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Integration and Regularity of Set-Valued Maps Represented by Generalized Steiner Points

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Abstract A family of probability measures on the unit ball in $\mathbb{R}^n$ generates a family of generalized Steiner (GS-)points for every convex compact set in $\mathbb{R}^n$. Such a "rich" family of probability measures determines a representation of a convex compact set by GS-points. In this way, a representation of a set-valued map with convex compact images is constructed by GS-selections (which are defined by the GS-points of its images).

The properties of the GS-points allow to represent Minkowski sum, Demyanov difference and Demyanov distance between sets in terms of their GS-points, as well as the Aumann integral of a set-valued map is represented by the integrals of its GS-selections. Regularity properties of set-valued maps (measurability, Lipschitz continuity, bounded variation) are reduced to the corresponding properties of its GS-selections.

This theory is applied to analyze the order of convergence of a discrete approximation of the Aumann integral.

Keywords generalized Steiner selections · Demyanov distance · Aumann integral · Castaing representation · set-valued maps · arithmetic set operations

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1 Introduction

We study representations of set-valued mappings with convex compact images in $\mathbb{R}^n$ by special single-valued selections constructed by generalized Steiner (GS-)points of their images. The idea of representing sets in $\mathcal{C}(\mathbb{R}^n)$, the space of nonempty convex compact subsets of $\mathbb{R}^n$, with "weighted Steiner points" obtained by various probability measures on the unit ball, was noticed in [20], and developed by Dentcheva [14] as an entire framework.

Every convex compact in $\mathbb{R}^n$ may be represented as

$$C = \overline{\bigcup_{\alpha \in \mathcal{M}_{\mathrm{sp}}} \{\mathrm{St}_\alpha(C)\}},$$

where $\mathcal{M}_{\mathrm{sp}}$ is a rich enough family of probability measures defined on the unit ball in $\mathbb{R}^n$, and $\mathrm{St}_\alpha(C)$ is the GS-point corresponding to such a measure α. For a set-valued map $F(\cdot)$, the family of generalized Steiner selections $\{ \mathrm{St}_\alpha(F(\cdot)) \mid \alpha \in \mathcal{M}_{\mathrm{sp}} \}$ is called here GS-representation of this map. The term GS-selection is used as an abbreviation for the selection $\mathrm{St}_\alpha(F(\cdot))$.

In [14, 12, 13], $\mathcal{M}_{\mathrm{sp}}$ consists of measures with C^1-density functions (we call them "smooth measures"), and a Castaing representation of set-valued maps by such GS-selections is constructed and applied to non-smooth analysis and stochastic optimization.

The following properties of the GS-points make them especially appropriate for the analysis of regularity and approximations of set-valued maps.

1. $\mathrm{St}_\alpha(C + D) = \mathrm{St}_\alpha(C) + \mathrm{St}_\alpha(D)$ for the Minkowski sum of sets C, D
2. $\mathrm{St}_\alpha(\lambda C) = \lambda \, \mathrm{St}_\alpha(C)$ for $\lambda \geq 0$
3. $\sup_{\alpha \in \mathcal{M}_{\mathrm{sp}}} \| \mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) \| = \mathrm{d_D}(C, D)$,
   where $\mathrm{d_D}$ is the Demyanov metric in $\mathcal{C}(\mathbb{R}^n)$

The well-known representation of convex sets and set-valued mappings by support functions [22] has properties analogous to the above, with the Hausdorff metric in the third property. Thanks to these properties, it is successfully applied to set-valued numerical approximation and integration (see e.g. [17, 8, 5, 18]) .

In this paper we replace the representing support functions by the GS-selections. To ensure uniform regularity properties of the selections, we have to replace the Hausdorff metric with the stronger Demyanov metric [11, 15].

The properties 1.–2. imply the reduction of positive linear operators on set-valued maps to the same operators on the GS-selections. Also by property 3., regularity properties of a set-valued map (e.g. bounded variation, Lipschitz continuity etc.) are reduced to the same properties of the parametrizing family of GS-selections. Parametrizations of regular set-valued map by smooth selections are highly applicable in various fields (see e.g. [4, 2, 25, 16]).

Here, we introduce new notions of Lipschitz continuity and bounded variation of set-valued mappings by their GS-selections, and demonstrate how these notions can be applied to the analysis of set-valued numerical integration.

The third property yields e.g. that the GS-selections in [14] inherit Lipschitz continuity of the set-valued map with respect to the Demyanov metric in a uniform way. Thus, generally, a Hausdorff Lipschitz continuous multifunction does not have a GS-representation with uniformly Lipschitz selections, while such a representation exists for mappings which are Lipschitz continuous with respect to the stronger Demyanov metric.

Motivated by numerical applications, we focus on GS-representations generated by smooth or special Dirac measures. To achieve the above properties also for Dirac measures, we modify slightly the definition of generalized Steiner points of [14], leaving it unchanged in the case of smooth measures.

The main features which distinguish the GS-representation from other known representations of set-valued maps by selections (see e.g. [19, 24, 4, 16]) are two: first, the parameter set $\mathcal{M}_{\text{sp}}$ is universal, not related to the considered multifunction, and second, we are not aware of any other selections satisfying (analogues of) the above three properties.

The paper is organized as follows: Section 2 contains the necessary notation and definitions as well as properties of the Steiner point. In the next section representations of convex compacts with GS-points are studied. In Section 4 the relations between arithmetic operations and representations are discussed. The connection of GS-points to the Demyanov difference and the Demyanov distance is also clarified there. New notions of variation and Lipschitz continuity of set-valued maps related to GS-selections are studied in Section 5. In Section 6 the Aumann integral is represented by integrals of GS-selections and its simple approximation by Riemannian sums is analyzed. An outline of future research is presented in the conclusions.

2 Preliminaries

First we introduce some notation. $\mathcal{C}(\mathbb{R}^n)$ denotes the set of all convex, compact, nonempty subsets of $\mathbb{R}^n$, $B_r(x)$ is the closed ball of radius r centered at $x \in \mathbb{R}^n$, B_1 is the closed unit ball in $\mathbb{R}^n$ and S_{n-1} its boundary, V_n denotes the Lebesgue volume of the unit ball.

Let $C \in \mathcal{C}(\mathbb{R}^n)$. Denote by $\delta^*(l, C) = \max\limits_{x \in C} \langle l, x \rangle$ the support function of C in direction $l \in \mathbb{R}^n$, and by

$$Y(l, C) := \{c \in C \mid \langle l, c \rangle = \delta^*(l, C)\}$$

the supporting face of C in this direction.

The Hausdorff distance between the sets A and B is denoted by $\mathrm{d_H}(A, B)$. The norm of a set $C \in \mathcal{C}(\mathbb{R}^n)$ is $\|C\| := \max_{x \in C} \|x\|$. $\mathrm{m}(C)$ is defined to be the (unique) norm-minimal point of C.

The point $x \in C$ is called exposed point of C (with an exposed direction $l \in \mathbb{R}^n$), if $Y(l, C) = \{x\}$. Then, we express the dependence on the direction as $x = y(l, C)$. The set of all exposed points of C is denoted by $\exp(C)$. Denote by $T_C \subset \mathbb{R}^n$ the set of all exposed directions l of C, $l \in \mathbb{R}^n$, for which $x = y(l, C)$ is an exposed point. T_C is a set of full measure in $\mathbb{R}^n$ (cf. [27]).

