An efficient algorithm for perturbed shortest path problems

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We present an efficient algorithm for perturbed shortest paths problems that arise, e.g., in dynamic games. Via a min-max version of Dijkstra's algorithm we compute piecewise constant upper and lower bounds on the upper value function of the game. Convergence of the bounds in the limit of vanishing cell diameter can be proved. The method is numerically demonstrated on a simple 2d dynamic game, the homicidal chauffeur.

1 Problem formulation

We consider a discrete time dynamical system

$$x(n+1) = f(x(n), u(n), w(n)), \quad k = 0, 1, \ldots, \tag{1}$$

controlled by two opposing players who chose controls $u(n) \in U$ and $w(n) \in W$ at each time instance. Player u tries to steer the state $x \in X$ of the system into a target set $T \subset X$ as fast as possible, while player w tries to prevent u from doing this. More precisely, at each time step, player u infers the nonnegative running cost $\ell(x, u) \geq 0$ and he tries to minimize the accumulated cost

$$J(\mathbf{x}_{u,w}) = \sum_{n=0}^{n_{\mathbf{x}_{u,w}}-1} \ell(x(n), u(n),$$

where $\mathbf{x}_{u,w}$ is the trajectory $(x(0), x(1), \ldots)$ generated by the two players and $n_{\mathbf{x}_{u,w}} = \inf\{n \geq 0 \mid x(n) \in T\}$ is the first time that the trajectory enters the target set T. We assume that player w has the advantage of knowing the control of player u at each time step, however, he cannot forsee future moves of u. Formally, player w chooses his control according to a nonanticipating strategy $\beta : U^{\mathbb{N}} \to W^{\mathbb{N}}$, i.e. we have $w(n) = \beta[u](n)$, with β satisfying $u(n) = u'(n)$ for all $n \leq N \Rightarrow \beta[u](n) = \beta[u'](n)$ for all $n \leq N$. The *upper value* of the dynamic game is then given by

$$V(x) = \sup_{\beta \in \mathcal{B}} \inf_{\mathbf{u} \in U^{\mathbb{N}}} J(\mathbf{x}_{u,\beta[u]}), \tag{2}$$

where $\mathcal{B}$ denotes the set of all nonanticipating strategies. The upper value function fulfills the discrete Hamilton-Jacobi-Isaacs equation

$$V(x) \quad = \quad \min_{u \in U} \max_{w \in W} \left\{ \ell(x, u) + V(f(x, u, w)) \right\}. \tag{3}$$

The right hand side of this equation defines a dynamic programming type operator L on functions on X.

2 Discretization

We are going to approximate V by functions which are piecewise constant on a finite partition $\mathcal{P}$ of X. Our dynamic game can then be reformulated as a perturbed shortest path problem on a weighted directed hypergraph, where the nodes are the cells of the partition and the edges are defined by possible transitions from one cell to another under the given dynamics f.

The approximate upper value function can efficiently be computed using a min-max-version if Dijkstra's algorithm, cf. [3]. This algorithm can formally be derived from the discrete Hamilton-Jacobi-Isaacs equation by projecting the dynamic programming operator into the space of $\mathcal{P}$-piecewise constant functions. Depending on how one incorporates the dependence on x in the discretization, the discrete approximation yields a lower or upper bound on the true upper value function. The finite Hamilton-Jacobi-Isaacs equations

$$\begin{aligned}
\tilde{V}_u(Q) \quad &= \quad \min_{u \in U} \max_{w \in W, x \in Q} \{\ell(x, u) + \tilde{V}_u(f(x, u, w))\} \\
\tilde{V}_l(Q) \quad &= \quad \min_{u \in U, x \in Q} \max_{w \in W} \{\ell(x, u) + \tilde{V}_l(f(x, u, w))\}
\end{aligned}$$

