Calculating the domain of attraction: Zubov's method and extensions

F. Camilli

Sez. di Matematica per l'Ingegneria, Dip. di Matematica Pura e Applicata, Università de l'Aquila, 67040 Roio Poggio (AQ), Italy
camilli@ing.univaq.it

L. Grüne

Mathematisches Institut, Fakultät für Mathematik und Physik, Universität Bayreuth, 95440 Bayreuth, Germany
lars.gruene@uni-bayreuth.de

F. Wirth

The Hamilton Institute, National University of Ireland Maynooth, Maynooth, Co. Kildare, Ireland
fabian.wirth@nuim.ie

Abstract

In a seminal work V.I. Zubov has described a constructive method to obtain Lyapunov functions. This method is the first to allow numerical construction of domains of attraction for general nonlinear systems. In this paper we describe recent generalizations of this method that is applicable for systems subject to perturbations or control inputs.

Key words: Domain of attraction, robust Lyapunov function, control Lyapunov function, Zubov's method.

1 Introduction

In his seminal thesis Lyapunov showed that a sufficient stability condition can be obtained in terms of a positive definite function that decreases along the trajectories of the system, or, as we say today, the existence of a Lyapunov function implies asymptotic stability. Converse theorems state that certain stability properties imply the existence of a Lyapunov function. Early results in this direction were obtained by Persidskii, see the discussion in [14, Chapter VI], Massera [19] and Kurcveĭl [17]. However, in recent times these results have been extended in several directions to cover perturbed systems and differential inclusions [18,10,24]. Further constructive approaches valid for C^2 systems and based on approximations by radial basis functions, respectively on a linear programming approach have recently been described in [12,13].

For nonlinear systems one of the interests in providing a Lyapunov function lies in the fact, that it automatically provides an estimation of the domain of attraction. This is discussed for instance in the books [14,15] and further information can be obtained in the articles [25,26,20]. However, a priori, a Lyapunov function only gives a conservative estimate for the domain of attraction in that a subset of the domain is described. It is often quite difficult to get an idea how good an approximation has been obtained.

In this paper we present a method that was originally developed by Zubov [27] which allows in principle for the exact calculation of the whole domain of attraction and we describe extension to perturbed and controlled systems. We begin by summarizing Zubov's result. Assume we are given a system of ordinary differential equations of the form

$$\dot{x} = f(x)\,, \quad x \in \mathbb{R}^n \tag{1}$$

with an asymptotically stable fixed point x^*. We assume throughout that $x^* = 0$ and that the vector field f is locally Lipschitz continuous on $\mathbb{R}^n$, so that existence and uniqueness of solutions is guaranteed. By $\varphi(t; x_0)$ we denote the solution at time t of (1) satisfying the initial condition $\varphi(0; x_0) = x_0$. The object of interest is now the domain of attraction (of the asymptotically stable fixed point $x^* = 0$), which is defined by

$$\mathcal{D}(0) := \{x \in \mathbb{R}^n \mid \varphi(t; x) \to 0, \text{ for } t \to \infty\}\,.$$

Unfortunately, there are few restrictions as to what a domain of attraction can look like. Any open subset of $\mathbb{R}^n$ that is diffeomorphic to $\mathbb{R}^n$ can be a domain of attraction, [14,26]. Thus the question of computation of such domains becomes crucial but also potentially difficult.

One of the milestones in the analysis of domains of attraction is Zubov's theorem [27] which links the domain of attraction of an asymptotically stable fixed point $x^* = 0$ of (1) to the first order partial differential equation

$$Dv(x) \cdot f(x) = -h(x)(1 - v(x))\sqrt{1 + \|f(x)\|^2}\,. \tag{2}$$

Namely, under suitable assumptions on h, the equation (2) has a unique solution v that vanishes in the origin. Furthermore, the set $v^{-1}([0,1))$ is equal to the domain of attraction $\mathcal{D}(0)$, or equivalently

$$\mathcal{D}(0) = \{x \in \mathbb{R}^n \mid 0 \le v(x) < 1\}\,.$$

The importance of this result is twofold. On the one hand it is an *existence* result, which states that a Lyapunov function exists that characterizes the whole domain of attraction. And in fact, there exist many of them since we have freedom in the choice of the auxiliary function h which will in general lead to different solutions of (2). On the other hand the equation also opens the door for computational approaches, as in order to find the Lyapunov function, we "only" have to solve equation (2).