As arithmetic operations in $\mathcal{C}(\mathbb{R}^n)$ we use the classical ones, namely the Minkowski sum and the scalar multiplication:

$$\begin{aligned} C + D &:= \{c + d \,|\, c \in C,\ d \in D\} \quad &&(\text{for } C, D \in \mathcal{C}(\mathbb{R}^n)), \\ \lambda C &:= \{\lambda c \,|\, c \in C\} &&(\text{for } C \in \mathcal{C}(\mathbb{R}^n),\ \lambda \in \mathbb{R}) \end{aligned}$$

Next we recall the notions of Demyanov difference and Demyanov distance of two convex compact sets $C, D \in \mathcal{C}(\mathbb{R}^n)$, which play a central role in our exposition (see [11, 27, 15]).

$$C \dot{-} D = \overline{\text{co}} \bigcup_{l \in T_C \cap T_D} \{y(l, C) - y(l, D)\} \in \mathcal{C}(\mathbb{R}^n)$$

is the Demyanov difference of C and D. The Demyanov distance between the sets $C, D \in \mathcal{C}(\mathbb{R}^n)$ is

$$\mathrm{d_D}(C, D) = \|C \dot{-} D\|. \tag{2.1}$$

We recall the definition of the Steiner point of a set $C \in \mathcal{C}(\mathbb{R}^n)$ [32, 33, 31]:

$$\mathrm{St}(C) := \frac{1}{V_n} \int_{B_1} \mathrm{m}(Y(p, C))\ dp \tag{2.2}$$

Here and further in the text, dp denotes the Lebesgue measure, $d\mu(p) = \frac{1}{V_n} dp$ is the normalized Lebesgue measure on B_1 (recall that $V_n = \mu(B_1)$). We note that the set-valued map $l \mapsto Y(l, C)$ is the marginal map of the support function of C and hence is upper semi-continuous (see e.g. [3, Chapter 1.2, Theorem 6]) and measurable w.r.t. the Borel σ-algebra (cf. [4, Theorem 8.2.9]). $\mathrm{m}(Y(\cdot, C))$ is measurable by [4, Corollary 8.2.13]. Since $Y(\cdot, C)$ is μ-almost everywhere single-valued and bounded, $\mathrm{m}(Y(p, C))$ coincides μ-a.e. with $y(p, C)$ and is μ-integrable. Hence,

$$\mathrm{St}(C) = \frac{1}{V_n} \int_{B_1 \cap T_C} y(p, C)\ dp = \int_{B_1 \cap T_C} y(p, C)\ d\mu(p).$$

Recall some important properties of the Steiner point.

Remark 2.1 Let $C, D \in \mathcal{C}(\mathbb{R}^n)$ and $\lambda, \nu \in \mathbb{R}$. Then, $\mathrm{St}(C) \in \operatorname{relint} C$ ([28], [30, (5.4.13)]) and

$$\mathrm{St}(\lambda C + \nu D) = \lambda\, \mathrm{St}(C) + \nu\, \mathrm{St}(D), \tag{2.3}$$

$$\mathrm{St}(RC) = R\, \mathrm{St}(C), \quad \text{if } R \text{ is an orthogonal } n \times n\text{-matrix}, \tag{2.4}$$

$$\mathrm{St}(\cdot) \quad \text{is Lipschitz continuous with respect to } \mathrm{d_H}(\cdot, \cdot). \tag{2.5}$$

Steiner points of sets in $\mathbb{R}^2$ are characterized uniquely by (2.3) with $\lambda = \nu = 1$ and (2.4) for R being a congruence transformation (composition of a rotation, symmetry and a translation) and only by continuity in (2.5), cf. [29, 31]. The optimal Lipschitz constant is recorded in [34].

Definition 2.2 Let $\mathcal{B}$ be the Borel σ-algebra on B_1. We use the notation

$$\alpha(A) = \int_A \alpha(dp) \quad (A \in \mathcal{B})$$

for a probability measure α on $\mathcal{B}$, i.e. a normalized (positive) measure with $\alpha(B_1) = 1$ (cf. [9, Chapter II.9], [10, 14]). $\mathcal{SM}$ denotes the set of probability measures α with $\mathcal{C}^1(B_1)$-density function $\theta(\cdot)$ with respect to μ. Thus,

$$\alpha(A) = \int_A \theta(p)\, d\mu(p) = \frac{1}{V_n} \int_A \theta(p)\, dp.$$

We will call measures from $\mathcal{SM}$ shortly "smooth measures".
The set $\mathcal{M}$ consists of all probability measures on $\mathcal{B}$, $\mathcal{AM}$ is the set of atomic (Dirac) measures $\alpha = \alpha_{[l]} \in \mathcal{M}$, concentrated in a point $l \in S_{n-1}$, i.e.

$$\alpha_{[l]}(A) = \begin{cases} 0, & \text{if } l \notin A,\ A \in \mathcal{B}, \\ 1, & \text{if } l \in A,\ A \in \mathcal{B}. \end{cases}$$

Introducing the arithmetic operations on measures $\alpha, \widetilde{\alpha} \in \mathcal{M}$ as

$$(\lambda\alpha)(A) := \lambda\alpha(A), \qquad (\alpha + \widetilde{\alpha})(A) := \alpha(A) + \widetilde{\alpha}(A)$$

for $A \in \mathcal{B}$ and $\lambda \geq 0$, we define the convex hull of $\mathcal{AM}$ in $\mathcal{M}$ as

$$\mathrm{co}(\mathcal{AM}) := \{\sum_{i=1}^{k} \lambda_i \alpha_i \mid \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0,\ \alpha_i \in \mathcal{AM}\ (i = 1, \ldots, k),\ k \in \mathbb{N}\}.$$

Atomic measures concentrated in non-zero vectors of the interior of B_1 are equivalent to measures of $\mathcal{AM}$ (cf. Remark 3.3).

Recall the definition of a generalized Steiner point of Dentcheva [14] for a measure $\alpha \in \mathcal{SM}$ with density function $\theta(\cdot)$:

$$\widetilde{\mathrm{St}}_\alpha(C) = \int_{B_1} \mathrm{m}(Y(p, C))\ \alpha(dp) = \frac{1}{V_n} \int_{B_1} \mathrm{m}(Y(p, C))\theta(p)\ dp. \tag{2.6}$$

Therefore, $\widetilde{\mathrm{St}}_\alpha(C) = \mathrm{St}(C)$ follows for the uniform measure α with density function $\theta(\cdot) \equiv 1$ by (2.2).

3 Representations of Sets by Generalized Steiner Points

First we define GS-points for a family of probability measures $\mathcal{M}_{\mathrm{sp}} \subset \mathcal{M}$. In this paper $\mathcal{M}_{\mathrm{sp}}$ is one of the classes $\mathcal{AM}$, $\mathrm{co}(\mathcal{AM})$ or $\mathcal{SM}$.

Definition 3.1 Let $C \in \mathcal{C}(\mathbb{R}^n)$ and $\alpha \in \mathcal{M}_{\mathrm{sp}}$. We define the generalized Steiner point (GS-point) of C by

$$\mathrm{St}_\alpha(C) := \int_{B_1} \mathrm{St}(Y(p, C))\ \alpha(dp). \tag{3.1}$$

The definition remains valid for a general measure $\alpha \in \mathcal{M}$, since the function $\mathrm{St}(Y(\cdot, C))$ is bounded and measurable and hence α-integrable (cf. [9, Section II.1]. Definition 3.1 coincides with (2.6) for smooth measures (see Lemma 3.2) and extends (2.6) to a broader class of measures. Furthermore, the advantage of (3.1) is that GS-points, defined for measures from $\mathcal{M}_{\mathrm{sp}} \setminus \mathcal{SM}$, also inherit basic properties of the classical Steiner point, which cannot be achieved with definition (2.6) (see Section 4).

Next, we give explicit formulae of GS-points for various measures α.

Lemma 3.2 *Let $C \in \mathcal{C}(\mathbb{R}^n)$. Then,*

$$\mathrm{St}_\alpha(C) = \begin{cases} \int\limits_{B_1} \mathrm{m}(Y(p,C))\theta(p)\ d\mu(p), & \textit{for}\ \ \alpha \in \mathcal{SM}\ \textit{with density}\ \theta(\cdot), \\ \mathrm{St}(Y(l,C)), & \textit{for}\ \ \alpha = \alpha_{[l]} \in \mathcal{AM},\ l \in S_{n-1}, \\ y(l,C), & \textit{for}\ \ \alpha = \alpha_{[l]} \in \mathcal{AM},\ l \in S_{n-1} \cap T_C, \\ \sum\limits_{i=1}^{k} \lambda_i \, \mathrm{St}_{\alpha_i}(C), & \textit{for}\ \ \alpha = \sum\limits_{i=1}^{k} \lambda_i \alpha_i \in \mathrm{co}(\mathcal{AM}). \end{cases}$$

Proof
(i) $\alpha \in \mathcal{SM}$
First, $\mathrm{St}(Y(\cdot, C))$ is μ-integrable by [20, Theorem 4.6]. The subintegral function in (3.1) differs to that in (2.6) only on a set of zero Lebesgue measure, thus

$$\mathrm{St}_\alpha(C) = \int\limits_{B_1 \cap T_C} y(p,C)\theta(p)\ d\mu(p) = \frac{1}{V_n} \int\limits_{B_1} \mathrm{m}(Y(p,C))\theta(p)\ dp. \tag{3.2}$$

(ii) $\alpha = \alpha_{[p^0]} \in \mathcal{AM}$ with $p^0 \in S_{n-1}$
The bounded, measurable function $\mathrm{St}(Y(\cdot, C))$ is α-integrable with $\mathrm{St}_\alpha(C) = \mathrm{St}(Y(p^0, C))$ which may be proved in a standard way (cf. [9, Section II.1]).