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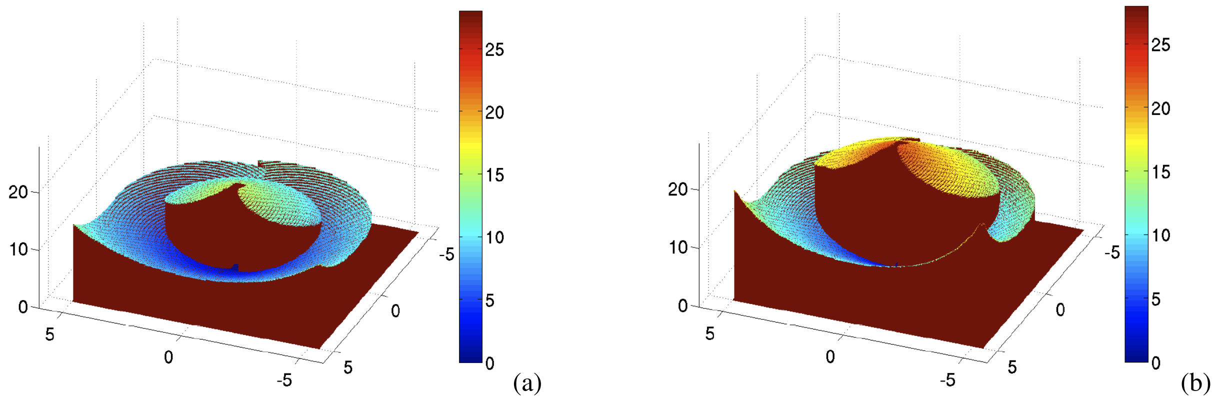

Fig. 1 (a) Lower and (b) upper approximation of the upper value function of the homicidal chauffeur game.

results in an upper, $\tilde{V}_u \geq V$, and in a lower approximation $\tilde{V}_\ell \leq V$ of V, respectively. These two bounds converge to V as the partition diameter goes to zero, cf. [3]. Based on $\tilde{V}_u$ and $\tilde{V}_\ell$, (approximately) optimal feedback laws can be be defined as

$$\begin{aligned} \tilde{F}^u(x) &= \operatorname*{argmin}_{u\in U} \max_{w\in W, x\in Q} \{\ell(x,u) + \tilde{V}_u(f(x,u,w))\}, \\ \tilde{F}^w(x,u) &= \operatorname*{argmax}_{w\in W} \{\ell(x,u) + \tilde{V}_l(f(x,u,w))\}. \end{aligned}$$

Theorem 2.1 *These feedbacks satisfy*

$$J(x, \tilde{F}^u, \beta) \leq \tilde{V}_u(x)$$

for all $\beta \in \mathcal{B}$ as well as $J(x, u, \tilde{F}^w) \geq V_\ell(x)$ for all $u \in U^{\mathbb{N}}$. Furthermore, $\tilde{F}^u$ is constant on each element Q of the partition.

For unperturbed systems, the fact that the feedback $\tilde{F}^u$ is constant on the cells enables a reinterpretation of the discretization as a quantization of the state space: all one needs to know in order to apply the feedback is the current cell that the system is in.

3 The homicidal chauffeur

As an example, we consider the following macabre scenario: a driver in a car that wants to catch a pedestrian. The car dynamics is given by

$$\dot{x}_{c,1} = v_c \sin\theta_c, \quad \dot{x}_{c,2} = v_c \cos\theta_c, \quad \dot{\theta}_c = u,$$

with $|u| \leq 1$, while the pedestrian walks at constant speed but can instantaneously change its direction:

$$\dot{x}_{p,1} = v_p \sin w, \quad \dot{x}_{p,2} = v_p \cos w,$$

with $w \in [0, 2\pi]$. In relative coordinates $x = x_p - x_c$ and for $v_c = 1$ and $v_p = 0.5$ the differential game can be written as

$$\dot{x}_1 = -x_2 u + \frac{1}{2}\sin w, \quad \dot{x}_2 = -1 + x_1 u + \frac{1}{2}\sin w.$$

We consider the discrete time system generated by integrating the differential equation for $h = 0.2$ time units, using constant functions u and w on the time interval of integration. The running cost is $\ell(x,u) = h$. Figure 1 shows $\tilde{V}_\ell$ (left) and $\tilde{V}_u$ (right) on a partition of 512^2 equally sized rectangles.

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