and generalize the corresponding shortest path algorithm. The resulting optimal value function yields a robustly stabilizing approximate optimal feedback law. This note is an abbreviated version of [5]. For the proofs of the statements here we refer to the full paper.

1 Problem formulation

We consider the problem of optimally stabilizing the discrete-time perturbed control system

$$x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \ldots, \tag{1}$$

where $f : X \times U \times W \to X$ is continuous, $x_k \in X$ is the state of the system, $u_k \in U$ is the control input and $w_k \in W$ is a perturbation parameter, chosen from sets $X \subset \mathbb{R}^d, U \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^\ell$. In addition to the evolution law, we are given a continuous cost function $g : X \times U \to [0, \infty)$, that assigns the cost $g(x_k, u_k)$ to any transition $x_{k+1} = f(x_k, u_k, w_k)$.

Our goal is to derive an (optimal) *feedback law* $u : X \to U$ that *stabilizes* the system in the sense that for a certain subset $S \subset X$ any trajectory starting in S tends to some prescribed set $O \subset X$, while the *worst case accumulated cost* is minimized.

For a given initial point $x \in X$, a control sequence $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \in U^{\mathbb{N}}$ and a perturbation sequence $\mathbf{w} = (w_k)_{k\in\mathbb{N}} \in W^{\mathbb{N}}$ yield the *trajectory* $\mathbf{x}(x, \mathbf{u}, \mathbf{w}) = (x_k(x, \mathbf{u}, \mathbf{w}))_{k\in\mathbb{N}}$, defined by $x_0 = x$ and $x_{k+1} = f(x_k(x, \mathbf{u}, \mathbf{w}), u_k, w_k)$, $k = 0, 1, \ldots$,while the associated accumulated cost is given by

$$J(x, \mathbf{u}, \mathbf{w}) = \sum_{k=0}^{\infty} g(x_k(x, \mathbf{u}, \mathbf{w}), u_k).$$

In order to formalize the interplay between the control and the perturbation we employ a *game theoretic viewpoint* (see, e.g., [3]): We assume that the controlling player has to choose the value u_k first and that the perturbing player has the advantage of knowing u_k when choosing the perturbation value w_k. However, the perturbing player is not able to forsee future choices of the controlling one. More formally, we restrict the choice of perturbation sequences $\mathbf{w} \in W^{\mathbb{N}}$ to those that result from applying a *nonanticipating strategy* $\beta : U^{\mathbb{N}} \to W^{\mathbb{N}}$ to a given control sequence $\mathbf{u} \in U^{\mathbb{N}}$, i.e. we have $\mathbf{w} = \beta(\mathbf{u})$, with β satisfying $u_k = u'_k \forall k \le K \Rightarrow \beta(\mathbf{u})_k = \beta(\mathbf{u}')_k \forall k \le K$ for any two control sequences $\mathbf{u} = (u_k)_k, \mathbf{u}' = (u'_k)_k \in U^{\mathbb{N}}$. Let $\mathcal{B}$ denote the set of all nonanticipating strategies $\beta : U^{\mathbb{N}} \to W^{\mathbb{N}}$.

As mentioned, our goal is to find a feedback law $u : X \to U$ such that with controls $u_k = u(x_k)$, x_k approaches a given set $O \subset X$, regardless of how the perturbation sequence $\mathbf{w}$ is chosen. Accordingly, we assume that we know a compact robust forward invariant set $O \subset X$, i.e. for all $x \in O$ there is a control $u \in U$ such that $f(x, u, W) \subset O$. Since we are done with controlling the system once we are on O, we assume that $g(x, u) = 0$ for all $x \in O$ and all $u \in U$ and $g(x, u) > 0$ for all $x \notin O$ and all $u \in U$.