(iv) $\alpha \in \mathrm{co}(\mathcal{AM})$
For a measure $\alpha = \sum\limits_{i=1}^{k} \lambda_i \alpha_i$ with $\alpha_i = \alpha_{[p^i]} \in \mathcal{AM}$ concentrated in $p_i \in S_{n-1}$, $\lambda_i \geq 0$, $\sum\limits_{i=1}^{k} \lambda_i = 1$, the α-integrability and the formula are straightforward:

$$\mathrm{St}_\alpha(C) = \sum_{i=1}^{k} \lambda_i \int\limits_{B_1} \mathrm{St}(Y(p,C))\ \alpha_i(dp) = \sum_{i=1}^{k} \lambda_i \, \mathrm{St}_{\alpha_i}(C).$$

□

Remark 3.3 Let $C \in \mathcal{C}(\mathbb{R}^n)$ and $\alpha = \alpha_{[l]} \in \mathcal{M}$ be the atomic measure concentrated in $l \in \mathrm{int}(B_1)$. Then,

$$\mathrm{St}_\alpha(C) = \begin{cases} \mathrm{St}_{\widetilde{\alpha}}(C), & \text{if } l \neq 0_{\mathbb{R}^n}. \text{ Hereby, } \widetilde{\alpha} \in \mathcal{AM} \text{ is concentrated} \\ & \quad \text{in } \eta = \frac{1}{\|l\|} \cdot l \in S_{n-1}, \\ \mathrm{St}(C), & \text{if } l = 0_{\mathbb{R}^n}, \text{ since } Y(0_{\mathbb{R}^n}, C) = C. \end{cases}$$

The following example illustrates the difference between the GS-points constructed with measures from $\mathcal{AM}$ and those obtained with measures from $\mathcal{SM}$.

Example 3.4 Let $C = [-1,1] \times \{1\}$, $l = \binom{\cos(\phi)}{\sin(\phi)}$ with $\phi \in [-\frac{\pi}{2}, \frac{3\pi}{2})$ and $\alpha = \alpha_{[l]} \in \mathcal{AM}$ be the atomic measure concentrated in l. Then,

$$\mathrm{St}_\alpha(C) = \begin{cases} \mathrm{St}(Y(l,C)) = \mathrm{St}(C) = \binom{0}{1} & (\phi \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}), \\ y(l,C) = \binom{1}{1} & (\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})), \\ y(l,C) = \binom{-1}{1} & (\phi \in (\frac{\pi}{2}, \frac{3\pi}{2})). \end{cases}$$

For a measure $\beta \in \mathcal{SM}$ with a density function $\theta(\cdot)$ and $\operatorname{supp}\theta \subset B_\varepsilon(m) \subset B_1$, by (3.2)

$$\mathrm{St}_\beta(C) = \frac{1}{V_2} \int\limits_{B_1} \mathrm{St}(Y(p,C))\theta(p)\, dp = \frac{1}{V_2} \int\limits_{B_1 \cap T_C} y(p,C)\theta(p)\, dp.$$

Denote by B_1^+ the right half of B_1 in $\mathbb{R}^2$ and by B_1^- its left half. Then,

$$\begin{aligned} \mathrm{St}_\beta(C) &= \int\limits_{B_1^+ \cap B_\varepsilon(m) \cap T_C} \theta(p) \begin{pmatrix} 1 \\ 1 \end{pmatrix} d\mu(p) + \int\limits_{B_1^- \cap B_\varepsilon(m) \cap T_C} \theta(p) \begin{pmatrix} -1 \\ 1 \end{pmatrix} d\mu(p) \\ &= \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (1-\lambda) \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \end{aligned}$$

where $\lambda = \beta(B_1^+ \cap B_\varepsilon(m)) \in [0,1]$. Thus, the GS-points for smooth probability measures cover the complete set C, including its exposed points, while the ones with measures of $\mathcal{AM}$ form a discrete set of three points. Here, the classical Steiner point $\mathrm{St}(C) = \binom{0}{1}$ may be represented, e.g. by a smooth measure β having a radially symmetric density function with $\varepsilon = 1$, $m = \binom{0}{0}$.

It follows from this example that the GS-points, in contrast with the classical Steiner point, are not necessary in the relative interior of the set, and even may be exposed points. Also, the exposed points here are represented as GS-points in a non-unique way, not only by measures from $\mathcal{AM}$ as in Lemma 3.2, but also by measures from $\mathcal{SM}$.

It is proved in [14, Lemma 5.4] that the set of GS-points obtained by smooth measures is dense in every convex compact subset of $\mathbb{R}^n$

$$C = \overline{\bigcup_{\alpha \in \mathcal{SM}} \{\mathrm{St}_\alpha(C)\}}. \tag{3.3}$$

Furthermore, C could be represented with countably many smooth measures $(\alpha_m)_{m\in\mathbb{N}}$ (cf. [12, Theorem 3.4]) as

$$C = \overline{\bigcup_{m \in \mathbb{N}} \{\mathrm{St}_{\alpha_m}(C)\}}. \tag{3.4}$$

Observe that the inequalities

$$\langle l, \mathrm{St}_\alpha(C)\rangle = \int\limits_{B_1} \langle l, \mathrm{St}(Y(p,C))\rangle \; \alpha(dp) \le \delta^*(l,C) \cdot \int\limits_{B_1} \alpha(dp) = \delta^*(l,C)$$

for all $l \in S_{n-1}$, yield that $\mathrm{St}_\alpha(C) \in C$ for every $\alpha \in \mathcal{M}_{\mathrm{sp}}$.

We will also show that one may replace $\mathcal{SM}$ by $\mathrm{co}(\mathcal{AM})$ in the GS-representation (3.3). The reason that we are interested in the atomic measures $\alpha \in \mathcal{AM}$ or $\alpha \in \mathrm{co}(\mathcal{AM})$ is that they are technically simpler to realize.

Lemma 3.2 yields the following representation of the set of exposed points of a given convex compact set $C \in \mathcal{C}(\mathbb{R}^n)$:

$$\exp(C) = \bigcup \{\mathrm{St}_\alpha(C) \,|\, \alpha = \alpha_{[l]} \in \mathcal{AM},\, l \in S_{n-1} \cap T_C\}. \tag{3.5}$$

The representation of a convex compact set as the closed convex hull of its exposed points (Theorem of Straszewicz) and (3.5) imply

Corollary 3.5 *Let $C \in \mathcal{C}(\mathbb{R}^n)$. If $\alpha \in \mathcal{AM}$, then $\mathrm{St}_\alpha(C) \in \partial C$ and*

$$C = \overline{\mathrm{co}} \bigcup_{\alpha \in \mathcal{AM}} \{\mathrm{St}_\alpha(C)\} = \overline{\bigcup_{\alpha \in \mathrm{co}(\mathcal{AM})} \{\mathrm{St}_\alpha(C)\}}. \tag{3.6}$$

The formula (3.6) gives another representation of a convex set different to the theorems of Minkowski and Straszewicz (cf. [30, Corollary 1.4.5 resp. Theorem 1.4.7]). This representation is non-minimal in general by (3.5), since the exposed points are a subset of all GS-points obtained by measures of $\mathcal{AM}$.

In the case of polytopes, all extremal points are exposed ones. Thus, the closure in (3.6) can be removed, since there is a finite number of atomic measures concentrated in exposed directions corresponding to all extremal points.

Corollary 3.6 *Let P be a convex polytope with M vertices. Then, there exists $\alpha_i \in \mathcal{AM}$, $i = 1, \dots, M$, concentrated in the vertices of P, with*

$$P = \mathrm{co}(\{\mathrm{St}_{\alpha_i}(P) \,|\, i = 1, \dots, M\}). \tag{3.7}$$

Furthermore, there exists $\beta_i \in \mathcal{SM}$, $i = 1, \dots, M$, such that (3.7) *holds with β_i replacing α_i.*

Indeed, every vertex $v \in P$ can be obtained as a GS-point with a smooth measure $\beta \in \mathcal{SM}$. For each vertex v there is a cone $K(v)$ of corresponding exposed directions with nonempty interior. Therefore, one may find a measure $\beta \in \mathcal{SM}$ with a density function θ such that $supp(\theta) \subset K(v)$. Then, $v = \mathrm{St}_\beta(P)$.