These results are presented in several books, see [14,15]. An overview over applications is given in [16]. For the case of real-analytic systems a constructive procedure is presented in [14,27] that allows for the approximation of the domain of attraction. This method was extended and simplified in [25].
In this note we present some recent results obtained in [4,6] which show an extension of Zubov's result for the case of perturbed systems. In the following Section 2 we present results in the spirit of Zubov's work that are applicable to perturbed systems. In Section 3 analogous results are derived for the case of controlled systems. We comment on the merits and the limitations of the method in Section 4.

2 Perturbed Systems

We consider perturbed systems of the form

$$\dot{x}(t) = f(x(t), a(t)), \tag{3}$$

where $a : \mathbb{R} \to A \subset \mathbb{R}^m$ is a measurable function taking values in the compact set A. Here $x \in \mathbb{R}^n$ denotes the state of the system and a represents an unknown time-varying perturbation acting on the system and $f : \mathbb{R}^n \times A \to \mathbb{R}^n$. In order to ensure existence and uniqueness of solutions we assume that f in (3) is continuous and locally Lipschitz continuous in x where the Lipschitz constants can be chosen independently of $a \in A$. We are interested in stability properties that are robust with respect to unknown perturbations taking values in A, or in other words in stability properties that hold uniformly for

$$\mathcal{A} := \{a : \mathbb{R} \to A \mid a \text{ is measurable}\}.$$

The solution of (3) corresponding to the initial condition $\varphi(t) = x_0$ and to a particular choice for the perturbation function $a \in \mathcal{A}$ is denoted by $\varphi(t; x_0, a)$. The Euclidean norm in $\mathbb{R}^n$ is denoted by $\| \cdot \|$.
As we are interested in domains of attraction we assume that the point $x^* = 0$ has a robust local stability property. To be precise, we need the following assumption.

Assumption 1 *Consider system* (3). *We assume that there is a ball of radius $r > 0$ around $x^* = 0$ and constants $M \geq 1, \beta < 0$ such that for all $a(\cdot) \in \mathcal{A}$ and all x with $\|x\| \leq r$ we have*

$$\|\varphi(t; x, a)\| \leq M e^{\beta t} \|x\|.$$

It may now seem that we have made two very strict assumptions that make the problem a not very interesting one. First of all we have implicitly assumed that the fixed point $x^* = 0$ is not perturbed by any of the perturbations, or in other words that

$$f(0, a) = 0, \quad \text{for all } a \in A,$$

a case that is rarely met in applications. Secondly, we have assumed local exponential decay which is maybe a more reasonable assumption, but still it gives rise to the question

if this assumption is really necessary.
Indeed, the problem can be studied in a more general context and the results that we present here remain essentially the same. We restrict ourselves to the present case for ease of exposition. For the more general case of compact attractors which includes the case that a fixed point is perturbed by the perturbations and for the case of general attraction rates we refer to [4,5,7]. Our main remark at this point is, that the results remain essentially the same after some obvious modifications.
We now define the domain of attraction robust with respect to perturbations.

Definition 2 *If the system* (3) *satisfies Assumption 1 we define the* uniform robust domain of attraction *of* $x^* = 0$ *by*

$$\mathcal{D}_0 = \left\{ x_0 \in \mathbb{R}^n \mid \begin{array}{l} \text{there exists a continuous function } \beta \text{ with } \beta(t) \to 0 \text{ as } t \to \infty \\ \text{such that } \|\varphi(t;x_0,a)\| \leq \beta(t) \text{ for all } t > 0,\ a \in \mathcal{A} \end{array} \right\}. \quad (4)$$

It may seem surprising, that in the definition of the robust domain of attraction we require a uniform attraction rate. This rate is given by the function β that is an upper bound for the decay for all perturbations. Indeed, it may be shown that this is not at all a severe restriction. It follows from [6, Prop. 2.3 (v)] that if x_0 is a point such that all trajectories $\varphi(t;x_0,a)$ converge to zero, but an upper bound β as in the definition of $\mathcal{D}_0$ does not exist, then $x_0 \in \partial\mathcal{D}_0$. Furthermore, such points cannot exist, if the set $\{f(x,a) \mid a \in A\}$ is convex for all $x \in \mathbb{R}^n$.
We now want to characterize the robust domain of attraction in a way that makes it accessible to theoretic considerations. The idea is to use methods from optimal control. To this end we define a function $g : \mathbb{R}^n \times A \to \mathbb{R}_+$ that associates to each point and perturbation value a certain cost. Then we can define the value function (which may take on the value ∞) $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$V(x) = \sup_{a \in \mathcal{A}} \int_0^{+\infty} g(\varphi(t;x,a), a(t))dt\,, \quad (5)$$