Our construction of the feedback law will be based on the *upper value function* $V : X \to [0, \infty]$,

$$V(x) = \sup_{\beta\in\mathcal{B}} \inf_{\mathbf{u}\in U^{\mathbb{N}}} J(x, \mathbf{u}, \beta(\mathbf{u})), \tag{2}$$

of the game (1), which fulfills the *optimality principle*

$$V(x) \quad = \quad \inf_{u\in U} \left[g(x, u) + \sup_{w\in W} V(f(x, u, w)) \right]. \tag{3}$$

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2 Multivalued games

A *multivalued game* is given by a multivalued map $F : X \times U \times W \rightrightarrows X$, where $X \subset \mathbb{R}^d$ is a closed set and $U \subset \mathbb{R}^m$, $W \in \mathbb{R}^\ell$ and the images of F are compact sets, together with a cost function $G : X \times X \times U \times W \to [0, \infty)$. We assume that $F(x, u, w) \neq \emptyset$ for all $x \in X$, $u \in U$, $w \in W$. Note that we have introduced a second state argument in G, which allows to associate different costs to the trajectories of the associated discrete inclusion.

For a given control sequence $\mathbf{u} = (u_k)_{k\in\mathbb{N}} \in U^\mathbb{N}$ and a given perturbation sequence $\mathbf{w} = (w_k)_{k\in\mathbb{N}} \in W^\mathbb{N}$, a *trajectory* of the game is given by any sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}} \in X^\mathbb{N}$ such that $x_{k+1} \in F(x_k, u_k, w_k)$, $k = 0, 1, 2, \ldots$. We denote by $\mathcal{X}_F(x, \mathbf{u}, \mathbf{w}) = \left\{(x_k)_k \in X^\mathbb{N} \mid x_0 = x, x_{k+1} \in F(x_k, u_k, w_k)\ \forall k \in \mathbb{N}\right\}$ the set of all trajectories of F associated to x, $\mathbf{u}$ and $\mathbf{w}$. The *accumulated cost* is given by $J_{(F,G)}(x, \mathbf{u}, \mathbf{w}) = \inf_{(x_k)_k\in\mathcal{X}_F(x,\mathbf{u},\mathbf{w})} \sum_{k=0}^\infty G(x_k, x_{k+1}, u_k, w_k)$. As in the previous section, we are interested in computing the upper value function

$$V_{(F,G)}(x) = \sup_{\beta\in\mathcal{B}} \inf_{\mathbf{u}\in U^\mathbb{N}} J_{(F,G)}(x, \mathbf{u}, \beta(\mathbf{u})), \quad x \in X, \tag{4}$$

of this game. By standard dynamic programming arguments [1] one sees that this function fulfills the optimality principle

$$V_{(F,G)}(x) = \inf_{u\in U} \sup_{w\in W} \inf_{x_1\in F(x,u,w)} \left\{G(x, x_1, u, w) + V_{(F,G)}(x_1)\right\}. \tag{5}$$

Observe that our original "single valued" game (1)–(2) can be recast in this multivalued setting by defining $F(x, u, w) := \{f(x, u, w)\}$ and $G(x, x_1, u, w) := g(x, u)$.

Definition 2.1 If (F_1, G_1) and (F_2, G_2) are two multivalued games such that $F_2(x, u, w) \subset F_1(x, u, w)$ for all x, u and w and $G_1(x, x', u, w) \leq G_2(x, x', u, w)$ for all $x, x' \in F_2(x, u, w)$ and all u and w, then (F_1, G_1) is called an *enclosure* of (F_2, G_2).

From this definition we immediately obtain the following proposition.

Proposition 2.2 *Let the game (F_1, G_1) be an enclosure of the game (F_2, G_2). Then $V_{(F_1,G_1)} \leq V_{(F_2,G_2)}$.*

The next proposition studies the convergence of the value functions $V_{(F_i,G_i)}$ of a sequence of games (F_i, G_i). In this proposition H denotes the Hausdorff distance for compact sets.