We note that the measures in Corollary 3.6 are non-unique, since a vertex in a polytope has many exposed directions.

The equality (3.3) and Corollary 3.5 show that every GS-point generated by a smooth measure can be approximated by a converging sequence of convex combinations of atomic measures and vice versa. We prove next that every exposed point may be approximated by GS-points obtained with measures from $\mathcal{SM}$ depending only on its exposed direction, which is important for the further theorems.

To do this, we introduce the notion of a normal Dirac sequence which slightly simplifies the one of [21, Chapter 4.3].

Definition 3.7 A sequence of $\mathcal{C}^1$-functions $\theta_m : \mathbb{R}^n \to \mathbb{R},\ m \in \mathbb{N}$, is called a normal Dirac sequence, if for all $m \in \mathbb{N}$

$$\theta_m(\cdot) \geq 0, \quad \int\limits_{\mathbb{R}^n} \theta_m(p)\ d\mu(p) = 1 \quad \text{and} \quad \operatorname{supp} \theta_m \subset B_{\frac{1}{m}}(0).$$

The existence of such a normal Dirac sequence is obvious and may be found e.g. in [21, Chapter 7.1C].

Lemma 3.8 *Let $C \in \mathcal{C}(\mathbb{R}^n)$ and $\alpha \in \mathcal{AM}$ be concentrated in $l \in S_{n-1} \cap T_C$. Consider a normal Dirac sequence $(\theta_m(\cdot))_{m\in\mathbb{N}}$. Then, the measures $\alpha_m \in \mathcal{SM}$ with density functions $\theta_m(\cdot - l^m)$, $l^m = \frac{m-1}{m}\ l$, $m \in \mathbb{N}$, fulfill*

$$\lim_{m\to\infty} \operatorname{St}_{\alpha_m}(C) = \operatorname{St}_\alpha(C) = y(l, C),$$

i.e. for all $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have

$$\| \operatorname{St}_{\alpha_m}(C) - \operatorname{St}_\alpha(C)\| \leq \varepsilon.$$

Proof Remark that $\operatorname{St}_\alpha(C) = y(l, C)$. As we have noted, the set-valued map $l \mapsto Y(l, C)$ is upper semi-continuous. Hence, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $p \in \mathbb{R}^n$ with $\|p - l\| < \delta$ we have $Y(p, C) \subset Y(l, C) + \epsilon B_1$, and if $p \in T_C$ with $\|p - l\| < \delta$, then $\|y(p, C) - y(l, C)\| \leq \epsilon$. Now, for $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$, we use that $\operatorname{supp} \theta_m(\cdot - l^m) \subset B_{\frac{1}{m}}(l^m) \subset B_1$ and that it generates a probability measure $\alpha_m \in \mathcal{SM}$:

$$\begin{aligned} &\| \operatorname{St}_{\alpha_m}(C) - y(l, C)\| \\ &= \| \int\limits_{B_1} \operatorname{St}(Y(p, C))\theta_m(p - l^m)\ d\mu(p) - y(l, C) \cdot \int\limits_{B_1} \theta_m(p - l^m)\ d\mu(p)\| \\ &\leq \int\limits_{B_1 \cap T_C} \|y(p, C) - y(l, C)\| \cdot \theta_m(p - l^m)\ d\mu(p) \\ &\leq \int\limits_{B_{\frac{1}{m}}(l^m)\cap T_C} \epsilon \cdot \theta_m(p - l^m)\ d\mu(p) = \epsilon. \end{aligned}$$

□

4 Generalized Steiner points and Arithmetic Set Operations

We start with some properties of GS-points.

Lemma 4.1 *Let $C, D \in \mathcal{C}(\mathbb{R}^n)$, $\alpha, \beta \in \mathcal{M}_{sp}$, R be an orthogonal matrix, and let $\lambda, \nu \geq 0$. Then,*

$$\operatorname{St}_{\lambda\alpha+\nu\beta}(C) = \lambda \operatorname{St}_{\alpha}(C) + \nu \operatorname{St}_{\beta}(C), \quad \text{if } \lambda + \nu = 1, \tag{4.1}$$

$$\operatorname{St}_{\alpha}(\lambda C + \nu D) = \lambda \operatorname{St}_{\alpha}(C) + \nu \operatorname{St}_{\alpha}(D), \tag{4.2}$$

$$\operatorname{St}_{\alpha}(RC) = R \operatorname{St}_{\widetilde{\alpha}}(C). \tag{4.3}$$

Hereby, $\widetilde{\alpha}(B) = \alpha(R \cdot B)$ for a measurable subset $B \in \mathcal{B}$ in (4.3). If $\alpha \in \mathcal{AM}$ is concentrated in l, $\widetilde{\alpha} \in \mathcal{AM}$ is concentrated in $R^t l$.

Proof The equality (4.1) follows from the definition of convex combination of measures and simple properties of integrals. (4.2) is shown in [14, Remarks after Theorem 3.6] and for atomic measures it is a consequence of (2.3) and the fact that $Y(l, \lambda C + \nu D) = \lambda Y(l, C) + \nu Y(l, D)$ for $\lambda, \nu \geq 0$.

Let R be an orthogonal matrix. For a measure $\alpha \in \mathcal{SM}$ with density function $\theta(\cdot)$, we substitute $z = R^t p$ and use the obvious relation $Y(p, RC) = RY(R^t p, C)$, the equality $\operatorname{St}(RB) = R \operatorname{St}(B)$ in (2.4) and $|\det R^t| = 1$:

$$\begin{aligned}
\operatorname{St}_{\alpha}(RC) &= \int\limits_{B_1} \operatorname{St}(Y(p, RC))\theta(p)\; d\mu(p) = \int\limits_{B_1} R \operatorname{St}(Y(R^t p, C))\theta(RR^t p)\; d\mu(p) \\
&= R \int\limits_{B_1} \operatorname{St}(Y(z, C))\widetilde{\theta}(z) |\det R^t| \; d\mu(z) = R \operatorname{St}_{\widetilde{\alpha}}(C),
\end{aligned}$$

where $\widetilde{\theta}(z) = \theta(Rz)$ and $\widetilde{\alpha} \in \mathcal{SM}$ is the corresponding measure.
If $\alpha \in \mathcal{AM}$ concentrated in $l \in S_{n-1}$, then $\widetilde{\alpha} \in \mathcal{AM}$ is concentrated in $R^t l$ and

$$\operatorname{St}_{\alpha}(RC) = \operatorname{St}(Y(l, RC)) = \operatorname{St}(RY(R^t l, C)) = R \operatorname{St}(Y(R^t l, C)) = R \operatorname{St}_{\widetilde{\alpha}}(C).$$

The last equalities may be easily adapted for $\alpha \in \operatorname{co}(\mathcal{AM})$. □

Remark 4.2 For $\alpha \in \mathcal{SM}$, (4.2) is known [14, Remarks after Theorem 3.6]. Note that if λ or ν are negative, equation (4.2) holds only for the Steiner point, and may be invalid for a generalized one. In particular, from (4.3) follows that $\operatorname{St}_{\alpha}(-A) = -\operatorname{St}_{\widetilde{\alpha}}(A)$, where $\widetilde{\alpha}$ is the "reflection" of α with respect to the origin, i.e. $\widetilde{\alpha}(B) = \alpha(-B)$ for any measurable set $B \subset B_1$. It is easy to prove (4.3) for any congruence transformation, showing that $\widetilde{\alpha}(B) = \alpha(RB)$ for any set $B \in \mathcal{B}$.

The following example shows that the equation (4.2) is not true for atomic measures with the definition (2.6) of GS-points. This explains, why we choose the Steiner point of the supporting faces in Definition 3.1 instead of the minimal norm element.

Example 4.3 Let $\lambda = \nu = 1$, $C := \operatorname{co}\{\binom{0}{0}, \binom{2}{0}\}$, $D := \operatorname{co}\{\binom{-3}{0}, \binom{-1}{0}\}$ and $\alpha \in \mathcal{AM}$ be concentrated in $l = \binom{0}{1}$. Then, $C + D = \operatorname{co}\{\binom{-3}{0}, \binom{1}{0}\}$ and

$$\mathrm{m}(Y(l,C)) + \mathrm{m}(Y(l,D)) = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix} = \mathrm{m}(Y(l,C+D))$$

which would lead to inequality in (4.2) for $\widetilde{\mathrm{St}}_\alpha(\cdot)$, in contrast to $\mathrm{St}_\alpha(\cdot)$ with

$$\mathrm{St}(Y(l,C)) + \mathrm{St}(Y(l,D)) = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} = \mathrm{St}(Y(l,C+D)).$$

The following GS-representations of Minkowski operations on sets are straightforward consequences of Corollary 3.5 and Lemma 4.1, or follow easily from (3.3) and [14] for $\mathcal{SM}$.