and in order to obtain a function that is finite everywhere we apply the transformation

$$v(x) = 1 - e^{-V(x)}. \quad (6)$$

The idea is now to define the function g and thus the optimal control problem in such a manner, that V, respectively v characterize the robust domain of attraction. The construction needed to do this is simple. The unique feature of the points in the domain of attraction is that all trajectories emanating from these points converge to zero. If $x = 0$ is the only zero of g and g is also bounded away from 0 outside of a bounded neighborhood of 0 then it is clear that for all trajectories that do not converge to zero the integral in (5) will diverge to infinity. If a suitable condition on g in a neighborhood is added then we obtain, that the value of the integral in (5) is finite if and only if the trajectory converges to zero. Hence a point x belongs to $\mathcal{D}_0$ if and only if $V(x)$ is finite or equivalently if and only if $v(x) < 1$. We now state the precise assumptions that lead to the desired result.

Assumption 3 *The running cost $g : \mathbb{R}^n \times A \to \mathbb{R}$ is continuous and satisfies*

(i) *For any $a \in A, g(0,a) = 0$ and $g(x,a) > 0$ for all $x \neq 0$.*
(ii) *There exists a constant $g_0 > 0$ such that $g(x,a) \geq g_0$ for all $a \in A, x_0 \geq r$.*
(iii) *The function g is locally Lipschitz continuous in x uniformly in a, i.e. for every $R > 0$ there exists a constant L_R such that*

$$\|g(x,a) - g(y,a)\| \leq L_R \|x - y\| \text{ for all } \|x\|, \|y\| \leq R, \text{ and all } a \in A\,.$$

Since g is nonnegative it is immediate that $V(x) \geq 0$ and $v(x) \in [0,1]$ for all $x \in \mathbb{R}^n$. Under the previous assumption the following result may be shown.

Proposition 4 ([6]) *Consider system* (3) *and a function $g : \mathbb{R}^n \times A \to \mathbb{R}$ such that the Assumptions 1 and 3 are satisfied. Then $v : \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying*

(i) *$v(x) < 1$ if and only if $x \in \mathcal{D}_0$.*
(ii) *$v(x) = 0$ if and only if $x = 0$.*
(iii) *$v(x) \to 1$ for $x \to x_0 \in \partial\mathcal{D}_0$ and for $\|x\| \to \infty$.*

Thus in the construction of v in the manner described above we have found a function that characterizes the robust domain of attraction just as the solution of Zubov's classical equation describes the domain of attraction. What is more is that if we consider the following straightforward generalization of Zubov's equation

$$\inf_{a \in A} \{-Dv(x)f(x,a) - (1 - v(x))g(x,a)\} = 0 \qquad x \in \mathbb{R}^n\,, \tag{7}$$

then we obtain the following theorem.

Theorem 5 ([6]) *Consider the system* (3) *and a function $g : \mathbb{R}^n \times A \to \mathbb{R}$ such that the Assumptions 1 and 3 are satisfied. Then the partial differential equation* (7) *has a unique bounded and continuous viscosity solution v on $\mathbb{R}^n$ satisfying $v(0) = 0$.*
This solution coincides with v defined in (6)*. In particular, we have the characterization*

$$\mathcal{D}_0 = \{x \in \mathbb{R}^n \,|\, v(x) < 1\}\,.$$

A few words of discussion are in order to interpret this result. First, the reader will note that the first order partial differential equation (7) is solved in the viscosity sense. Indeed, an example in [6] shows that classical solutions may not exist, so that we have to resort to a solution sense that allows for solutions that are not (everywhere) differentiable. A good introduction to the corresponding theory is provided by the book [2].
Secondly, as in the case of Zubov's original result the importance of the result is twofold. First of all it provides an existence result and then it also provides way of calculating domains of attraction. It has to be mentioned however, that the existence result was already shown to be true in [18,24]. On the other hand, these papers do not propose methods for the calculation of robust domains of attraction. It can be seen from examples

in [4] that Theorem 5 indeed provides a way for the calculation of domains of attraction. Numerical details of the method are also discussed in [4].