Proposition 2.3 *Let the sequence of games (F_i, G_i), $i \in \mathbb{N}$, be enclosures of the game (F, G) and assume $\sup_{x\in X,u\in U,w\in W} H(F_i(x, u, w), F(x, u, w)) \to 0$ and $\sup_{x,x_1\in X,u\in U,w\in W} |G_i(x, x_1, u, w) - G(x, x_1, u, w)| \to 0$ as $i \to \infty$. Assume furthermore that F is upper semi–continuous in x and that G is continuous in x and x_1, both uniformly in u and w and on compact subsets of X. In addition, we assume that there exists $\alpha \in \mathcal{K}_\infty$[1] with $G(x, x_1, u, w) \geq \alpha(d(x, O) + d(x_1, O))$ and $G_i(x, x_1, u, w) \geq \alpha(d(x, O) + d(x_1, O))$ for all $i \in \mathbb{N}$, $u \in U$, $w \in W$, and that $V_{(F,G)}$ is continuous on ∂O. Then for each compact set $K \subset X$ for which $\sup_{x\in K} V_{(F,G)}(x) < \infty$ we have*

$$\sup_{x\in K} |V_{(F_i,G_i)}(x) - V_{(F,G)}(x)| \to 0 \quad \textit{as } i \to \infty,$$

i.e., uniform convergence on compact sets in the domain of $V_{(F,G)}$.

3 Discretization of the game

We now describe the set oriented discretization technique which transforms our problem into a graph theoretic problem. Let $\mathcal{P}$ be a finite partition of X, i.e. $\mathcal{P}$ is a finite set of mutually disjoint subsets $P \subset X$. Define the map $\pi : X \to \mathcal{P}$, $\pi(x) = P$, $x \in P$, as well as $\rho : X \rightrightarrows X$, $\rho = \pi^{-1} \circ \pi$. Consider the multivalued game (F, G) with $F(x, u, w) = \rho(f(x, u, w))$ and $G(x, x_1, u, w) = g(x, u)$. From the optimality principle (5) we obtain

$$V_{(F,G)}(x) \quad = \quad \inf_{u\in U} \left\{ g(x, u) + \sup_{w\in W} \inf_{x_1\in F(x,u,w)} V_{(F,G)}(x_1) \right\}.$$

The right hand side defines an operator on real valued functions on X, the *dynamic programming operator* $L : \mathbb{R}^X \to \mathbb{R}^X$, $L[v](x) = \inf_{u\in U} \left\{g(x, u) + \sup_{w\in W} \inf_{x_1\in F(x,u,w)} v(x_1)\right\}$. Note that the optimal value function $V_{(F,G)}$ is, by definition, a fixed point of L, i.e. $L[V_{(F,G)}] = V_{(F,G)}$. Abusing notation, we identify the space $\mathbb{R}^\mathcal{P}$ with the subspace of real valued functions on X that are piecewise constant on the elements of the partition $\mathcal{P}$ (in fact, we view $v \in \mathbb{R}^\mathcal{P}$ as the function $v \circ \pi \in \mathbb{R}^X$). We define the projection $\varphi : \mathbb{R}^X \to \mathbb{R}^\mathcal{P} \subset \mathbb{R}^X$, $\varphi[v](x) = \inf_{x'\in\rho(x)} v(x')$, and the corresponding *discretized dynamic programming operator* $L_\mathcal{P} : \mathbb{R}^\mathcal{P} \to \mathbb{R}^\mathcal{P}$, $L_\mathcal{P} = \varphi \circ L$. Correspondingly, the discretized operator is given by $L_\mathcal{P}[v](x) = \inf_{x'\in\rho(x),u\in U} \left\{g(x', u) + \sup_{x_1\in F(x',u,W)} v(x_1)\right\}$, since $v \in \mathbb{R}^\mathcal{P}$ is constant on each element of $\mathcal{P}$, i.e. on

[1] A function $\gamma : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class $\mathcal{K}_\infty$, if, in addition, it is unbounded.