Corollary 4.4 *Let $C, D \in \mathcal{C}(\mathbb{R}^n)$ and $\lambda, \nu \geq 0$. Then,*

$$\lambda C + \nu D = \begin{cases} \operatorname{co} \overline{\bigcup\limits_{\alpha \in \mathcal{AM}} \{\lambda \mathrm{St}_\alpha(C) + \nu \mathrm{St}_\alpha(D)\}}, & \text{if } \mathcal{M}_{sp} = \mathcal{AM}, \\ \overline{\bigcup\limits_{\alpha \in \mathcal{M}_{sp}} \{\lambda \mathrm{St}_\alpha(C) + \nu \mathrm{St}_\alpha(D)\}}, & \text{if } \mathcal{M}_{sp} \in \{\mathcal{SM}, \operatorname{co}(\mathcal{AM})\}. \end{cases}$$

The next theorems provide explicit forms of the Demyanov difference of two sets in terms of their GS-representations.

Theorem 4.5 *Let $C, D \in \mathcal{C}(\mathbb{R}^n)$. Then,*

$$C \dot{-} D = \overline{\bigcup_{\alpha \in \mathcal{SM}} \{\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D)\}}. \tag{4.4}$$

Proof Denote $U := \bigcup\limits_{\alpha \in \mathcal{SM}} \{\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D)\}$. The convexity of the set U follows easily from the convexity of $\mathcal{SM}$ and (4.1). We prove now that $C \dot{-} D = \overline{U}$.
"$\subset$": Let $y(l,C) - y(l,C) \in C \dot{-} D$ with $l \in S_{n-1} \cap T_C \cap T_D$. By Lemma 3.2, $y(l,C) - y(l,D) = \mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D)$ with $\alpha \in \mathcal{AM}$ concentrated in such l. According to Lemma 3.8, there exists a normal Dirac sequence $(\theta_m(\cdot))_{m \in \mathbb{N}} \subset \mathcal{C}^1(B_1)$ with corresponding measures $(\alpha_m)_{m \in \mathbb{N}} \subset \mathcal{SM}$ and

$$\mathrm{St}_{\alpha_m}(C) \xrightarrow[m\to\infty]{} y(l,C), \quad \mathrm{St}_{\alpha_m}(D) \xrightarrow[m\to\infty]{} y(l,D).$$

Since $\mathrm{St}_{\alpha_m}(C) - \mathrm{St}_{\alpha_m}(D) \in U$ for every $m \in \mathbb{N}$, we have

$$y(l,C) - y(l,D) = \lim_{m\to\infty} (\mathrm{St}_{\alpha_m}(C) - \mathrm{St}_{\alpha_m}(D)) \in \overline{U}.$$

Taking the closed convex hull on both sides over all considered l, we complete the proof of this inclusion.

"⊃": Let $\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) \in U$ with $\alpha \in \mathcal{SM}$ and density function $\theta(\cdot) \in \mathcal{C}^1(B_1)$. Then,

$$\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) = \int_{B_1} (\mathrm{St}(Y(p,C)) - \mathrm{St}(Y(p,D)))\theta(p)\, d\mu(p)$$
$$= \int_{B_1 \cap T_C \cap T_D} (y(p,C) - y(p,D))\theta(p)\, d\mu(p).$$

By the convexity and compactness of $C \dot{-} D$ we have

$$\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) \in C \dot{-} D, \tag{4.5}$$

since $y(p,C) - y(p,D) \in C \dot{-} D$ and $C \dot{-} D$ equals the Aumann integral $\int_{B_1} \theta(p)(C \dot{-} D) d\mu(p)$. Taking the closed union over all $\alpha \in \mathcal{SM}$ in (4.5) completes the proof. □

In the next theorem we replace the set $\mathcal{SM}$ in (4.4) by $\mathrm{co}(\mathcal{AM})$.

Theorem 4.6 *Let $C, D \in \mathcal{C}(\mathbb{R}^n)$. Then,*

$$C \dot{-} D = \overline{\bigcup_{\alpha \in \mathrm{co}(\mathcal{AM})} \{\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D)\}} = \overline{\mathrm{co} \bigcup_{\alpha \in \mathcal{AM}} \{\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D)\}}. \tag{4.6}$$

Proof Denote

$$U := \bigcup_{\alpha \in \mathrm{co}(\mathcal{AM})} \{\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D)\}, \tag{4.7}$$

$$V := \mathrm{co} \bigcup_{l \in S_{n-1}} \{\mathrm{St}(Y(l,C)) - \mathrm{St}(Y(l,D))\}. \tag{4.8}$$

The convexity of the set U follows from the convexity of $\mathrm{co}(\mathcal{AM})$.
First we show that $\overline{V} = C \dot{-} D$.
The inclusion "⊃" is obvious. To prove "⊂", we use Theorem 4.5 for the sets $Y(l,C)$ and $Y(l,D)$ and the normalized Lebesgue measure, thus we get $\mathrm{St}(Y(l,C)) - \mathrm{St}(Y(l,D)) \in Y(l,C) \dot{-} Y(l,D)$. Since $Y(l,C) \dot{-} Y(l,D)$ is a subset of $C \dot{-} D$ ([15, Lemma 3.2]), it follows that $\overline{V} = C \dot{-} D$.

Now, we show that $\overline{U} = \overline{V}$.
"⊃" follows from Lemma 3.2 and the convexity of U.
"⊂": For $\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) \in U$ with $\alpha \in \mathcal{AM}$ concentrated in $l \in S_{n-1}$,

$$\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) = \mathrm{St}(Y(l,C)) - \mathrm{St}(Y(l,D)) \in V.$$

For a convex combination $\alpha = \sum_{i=1}^k \lambda_i \alpha_i \in \mathrm{co}(\mathcal{AM})$ we have by Lemma 3.2

$$\mathrm{St}_\alpha(C) - \mathrm{St}_\alpha(D) = \sum_{i=1}^k \lambda_i (\mathrm{St}_{\alpha_i}(C) - \mathrm{St}_{\alpha_i}(D)) \in \mathrm{co}(V) = V.$$

Taking the union over all $\alpha \in \mathrm{co}(\mathcal{AM})$ and the closure of both sides, we get the needed inclusion. □

Theorem 4.6 also motivates the use of the Steiner point instead of the minimal norm element in Definition 3.1. The following example shows that the equality (4.6) is not true with definition (2.6) for GS-points with $\alpha \in \mathcal{AM}$.

Example 4.7 Take $C := \operatorname{co}\{\binom{0}{0}, \binom{2}{0}\},\ D := \operatorname{co}\{\binom{-3}{0}, \binom{-1}{0}\}$.
Clearly, $C \dot{-} D = \{\binom{3}{0}\}$ and

$$\operatorname{co}\overline{\bigcup_{\alpha\in\mathcal{AM}} \{\widetilde{\operatorname{St}}_\alpha(Y(l,C)) - \widetilde{\operatorname{St}}_\alpha(Y(l,D))\}} = \operatorname{co}\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\} \underset{\neq}{\supset} C \dot{-} D.$$

The following representations of the Demyanov distance on $\mathcal{C}(\mathbb{R}^n)$ hold due to Theorems 4.5 and 4.6.