3 Control Lyapunov Functions

In this section we investigate the question of existence of Lyapunov functions for systems with inputs or control systems. While for (perturbed) ordinary differential equations the property of interest is stability, for systems with control inputs a basic question concerns the existence of control functions steering the system to a desired target. Consider a control system

$$\dot{x}(t) = f(x(t), u(t)) \,, \tag{8}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ denotes the input, and where f is sufficiently regular with $f(0,0) = 0$. The set of control values $U \subset \mathbb{R}^m$ is a closed set and the space of admissible control functions is given by

$$u \in \mathcal{U} := L^\infty([0,\infty), U) \,.$$

Solutions corresponding to an initial value x and a control $u \in \mathcal{U}$ at time t are denoted by $\varphi(t,x,u)$, which are defined on a maximal positive interval of definition $[0, T_{\max}(x,u))$, where we do not exclude the case that $T_{\max}(x,u)) < \infty$. i.e. that solutions explode. We call a point $x_0 \in \mathbb{R}^n$ *asymptotically controllable* to 0 if there exists a measurable, essentially bounded function $u_0 : \mathbb{R}_+ \to \mathbb{R}^m$ such that the corresponding solution $\varphi(t, x_0, u_0)$ of (8) satisfies $\varphi(t, x_0, u_0) \to 0$ for $t \to \infty$. The *domain of asymptotic null-controllability* is the collection of all points that are asymptotically controllable to 0, i.e.

$$\mathcal{D}_c := \{x \in \mathbb{R}^n \,|\, \text{there exists } u \in \mathcal{U} \text{ with } \|\varphi(t,x,u)\| \to 0 \text{ for } t \to \infty\} \,.$$

Also for this concept there is a Lyapunov theory which has received widespread attention in recent years. In contrast to the case of ordinary differential equations, where smooth Lyapunov functions always exist for asymptotically stable systems, it is not reasonable to require too many regularity properties of Lyapunov functions for controllability questions. For this reason it is now standard to formulate the concept of a *control Lyapunov function* in nondifferential terms. To formulate this recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite, if $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(x) = 0$ iff $x = 0$. The function V is proper if preimages of compact sets are compact. A positive definite, proper function V is called a control-Lyapunov function (CLF) for (8) if there is a positive definite function W such that for every compact set $X \subset \mathbb{R}^n$ there is a compact set U_X of control values so that V is a continuous viscosity supersolution of

$$\max_{u \in U_X} -DV(x) f(x,u) \geq W(x) \,, \quad x \in X \,. \tag{9}$$

For the definition of viscosity solutions we refer to [2].
While design techniques using Lyapunov functions have been popular in applied control theory for a long time, the systematic study of control Lyapunov functions only started

with Artstein [1], who proved for the case of systems affine in the control term u that the existence of a smooth CLF is equivalent to stabilizability by continuous state feedback. For general systems of the form (8) the existence of a global continuous CLF is equivalent to global asymptotic null controllability [22]. Since then a rich body of literature has been created in this field. We do not have the space here to discuss the various contributions in sufficient detail and refer the reader to [11,9,21] for an overview and further pointers to the literature.

Our aim is to determine a CLF as (i) an optimal value function of a suitable control problem and (ii) as unique viscosity solution to a suitable Hamilton-Jacobi equation which is a generalization of Zubov's equation.

To this end we need the following regularity assumption on f [1]. We assume that there exists $\gamma \in \mathcal{K}_\infty$ such that for any $R > 0$ there is $C_R > 0$ with

$$\|f(x,u) - f(y,u)\| \leq C_R(1 + \gamma(\|u\|))\|x - y\|\,, \tag{10}$$

for all x, y with $\|x\|$, $\|y\| \leq R$.

Furthermore, we assume a local controllability assumption, similar to the approach for perturbed systems. There exists an open ball $B(0,r)$, a constant $\bar{u} > 0$, and $\beta \in \mathcal{KL}$ such that for any $x \in B(0,r)$ there exists $u_x \in \mathcal{U}$ with $\|u_x\|_\infty \leq \bar{u}$, $T_{\max}(x, u_x) = \infty$ and

$$\|\varphi(t, x, u_x)\| \leq \beta(\|x\|, t)\,, \quad \forall t \geq 0\,. \tag{11}$$

It is known [23] that for any $\beta \in \mathcal{KL}$ there exist two functions $\alpha_1, \alpha_2 \in \mathcal{K}^\infty$ such that $\beta(r,t) \leq \alpha_2(\alpha_1(r)e^{-t})$. For ease of presentation we will work with these two functions. Under our standing assumptions it may be shown that the domain $\mathcal{D}_c$ is open, connected and for every $x \in \mathcal{D}_c$ there is a $u \in \mathcal{U}$ such that $\varphi(t,x,u) \in \mathcal{D}_c$ for all $t \geq 0$. Note that the domain of nullcontrollability $\mathcal{D}D_0$ is in general not diffeomorphic to $\mathbb{R}^n$. This is in contrast to the theory of domains of attraction of (perturbed) ordinary differential equations. In the case of asymptotically stable fixed points the domain of attraction is diffeomorphic to $\mathbb{R}^n$ even for perturbed systems, see e.g. [6,26].