each set $F(x', u, w)$. We define the discretized optimal value function $V_{\mathcal{P}} \in \mathbb{R}^{\mathcal{P}}$ as the unique fixed point of $L_{\mathcal{P}}$ with $V_{\mathcal{P}}(P) = 0$ for all partition elements $P \in \mathcal{P}$ with $\overline{\pi^{-1}(P)} \cap O \neq \emptyset$. Then $V_{\mathcal{P}}$ satisfies the optimality principle

$$V_{\mathcal{P}}(x) = \inf_{x' \in \rho(x), u \in U} \left\{ g(x', u) + \sup_{x_1 \in F(x', u, W)} V_{\mathcal{P}}(x_1) \right\}. \tag{6}$$

We can formulate (6) equivalently in terms of an associated graph. To this end note that for any pair $(x, u) \in X \times U$, the set $F(x, u, W) \subset X$ is the union of a finite set of elements from the partition $\mathcal{P}$. In particular, the family $\{F(x', u, W) : (x', u) \in \rho(x) \times U\}$ of subsets of X is finite for any $x \in X$. Putting this in terms of a corresponding map on $\mathcal{P}$: each partition element P is mapped to a finite family $\{\mathcal{N}_i\}_{i=1,\ldots,i(P)}$, $\mathcal{N}_i \subset \mathcal{P}$, of subsets of $\mathcal{P}$ under all perturbations. Formally, we have a directed hypergraph $(\mathcal{P}, E)$ with the set $E \subset \mathcal{P} \times 2^{\mathcal{P}}$ of hyperedges given by $E = \{(P, \mathcal{N}) \mid \pi(F(x, u, W)) = \mathcal{N}$ for some $(x, u) \in P \times U\}$, or, equivalently, the multivalued map $\mathcal{F} : \mathcal{P} \rightrightarrows 2^{\mathcal{P}}$, $\mathcal{F}(P) = \{\pi(F(x, u, W)) : (x, u) \in P \times U\}$. If we define weights on the edges of this hypergraph by $\mathcal{G}(P, \mathcal{N}) = \inf\{g(x, u) : (x, u) \in P \times U, F(x, u, W) = \mathcal{N}\}$, then we can write (6) equivalently as

$$V_{\mathcal{P}}(P) = \inf_{\mathcal{N} \in \mathcal{F}(P)} \left\{ \mathcal{G}(P, \mathcal{N}) + \sup_{N \in \mathcal{N}} V_{\mathcal{P}}(N) \right\}. \tag{7}$$

As in the case of an unperturbed system (with associated directed weighted graph), one can directly derive a shortest path algorithm (cf. [2]) for the computation of $V_{\mathcal{P}}$ from this optimality principle, see [5].

4 Convergence Analysis

We now show in which sense the approximate optimal value function constructed in the preceeding section converges to the true one as the underlying partitions are refined.

Proposition 4.1 *Consider the discretized optimal value function $V_{\mathcal{P}}$ and the optimal value function $V_{(F,G)}$ from (4). If $V_{(F,G)}$ is continuous on ∂O, then these functions are related by $V_{\mathcal{P}}(x) = \inf_{x' \in \rho(x)} V_{(F,G)}(x')$.*

Note that in order to obtain this assertion it is sufficient that the union of those partition elements that have nonempty intersection with O form a neighborhood of O. If this is true, one can actually drop the assumption on the continuity of $V_{(F,G)}$ on ∂O. We now consider a sequence of increasingly finer partitions of X and ask under which conditions the corresponding approximate optimal value functions converge to the value function of the game (f, g). In a *nested* sequence of partitions, each element of a partition is contained in an element of the preceding partition.