Corollary 4.8 *Let $C, D \in \mathcal{C}(\mathbb{R}^n)$. Then,*

$$\begin{aligned}\mathrm{d}_D(C,D) &= \sup_{\alpha\in\mathcal{SM}} \|\operatorname{St}_\alpha(C) - \operatorname{St}_\alpha(D)\| = \sup_{\alpha\in\operatorname{co}(\mathcal{AM})} \|\operatorname{St}_\alpha(C) - \operatorname{St}_\alpha(D)\| \\ &= \sup_{\alpha\in\mathcal{AM}} \|\operatorname{St}_\alpha(C) - \operatorname{St}_\alpha(D)\| = \sup_{l\in S_{n-1}} \|\operatorname{St}(Y(l,C)) - \operatorname{St}(Y(l,D))\|.\end{aligned}$$

Proof The first two equalities are due to Theorems 4.5 and 4.6 as well as (2.1). For the last two equalities, we note that due to Lemma 3.2 and the triangle inequality of the norm,

$$\sup_{\alpha\in\mathcal{AM}} \|\operatorname{St}_\alpha(C) - \operatorname{St}_\alpha(D)\| = \sup_{\alpha\in\operatorname{co}(\mathcal{AM})} \|\operatorname{St}_\alpha(C) - \operatorname{St}_\alpha(D)\|. \tag{4.9}$$

Again, Lemma 3.2 yields $\operatorname{St}_\alpha(C) = \operatorname{St}(Y(l,C))$ for $\alpha \in \mathcal{AM}$ concentrated in $l \in S_{n-1}$ which completes the proof. □

In the next section we consider set-valued maps defined on a finite interval $I = [t_0, T]$ with images in $\mathcal{C}(\mathbb{R}^n)$. By (3.3) and (3.6) such maps have representations by GS-selections with smooth measures resp. convex combinations of atomic measures. For computational purposes a finite number of directions $\mathcal{G}_M = \{l^1, \dots, l^M\} \subset S_{n-1} \cap T_C$ approximating $S_{n-1} \cap T_C$ are chosen and the convex hull of the corresponding atomic measures are used to generate finitely many GS-selections.

5 Regularity Properties of GS-Selections

In this section, we aim to characterize regularity properties of set-valued maps by the same properties possessed uniformly by a family of GS-selections obtained by atomic and smooth measures.

Given a measurable set-valued map $F(\cdot)$ with images in $\mathcal{C}(\mathbb{R}^n)$ on I and a probability measure $\alpha \in \mathcal{M}_{\mathrm{sp}}$, a GS-selection of this map is defined by $t \mapsto \operatorname{St}_\alpha(F(t))$, i.e. the GS-point of the image $F(t)$, $t \in I$.

The equations (3.3) and (3.4) immediately imply the equalities

$$F(t) = \overline{\bigcup_{\alpha \in \mathcal{SM}} \{\mathrm{St}_\alpha(F(t))\}} = \overline{\bigcup_{m \in \mathbb{N}} \{\mathrm{St}_{\alpha_m}(F(t))\}} \quad (t \in I) \tag{5.1}$$

with $(\alpha_m)_{m \in \mathbb{N}} \subset \mathcal{SM}$. The latter equality forms a Castaing representation of the set-valued map (cf. [12, Theorem 3.4]).

In [14, 12] results of the following type were obtained: If the set-valued map $F : I \to \mathcal{C}(\mathbb{R}^n)$ has a regularity property (e.g. continuous, Lipschitz, pseudo-Lipschitz and semi-differentiable) w.r.t. Hausdorff distance, then each GS-selection is regular, but this regularity is not necessarily uniform in the family of all GS-selections.

Recall that $F(\cdot)$ is Lipschitz with respect to the Demyanov metric (D-Lipschitz) on the domain I with a constant L, if

$$\mathrm{d_D}(F(s), F(t)) \le L \cdot |s - t|, \qquad s, t \in I. \tag{5.2}$$

The following equivalence between the D-Lipschitz continuity of a multimap and the uniform Lipschitz continuity of its GS-selections follows directly from Corollary 4.8.

Proposition 5.1 *The mapping $F : I \to \mathcal{C}(\mathbb{R}^n)$ is D-Lipschitz with a constant L, if and only if the GS-selections $(\mathrm{St}_\alpha(F(\cdot)))_{\alpha \in \mathcal{M}_{sp}}$ are uniformly Lipschitz with the same constant L, i.e.*

$$\sup_{\alpha \in \mathcal{M}_{sp}} \| \mathrm{St}_\alpha(F(s)) - \mathrm{St}_\alpha(F(t)) \| \le L \cdot |s - t|, \qquad s, t \in I. \tag{5.3}$$

In [14, Theorem 3.6] and [12, Proposition 2.4] it is shown that the Lipschitz constant $\widetilde{L}$ of a GS-selection of a Lipschitz (w.r.t. the Hausdorff metric) multifunction, generated by a measure $\alpha \in \mathcal{SM}$, equals

$$\widetilde{L} = n \cdot \max_{p \in S_{n-1}} \theta(p) + \max_{p \in B_1} \|\nabla \theta(p)\|$$

and depends linearly on the sup-norms of the density $\theta(\cdot)$ of α and of the gradient of $\theta(\cdot)$. If the multifunction is Lipschitz w.r.t. the Demyanov metric, a uniform Lipschitz constant for its GS-selections in (5.1) (i.e. a uniform bounded derivative of the corresponding densities) exists.

Moreover, the D-continuity of a multifunction $F(\cdot)$ is equivalent to the uniform continuity of the family of GS-selections $(\mathrm{St}_\alpha(F(\cdot)))_{\alpha \in \mathcal{M}_{\mathrm{sp}}}$.

Note that D-continuity is stronger than Hausdorff-continuity. The next example modifiying Example [15, Example 3.1] presents a multifunction which is Lipschitz continuous with respect to the Hausdorff metric, but not with respect to the Demyanov metric and has discontinuous GS-selections generated by atomic measures.

Example 5.2 Consider $I = [-\frac{\pi}{2}, \frac{3\pi}{2}]$ and define the set-valued map $F : I \to \mathcal{C}(\mathbb{R}^2)$ with $F(t) = \operatorname{co}\{\binom{0}{0}, \binom{\cos(t)}{\sin(t)}\}$. Consider the atomic measure $\alpha_{[l]} \in \mathcal{AM}$

concentrated in $l = \binom{1}{0}$. Then, Lemma 3.2 shows:

$$\mathrm{St}_{\alpha_{[l]}}(F(t)) = \begin{cases} \binom{\cos(t)}{\sin(t)}, & t \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ \frac{1}{2}\binom{0}{1}, & t = \frac{\pi}{2}, \\ \binom{0}{0}, & t \in (\frac{\pi}{2}, \frac{3\pi}{2}), \\ \frac{1}{2}\binom{0}{-1}, & t = \frac{-\pi}{2}, \frac{3\pi}{2}. \end{cases}$$

Clearly, $\mathrm{St}_{\alpha_{[l]}}(F(\cdot))$ is discontinuous, and the same is true for any measure $\alpha \in \mathcal{AM}$.

Definition 5.3 Consider $I = [t_0, T]$ and a multimap $F : I \to \mathcal{C}(\mathbb{R}^n)$. F has bounded D-variation, if there exists a constant $V < \infty$ such that for any partition $t_0 < t_1 < ... < t_N = T$, $N \in \mathbb{N}$,

$$\sum_{i=0}^{N-1} \mathrm{d_D}(F(t_{i+1}), F(t_i)) \le V.$$

The minimal of such numbers V is called the D-variation of F and denoted by $\bigvee_I^D F$.

For the family of measures $\mathcal{M}_{\mathrm{sp}}$ we say that $F(\cdot)$ has bounded $\mathcal{M}_{\mathrm{sp}}$-variation on I, if there exists a constant $V < \infty$ such that for all partitions $t_0 < t_1 < ... < t_N = T$ with $N \in \mathbb{N}$,

$$\sup_{\alpha \in \mathcal{M}_{\mathrm{sp}}} \sum_{i=0}^{N-1} \| \mathrm{St}_\alpha(F(t_{i+1})) - \mathrm{St}_\alpha(F(t_i)) \| \le V.$$

The minimal of such constants V will be called $\mathcal{M}_{\mathrm{sp}}$-variation of F and will be denoted by $\bigvee_I^{\mathcal{M}_{\mathrm{sp}}} F$.

Note that, by the convexity, the above variation will not change, if one takes the set $\mathcal{AM}$ instead of the index set $\mathrm{co}(\mathcal{AM})$. By Lemma 3.2,

$$\bigvee_I^{\mathcal{AM}} F = \sup_{\substack{l \in S_{n-1} \\ t_0 < t_1 < ... < t_N = T \\ N \in \mathbb{N}}} \sum_{i=0}^{N-1} \| \mathrm{St}(Y(l, F(t_{i+1}))) - \mathrm{St}(Y(l, F(t_i))) \|.$$

For the purpose of numerical set-valued integration, the weaker $\mathcal{M}_{\mathrm{sp}}$-variation is applied instead of the D-variation.