We describe how to characterize the domain of asymptotic null-controllability via an optimal control problem. In order to set up the problem we need a running cost $g : \mathbb{R}^n \times U \to \mathbb{R}$. The assumptions on g are as follows:

The function $g : \mathbb{R}^n \times U \to \mathbb{R}$ is continuous and satisfies (10) with the same $\gamma \in \mathcal{K}_\infty$ as f. Furthermore, for all $c > 0$ we have

$$\inf\{g(x,u) \mid \|x\| \geq c, u \in U\} =: g_c > 0\,.$$

[1] As usual we call a function α of class $\mathcal{K}_\infty$ if it is a homeomorphism of $[0,\infty)$, a continuous function β in two real nonnegative arguments is called of class $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first and decreasing to zero in the second argument.

We need to ensure convergence of the integral cost that is introduced shortly for the "right" stabilizing solutions. To this end using the simplification $\beta(r,t) \le \alpha_2(\alpha_1(r)e^{-t})$ for β from (11) and some arbitrary $\eta > 0$ we assume that there is a constant $C > 0$ such that

$$g(x,u) \le C(\alpha_2^{-1}(\|x\|))^\eta \text{ for all } (x,u) \in B(0,r) \times B(0,\bar{u}) .$$

Finally, we need the following technical assumption. It is used in the proof of continuity of v as well as in showing that v vanishes only in 0.

$$g(x,u) \ge \|f(x,u)\| + \gamma(\|u\|) \text{ whenever } \|x\| \ge 2r \text{ or } \|u\| \ge 2\bar{u} .$$

We now define the functional

$$J(x,u) := \begin{cases} \int_0^\infty g(\varphi(t,x,u),u(t))dt\,, & \text{if } T_{\max}(x,u) = \infty \\ \infty & \text{else} \end{cases}, \tag{12}$$

and the (extended real valued) optimal value functions

$$V(x) := \inf_{u \in \mathcal{U}} J(x,u)\,, \quad v(x) := 1 - e^{-V(x)}\,, \quad x \in \mathbb{R}^n . \tag{13}$$

Note that both V and v satisfy appropriate dynamic programming principles, e.g., for each $T > 0$ we have

$$v(x) = \inf_{u \in \mathcal{U}} \left\{1 + G(x,T,u)(v(\varphi(T,x,u)) - 1)\right\} , \tag{14}$$

where

$$G(x,T,u) := \exp\left(-\int_0^T g(\varphi(t,x,u),u(t))dt\right) .$$

From the dynamic programming principle it appears reasonable to study the following first order partial differential equation. Again we see, that it is a straightforward extension of Zubov's original equation. This time, however, the optimization problem is a minimization not a maximization problem.

$$\sup_{u \in U}\{-Dv(x)f(x,u) - (1 - v(x))g(x,u)\} = 0, \tag{15}$$

The following result shows that Zubov's approach can also be extended to the the calculation of control Lyapunov functions.

Theorem 6 ([8]) *Assume that f and g satisfy the assumptions described above. Then*

(i) The function v from (13) *is the unique bounded viscosity solution of* (15) *with* $v(0) = 0$

(ii) The function v characterizes the domain of asymptotic controllability via

$$\mathcal{D}_c = \{x \in \mathbb{R}^n \,|\, v(x) < 1\}.$$

4 Conclusions

We have discussed a method for the calculation of domains of attraction for perturbed systems. The method relies on the ability to numerically solve for a viscosity solution of a first order PDE. This has been an active research area over the past decade and quite efficient methods are available, see for instance the appendix by M. Falcone in [2]. For lower dimensional systems this provides an efficient method to obtain estimates for domains of attraction.
It is clear that the approach is restricted to low dimensions because for the calculations a region of the state space has to be gridded which requires exponential cost in the dimension of the problem. However, to the best of our knowledge this is the first possibility to calculate domains of attractions for systems subject to perturbations available in the literature.
Finally, it should be noted that Zubov's idea can also be used in the context of stochastic systems, see [3]. Overall, this shows that "Zubov's method" brings deeper insight to a broad range of applications: perturbed systems, controlled systems and stochastic systems. The ideas behind Zubov's work reach well beyond the original result and motivate new research to this day. This is what distinguishes a great achievement.