Theorem 4.2 *Let $(\mathcal{P}_i)_{i \in \mathbb{N}}$ be a nested sequence of partitions of X such that $\sup_{x \in X} H(\rho_i(x), \{x\}) \to 0$ as $i \to \infty$. Assume that $g(x, u)$ is continuous, that $g(x, u) > 0$ for $x \notin O$ and that $V_{(f,g)}$ is continuous on ∂O. Then*

$$\|V_{\mathcal{P}_i}|_{K_i} - V_{(f,g)}|_{K_i}\|_\infty \to 0 \quad \text{as } i \to \infty$$

for every compact set $K \subseteq X$ on which $V_{(f,g)}$ is continuous and $K_i = \bigcup_{P \in \mathcal{P}_i,\, \pi^{-1}(P) \subset K} \pi^{-1}(P)$ being the largest subset of K which is a union of partition elements $P \in \mathcal{P}_i$.

If we assume furthermore that the set of discontinuities of $V_{(f,g)}$ has zero Lebesgue measure, then

$$\|V_{\mathcal{P}_i}|_K - V_{(f,g)}|_K\|_{L^1} \to 0 \quad \text{as } i \to \infty$$

on every compact set $K \subseteq X$ with $\sup_{x \in K} V_{(f,g)}(x) < \infty$.

Corollary 4.3 *Under the assumptions of Theorem 4.2 we have $V_{\mathcal{P}_i}(x) \to V_{(f,g)}(x)$ as $i \to \infty$ for Lebesgue-almost all $x \in K$, where K is any compact subset of the domain of $V_{(f,g)}$.*

5 Feedback Construction

We use the approximate optimal value function $V_{\mathcal{P}}$ and the optimality principle (3) in order to construct an approximate optimal feedback. More precisely, for any point $x \in S_0$, $S_0 := \{x \in X : V_{(f,g)}(x) < \infty\}$, we define

$$u_{\mathcal{P}}(x) = \operatorname{argmin}_{u \in U} \max_{w \in W} \{g(x, u) + V_{\mathcal{P}}(f(x, u, w))\}.$$

We can immediately adapt Theorem 3 from [6] in order to obtain a statement about the performance of this feedback. The following result in particular shows that the feedback is robust with respect to arbitrary perturbations of the system.

Theorem 5.1 *Let the assumptions of Theorem 4.2 be satisfied. Let* $D \subset S_0$ *be an open set with compact closure, such that* $\overline{D} \subset S_0$*,* $O \subset D$ *and on which* $V_{(f,g)}$ *is continuous. Let* $c > 0$ *be such that the inclusion* $D_c(i_0) := V_{\mathcal{P}_{i_0}}^{-1}([0,c]) \subset D$ *holds for some* $i_0 \in \mathbb{N}$*. Then there exists a function* $\delta : \mathbb{R} \to \mathbb{R}$ *with* $\lim_{\alpha\to 0}\delta(\alpha) = 0$ *such that for all sufficiently small* ε*, all sufficiently large* i*, all* $\eta \in (0,1)$*, all* $x_0 \in D_c(i)$ *and all perturbation sequences* $(w_k)_k \in W^{\mathbb{N}}$*, the trajectory generated by* $x_{k+1} = f(x_k, u_{\mathcal{P}_i}(x_k), w_k)$ *satisfies*

$$V(x_k) \le \max\left\{ V(x_0) - (1-\eta)\sum_{j=0}^{k-1} g(x_j, u_{\mathcal{P}_i}(x_j)), \delta(\varepsilon/\eta) + \varepsilon \right\}.$$

Remark 5.2 A particular application of our result is to robustify the feedback construction from [6] with respect to small perturbations which may be due, e.g., to discretization errors resulting from the numerical computation of the discrete time system from an ordinary differential equation. For this purpose, a particularly convenient way is to consider an "ε-inflated" system related to the original unperturbed system. More precisely, given an unperturbed control system $f : X \times U \to X$, one considers the perturbed system $x_{k+1} = f(x_k, u_k) + \varepsilon w_k$, $k = 0, 1, \ldots$, with $w_k \in [-1,1]^d$ for some (small) $\varepsilon > 0$. In the numerical realization, the sets $F(x,u,W) = f(x,u) + \varepsilon[-1,1]^d$ are easy to construct using ideas from rigorous discretization, see [7, 4].

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