Proposition 5.4 *If $F : I \to \mathcal{C}(\mathbb{R}^n)$, then*

$$\bigvee_I^{\mathcal{AM}} F = \bigvee_I^{\mathrm{co}(\mathcal{AM})} F \le \bigvee_I^{D} F, \quad \bigvee_I^{\mathcal{SM}} F \le \bigvee_I^{D} F.$$

Proof Clearly, $\bigvee_I^{\mathcal{AM}} F \le \bigvee_I^{\mathrm{co}(\mathcal{AM})} F$. Take a convex combination of measures $\alpha = \sum_{j=1}^k \lambda_j \alpha_j$, $\alpha_j \in \mathcal{AM}$. Then, for an arbitrary partition $(t_i)_{i=0,\ldots,N}$ of I we have by (4.1) and the convexity of the norm

$$\begin{aligned}
&\sum_{i=0}^{N-1} \| \operatorname{St}_\alpha(F(t_{i+1})) - \operatorname{St}_\alpha(F(t_i))\| \\
\le &\sum_{j=1}^{k} \lambda_j \sum_{i=0}^{N-1} \| \operatorname{St}_{\alpha_j}(F(t_{i+1})) - \operatorname{St}_{\alpha_j}(F(t_i))\| \\
\le &\sup_{\alpha \in \mathcal{AM}} \sum_{i=0}^{N-1} \| \operatorname{St}_\alpha(F(t_{i+1})) - \operatorname{St}_\alpha(F(t_i))\|,
\end{aligned}$$

which shows that $\bigvee_I^{\mathrm{co}(\mathcal{AM})} F \le \bigvee_I^{\mathcal{AM}} F$.

On the other hand, Corollary 4.8 shows $\| \operatorname{St}_\alpha(F(t_{i+1})) - \operatorname{St}_\alpha(F(t_i))\| \le \mathrm{d_D}(F(t_{i+1}), F(t_i))$ for $\alpha \in \mathcal{M}_{\mathrm{sp}}$, $i = 0, \ldots, N-1$. With this fact the inequality $\bigvee_I^{\mathcal{M}_{\mathrm{sp}}} F \le \bigvee_I^{D} F$ follows easily.

We conjecture that the $\mathcal{SM}$-variation of F equals its $\mathcal{AM}$-variation, but this is still an open question. At least, the following criterion is obvious.

Corollary 5.5 *If $F : I \to \mathcal{C}(\mathbb{R}^n)$ has bounded D-variation, then $F(\cdot)$ has bounded $\mathcal{AM}$-variation and bounded $\mathcal{SM}$-variation.*

6 Approximate Set-Valued Integration

The commutation between the classical Steiner selection and the integral operator is well-known (cf. [20]) and is based on the well-known formula of the support function of the Aumann integral [1]:

$$\delta^*(p, \int_I F(t)\ )dt = \int_I \delta^*(p, F(t))\ dt, \qquad p \in \mathbb{R}^n. \tag{6.1}$$

Let us recall some classical notions for a set-valued map $F : I \to \mathcal{C}(\mathbb{R}^n)$. It is called measurable, if the inverse image of each open set is a measurable set (cf. [4, Definition 8.1.1]). In [23, Corollary 2.5] it is proved that a measurable map can be approximated by measurable, simple maps. Furthermore, $F(\cdot)$ is integrably bounded, if $\|F(\cdot)\|$ is bounded by an integrable function (cf. [4, Section 8.6]).

Proposition 6.1 ([20, Proposition 4.5]) *Let $F : I \to \mathcal{C}(\mathbb{R}^n)$ be measurable and integrably bounded. Then,*

$$\operatorname{St}(\int_I F(t)\ dt) = \int_I \operatorname{St}(F(t))\ dt.$$

We will provide in this section generalizations of this commutation to GS-selections w.r.t. smooth and atomic measures.

Proposition 6.2 *Let* $F : I \to \mathcal{C}(\mathbb{R}^n)$ *be measurable and integrably bounded and* $\alpha \in \mathcal{SM}$*. Then,*

$$\mathrm{St}_\alpha(\int_I F(t)\ dt) = \int_I \mathrm{St}_\alpha(F(t))\ dt.$$

Proof For $\alpha \in \mathcal{SM}$, $\mathrm{St}_\alpha(F(\cdot))$ is integrable as a composition of the Lipschitz continuous $\mathrm{St}_\alpha(\cdot)$ (cf. [14, Theorem 3.6]) and the measurable and integrably bounded $F(\cdot)$ (cf. [4, Theorem 8.2.8]). Denote $C := \int_I F(t)dt \in \mathcal{C}(\mathbb{R}^n)$. It follows from [14, Theorem 3.6] and Lemma 3.2 that

$$\mathrm{St}_\alpha(C) = \frac{1}{V_n}\Big(\int_{S_{n-1}} p \cdot \delta^*(p, C)\theta(p)\ \omega(dp) - \int_{B_1} \delta^*(p, C)\nabla\theta(p)\ dp\Big) \tag{6.2}$$

with $\omega(\cdot)$ being the Lebesgue surface measure on S_{n-1}. Replacing $\delta^*(p, C)$ by (6.1) in the last equation, and using Fubini's theorem and (6.2) for $\mathrm{St}_\alpha(F(t))$, we get

$$\begin{aligned}\mathrm{St}_\alpha(C) &= \int_I \frac{1}{V_n}\Big(\int_{S_{n-1}} p \cdot \delta^*(p, F(t))\theta(p)\ \omega(dp) - \int_{B_1} \delta^*(p, F(t))\nabla\theta(p)\ dp\Big)\ dt \\ &= \int_I \mathrm{St}_\alpha(F(t))\ dt.\end{aligned}$$

□

For later use, we prove that the supporting face of the Aumann integral is the Aumann integral of the supporting face of the integrand in the same direction l.

Proposition 6.3 *Let* $F : I \rightrightarrows \mathbb{R}^n$ *be measurable and integrably bounded with images in* $\mathcal{C}(\mathbb{R}^n)$ *and let* $l \in S_{n-1}$*. Then,* $Y(l, F(\cdot))$ *is measurable, integrably bounded and fulfills*

$$Y(l, \int_I F(t)\ dt) = \int_I Y(l, F(t))\ dt.$$

Proof First of all, $Y(l, F(\cdot))$ has images in $\mathcal{C}(\mathbb{R}^n)$ and is integrably bounded by the same function as $F(\cdot)$ itself. Since $Y(l, F(\cdot)) = \mathrm{argmax}_{x \in F(t)} \langle l, x\rangle$ is the marginal map of $f(x) = \langle l, x\rangle$, it follows that $Y(l, F(\cdot))$ is measurable by [4, Theorem 8.2.11].
"$\subset$": Let $z \in Y(l, \int_I F(t)dt)$. Then, $z \in \int_I F(t)dt$ and

$$\langle l, z\rangle = \delta^*(l, \int_I F(t)\ dt) = \int_I \delta^*(l, F(t))\ dt.$$

From the definition of the Aumann integral, an integrable selection $f(\cdot)$ of $F(\cdot)$ must exist with $z = \int_I f(t)dt$. Assume that there exists a measurable subset $J \subset I$ with Lebesgue measure greater 0 and $f(t) \notin Y(l, F(t))$ on J. Then,

$$\int\limits_I \delta^*(l, F(t))\ dt = \langle l, z \rangle = \int\limits_I \langle l, f(t) \rangle\ dt$$
$$< \int\limits_{I \setminus J} \delta^*(l, F(t))\ dt + \int\limits_J \delta^*(l, F(t))\ dt = \int\limits_I \delta^*(l, F(t))\ dt$$

which is a contradiction. Hence, $f(t) \in Y(l, F(t))$ for almost every $t \in I$ and thus, $z \in \int_I Y(l, F(t))dt$.

"$\supset$": Let $g(\cdot)$ be an integrable selection of $Y(l, F(\cdot))$. Then,

$$\langle l, g(t) \rangle = \delta^*(l, F(t)) \quad \text{(a.e. in } I\text{)},$$
$$\langle l, \int\limits_I g(t)\ dt \rangle = \int\limits_I \langle l, g(t) \rangle\ dt = \int\limits_I \delta^*(l, F(t))\ dt = \delta^*(l, \int\limits_I F(t)\ dt).$$

Since $g(\cdot)$ is also an integrable selection of $F(\cdot)$, we have from above that

$$\int\limits_I g(t)\ dt \in Y(l, \int\limits_I F(t)\ dt).$$

□

The following proposition is the generalization of Proposition 6.1 to GS-selections for (convex combinations of) atomic measures.