References

[1] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Anal., 7:1163–1173, 1983.

[2] M. Bardi and I. Capuzzo Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, 1997.

[3] F. Camilli, L.Grüne, *Characterizing attraction probabilities via the stochastic Zubov method*, Discrete Contin. Dynam. Systems B 3(3):457–468, 2003.

[4] F. Camilli, L. Grüne, and F. Wirth. *A regularization of Zubov's equation for robust domains of attraction.* in: *Nonlinear Control in the Year 2000*, A. Isidori et al., eds., Lecture Notes in Control and Information Sciences, Vol. 258, pp. 277 - 290, Springer-Verlag, London, 2000.

[5] F. Camilli, L. Grüne, and F. Wirth, *Zubov's method for perturbed differential equations* Proc. Symposium on Mathematical Theory of Networks and Systems MTNS-2000, Perpignan, France, July 2000. (CD-Rom) paper B100.

[6] F. Camilli, L. Grüne, and F. Wirth, *A generalization of Zubov's method to perturbed systems.* SIAM J. Control Optimization 40(2):496–515, 2002.

[7] F. Camilli, L. Grüne, F. Wirth. A generalization of Zubov's method to perturbed systems. Proc. 41st IEEE Conference on Decision and Control, CDC2002, Las Vegas, NV, US, December 2002.

[8] F. Camilli, L. Grüne, and F. Wirth, *Control Lyapunov Functions and Zubov's Method.* submitted, 2005.

[9] F. H. Clarke, Y. S. Ledyaev, L. Rifford, and R. J. Stern, *Feedback stabilization and Lyapunov functions*, SIAM J. Control Optim., 39:25–48 2000.

[10] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern. *Asymptotic stability and smooth Lyapunov functions.* J. Differential Equations 149(1):69–114, 1998.

[11] R. A. Freeman and P. V. Kokotovic, *Inverse optimality in robust stabilization*, SIAM J. Control Optim., 34:1365–1391, 1996.

[12] P. Giesl. *Approximation of domains of attraction and Lyapunov functions using radial basis functions.* Proc. Nolcos04, IFAC Symposium on Nonlinear Control Systems. Stuttgart, Germany, 2004.

[13] S. F. Hafstein. *A constructive converse Lyapunov theorem on exponential stability.* Discrete Contin. Dynam. Systems. 10(3):657–678, 2004.

[14] W. Hahn, *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[15] H. K. Khalil. *Nonlinear Systems.* 2nd ed. Prentice-Hall, Upper Saddle River, NJ, 1996.

[16] N.E. Kirin, R.A. Nelepin, and V.N. Bajdaev, *Construction of the attraction region by Zubov's method.* Differ. Equations **17** (1982), 871–880.

[17] Kurcveĭl, Y. (1956). *On the inversion of the second theorem of Lyapunov on stability of motion.* Czechoslovak Math. J. **6(81)**, 217–259, 455–484.

[18] Y. Lin, E. D. Sontag, and Y. Wang. *A smooth converse Lyapunov theorem for robust stability.* SIAM J. Control Optim. 34:124–160, 1996.

[19] J. L. Massera. *On Liapunov's condition of stability.* Annals of Math., 50:705–721, 1949.

[20] A. Papachristodoulou, S. Prajna. *On the construction of Lyapunov functions using the sum of squares decomposition.* Proc. 41st IEEE Conference on Decision and Control (CDC 2002), Las Vegas, NV, December 2002, pp. 3482–3487.

[21] L. Rifford. *Semiconcave control-Lyapunov functions and stabilizing feedbacks*, SIAM J. Control Optim., 41:659–681 2002.

[22] E. D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.

[23] E. D. Sontag, *Comments on integral variants of ISS*, Systems Control Lett., 34 (1998), pp. 93–100.

[24] A. Teel and L. Praly. *A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions.* ESAIM Control, Optimization and Calculus of Variations 5:313-367, 2000.

[25] A. Vannelli and M. Vidyasagar, *Maximal Lyapunov functions and domains of attraction for autonomous nonlinear systems.* Automatica 21(1):69–80, 1985.

[26] F.W. Wilson, *The structure of the level surfaces of a Lyapunov function.* J. Diff. Equations 3:323–329, 1967.

[27] V.I. Zubov. *Methods of A.M. Lyapunov and their Application*, P. Noordhoff, Groningen, 1964.