Proposition 6.4 *Let* $F : I \to \mathcal{C}(\mathbb{R}^n)$ *be measurable and integrably bounded and* $\alpha = \alpha_{[l]} \in \mathcal{AM}$, $l \in S_{n-1}$, *or* $\alpha \in \mathrm{co}(\mathcal{AM})$. *Then,*

$$\mathrm{St}_\alpha(\int\limits_I F(t)\ dt) = \int\limits_I \mathrm{St}_\alpha(F(t))\ dt.$$

Proof Proposition 6.3 shows that $Y(l, F(\cdot))$ is measurable and integrably bounded with images in $\mathcal{C}(\mathbb{R}^n)$. By Proposition 6.1,

$$\mathrm{St}(\int\limits_I Y(l, F(t))\ dt) = \int\limits_I \mathrm{St}(Y(l, F(t)))\ dt.$$

Applying once more Proposition 6.3 we arrive at

$$\mathrm{St}(\int\limits_I Y(l, F(t))\ dt) = \mathrm{St}(Y(l, \int\limits_I F(t)\ dt)).$$

Lemma 3.2 shows that for $\alpha = \alpha_{[l]} \in \mathcal{AM}$

$$\begin{aligned}\mathrm{St}_{\alpha_{[l]}}(\int_I F(t)\ dt) &= \mathrm{St}(Y(l, \int_I F(t)\ dt)) \\ &= \int_I \mathrm{St}(Y(l, F(t)))\ dt = \int_I \mathrm{St}_{\alpha_{[l]}}(F(t))\ dt.\end{aligned}$$

The equality for $\alpha \in \mathrm{co}(\mathcal{AM})$ is obvious. □

The last propositions and (3.3) yield the following GS-representation of the Aumann integral.

Corollary 6.5 *Let $F : I \to \mathcal{C}(\mathbb{R}^n)$ be measurable and integrably bounded. Then,*

$$\int_I F(t)\ dt = \begin{cases} \mathrm{co}\,\overline{\bigcup\limits_{\alpha \in \mathcal{AM}} \{\int_I \mathrm{St}_\alpha(F(t))\ dt\}} & \text{for } \mathcal{M}_{sp} = \mathcal{AM}, \\ \overline{\bigcup\limits_{\alpha \in \mathcal{M}_{sp}} \{\int_I \mathrm{St}_\alpha(F(t))\ dt\}} & \text{for } \mathcal{M}_{sp} = \mathrm{co}(\mathcal{AM}), \mathcal{SM}. \end{cases}$$

In order to give error estimates for the approximation of the integral $\int_I F(t)dt$ by Riemann sums, we first study a property for the Demyanov distance and then a convergence result with the D-variation of F.

Proposition 6.6 *Let $F : I \to \mathcal{C}(\mathbb{R}^n)$ be bounded, measurable and has bounded $\mathcal{M}_{sp}$-variation. Then,*

$$\| \mathrm{St}_\alpha(\int_I F(t)\ dt) - h \sum_{i=0}^{N-1} \mathrm{St}_\alpha(F(t_i)) \| \leq h \bigvee_I^{\mathcal{M}_{sp}} F$$

for $N \in \mathbb{N}$, $h = \frac{T - t_0}{N}$.

Proof The proof is straight forward and uses Propositions 6.2 and 6.4:

$$\begin{aligned}\Delta :=& \| \mathrm{St}_\alpha(\int_I F(t)\ dt) - h \sum_{i=0}^{N-1} \mathrm{St}_\alpha(F(t_i)) \| \\ =& \| \sum_{i=0}^{N-1} \mathrm{St}_\alpha(\int_{[t_i, t_{i+1}]} F(t)\ dt) - \sum_{i=0}^{N-1} h \cdot \mathrm{St}_\alpha(F(t_i)) dt \| \\ \leq& \sum_{i=0}^{N-1} \| \int_{[t_i, t_{i+1}]} (\mathrm{St}_\alpha(F(t)) - \mathrm{St}_\alpha(F(t_i)))\ dt \|\end{aligned}$$

The estimations

$$\begin{aligned}
&\| \int\limits_{[t_i,t_{i+1}]} (\mathrm{St}_\alpha(F(t)) - \mathrm{St}_\alpha(F(t_i)))\ dt\| \\
&\leq \int\limits_{[t_i,t_{i+1}]} \| \mathrm{St}_\alpha(F(t)) - \mathrm{St}_\alpha(F(t_i)) dt\|\ dt \leq \int\limits_{[t_i,t_{i+1}]} \bigvee_{[t_i,t]} \mathrm{St}_\alpha(F(\cdot))\ dt \\
&\leq \int\limits_{[t_i,t_{i+1}]} \bigvee_{[t_i,t_{i+1}]} \mathrm{St}_\alpha(F(\cdot))\ dt = h \bigvee_{[t_i,t_{i+1}]} \mathrm{St}_\alpha(F(\cdot))
\end{aligned}$$

show

$$\Delta \leq h \sum_{i=0}^{N-1} \bigvee_{[t_i,t_{i+1}]} \mathrm{St}_\alpha(F(\cdot)) = h \bigvee_I \mathrm{St}_\alpha(F(\cdot)).$$

□

Remark 6.7 Let $F, G : I \to \mathcal{C}(\mathbb{R}^n)$ be measurable and integrably bounded and $C_i, D_i \in \mathcal{C}(\mathbb{R}^n)$, $i = 1, \ldots, k$. Then,

$$\mathrm{d_D}(\sum_{i=1}^k C_i, \sum_{i=1}^k D_i) \leq \sum_{i=1}^k \mathrm{d_D}(C_i, D_i) \quad \text{(cf. [15, Lemma 6.2])},$$

$$\mathrm{d_D}(\int\limits_I F(t)\ dt, \int\limits_I G(t)\ dt) \leq \int\limits_I \mathrm{d_D}(F(t), G(t))\ dt,$$

where it is easy to check that $\mathrm{d_D}(F(\cdot), G(\cdot))$ is integrable.

Taking the supremum over $\alpha \in \mathcal{M}_{\mathrm{sp}}$ in Proposition 6.6 or repeat the idea of the proof and use the estimations of Remark 6.7, we obtain the following result in view of Corollary 4.8.

Corollary 6.8 *If $F : I \to \mathcal{C}(\mathbb{R}^n)$ has bounded $\mathcal{M}_{sp}$-variation, then,*

$$\mathrm{d}_D(\int\limits_I F(t)\ dt, h \sum_{i=0}^{N-1} F(t_i)) \leq h \bigvee_I^{\mathcal{M}_{sp}} F \leq \bigvee_I^{D} F.$$

The last two estimates are analogous to the estimate obtained in [5, Korollar 1.3.5] and [17] in terms of the Hausdorff distance, supposing that F has bounded $\mathrm{d_H}$-variation. Please notice that here we obtain an estimate in the stronger Demyanov metric under the stronger condition on the variation. One advantage is that Corollary 6.5 shows that the Aumann integral can be densely represented with the help of GS-selections (by smooth resp. convex combinations of atomic measures). Hence, the set-valued approximation of the Aumann integral is reduced to the approximation of the pointwise integrals of the GS-selections. Moreover, Proposition 6.6 and Corollary 4.8 imply the uniform convergence of

$$h \sum_{i=0}^{N-1} \mathrm{St}_\alpha(F(t_i)) \xrightarrow[N \mapsto \infty]{} \int_I \mathrm{St}_\alpha(F(t))\, dt$$

in $\alpha \in \mathcal{M}_{\mathrm{sp}}$, if $F(\cdot)$ has bounded D-variation or (slightly weaker) bounded $\mathcal{M}_{\mathrm{sp}}$-variation. Thus, the integrals of piecewise constant approximations of GS-selections of F with measures in $\mathcal{M}_{\mathrm{sp}}$ approximate the integral of the GS-selections of F of order 1 uniformly in $\mathcal{M}_{\mathrm{sp}}$.

7 Conclusions

Generalized Steiner points are a proper tool for the representation of convex compact sets, because of the arithmetic and metric properties 1.–3. (cf. the introduction). Moreover, the differences of GS-points corresponding to the same measure generate a dense representation of the Demyanov difference and the supremum norm of these differences yields the Demyanov distance. In a forthcoming paper, we will study an embedding of the cone of the convex, compact subsets in $\mathbb{R}^n$ into a linear metric space with the help of the GS-points, in the spirit of [26, 22, 6, 7].

Further research will also be directed towards the study of regularity and approximations of set-valued maps in terms of their GS-selections. We have demonstrated here how this approach works for the notions of Lipschitz continuity and bounded variation of a multifunction, and for one numerical set-valued quadrature method.

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