Computing minimal optimization horizons for stabilizing unconstrained MPC schemes

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We present a method for the computation of minimal stabilizing optimization horizons in MPC schemes without stabilizing terminal constraints. The method applies to general nonlinear control systems satisfying a controllability assumption. Key idea is the formulation of a small linear program whose solution determines the minimal horizon.

1 Introduction

Model predictive control (MPC) is a popular technique for the optimization based stabilization of nonlinear and/or constrained control systems. It relies on the well known fact, that for a nonlinear discrete time control system

$$x(n+1) = f(x(n), u(n)), \quad x(0) = x_0 \tag{1}$$

with $x(n) \in X$ and $u(n) \in U$ a stabilizing feedback law — i.e., a map $\mu : X \to U$ such that the origin is asymptotically stable for the closed loop system $x(n+1) = f(x(n), \mu(x(n)))$ — can be obtained by solving an infinite horizon optimal control problem. However, since such problems are computationally hard to solve, in MPC schemes one typically uses a receding horizon technique, in which the infinite horizon problem is replaced by a sequence of finite horizon problems to be solved online. A key parameter for the efficiency of this technique is the needed horizon length for the finite horizon problem, because the numerical effort for its solution crucially depends on this length. In this paper we sketch a method for estimating the minimal stabilizing horizon for MPC schemes without stabilizing terminal constraints and costs.

2 Preliminaries

In order to find a stabilizing feedback law for (1), one can define the infinite horizon cost function

$$J_\infty(x_0, u) := \sum_{n=0}^{\infty} l(x(n), u(n)), \tag{2}$$

with running cost $l : X \times U \to \mathbb{R}_{\geq 0}$. We define the optimal value function for this problem by

$$V_\infty(x_0) := \inf_{u \in \mathcal{U}} J_\infty(x_0, u)$$

with $\mathcal{U}$ denoting the space of control sequences $u : \mathbb{N}_0 \to U$. Using the dynamic programming principle one can prove that the optimal feedback law for this problem is given by

$$\mu_\infty(x) := \operatorname*{argmin}_{u \in U} \{l(x,u) + V_\infty(f(x,u))\} \qquad \text{and satisfies} \qquad V_\infty(x) = l(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x))). \tag{3}$$

Under appropriate upper and lower boundedness conditions on l (see, e.g., [1]), which we assume to hold henceforth, the second inequality in (3) shows that V_∞ is a Lyapunov function for the closed loop system $x(n+1) = f(x(n), \mu_\infty(x(n)))$ which implies asymptotic stability.

Unfortunalety, in practice this theoretically elegant approach is often infeasible because both V_∞ and μ_∞ are in general hard to compute. Thus, we replace the problem by the iterative online solution of the truncated finite horizon problem

$$J_N(x_0, u) := \sum_{n=0}^{N-1} l(x(n), u(n)) \tag{4}$$

for $N \in \mathbb{N}_0$ (using $\sum_{n=0}^{-1} = 0$) with optimal value function $V_N(x_0) := \inf_{u \in \mathcal{U}} J_N(x_0, u)$. Denoting the optimal control sequence for this problem and some $x_0 \in X$ by $u^*(0), \ldots, u^*(N-1)$, we obtain the receding horizon feedback law μ_N for this x_0 by setting

$$\mu_N(x_0) := u^*(0).$$

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Now the question we want to answer is: How large does the horizon N need to be such that we can guarantee asymptotic stability of the closed loop system $x(n+1) = f(x(n), \mu_N(x(n)))$?

Our analysis is based on the observation that if there exists $\alpha \in (0,1]$ such that the inequality

$$V_N(x) \geq V_N(f(x, \mu_N(x))) + \alpha l(x, \mu_N(x))) \qquad (5)$$

holds for all $x \in X$, then — under the same conditions on l as for the infinite horizon problem, above — the function V_N is a Lyapunov function for the closed loop system and thus asymptotic stability follows.

3 Main result

In order to obtain an estimate for N, clearly some conditions on f and l need to be imposed. Different conditions of this type have been used, e.g., in [2, 3, 4, 5]. Here we utilize a controllability condition which is a special case of the one used in [5]:

Assumption 3.1 There are $C > 0$, $\sigma \in [0,1)$ such that for each $x_0 \in X$ there exists a control sequence $u_{x_0} \in \mathcal{U}$ satisfying

$$l(x(n), u_{x_0}(n)) \leq C\,\sigma^n \min_{u \in U} l(x(0), u)$$

for all $n \in \mathbb{N}_0$.

Using C and σ from this condition we define the following linear optimization problem.

Problem 3.2 Given $N \in \mathbb{N}$, $C > 0$ and $\sigma \in [0,1)$, compute

$$\alpha := \min_{\lambda_0, \lambda_1, \ldots, \lambda_{N-1}, \nu \geq 0} \sum_{n=1}^{N-1} \lambda_n - \nu$$

subject to the constraints

$$\sum_{n=k}^{N-1} \lambda_n \leq C \frac{1-\sigma^{N-k}}{1-\sigma} \lambda_k, \quad k = 0, \ldots, N-2 \quad \text{and} \quad \nu \leq \sum_{n=0}^{j-1} \lambda_{n+1} + C \frac{1-\sigma^{N-j}}{1-\sigma} \lambda_{j+1}, \quad j = 0, \ldots, N-2.$$

For this problem we can obtain the following main result:

Theorem 3.3 *Let $N \in \mathbb{N}$, $C > 0$ and $\sigma \in [0,1)$ be given and let α be the optimal value from Problem 3.2. Then the following assertions hold.*
(i) If $\alpha > 0$, then every control system (1) *with running cost l satisfying Assumption 3.1 is stabilized by the MPC feedback μ_N.*
(ii) Conversely, if $\alpha < 0$, then there exists a control system (1) *and a running cost l satisfying Assumption 3.1 which is not stabilized by the MPC feedback μ_N.*

Idea of the proof: We only sketch the main ideas of the proof and refer to [5] for details.
(i) Let $x^*(k), u^*(k)$ be an optimal trajectory and a corresponding optimal control for the problem with horizon N. Then, using the Controllability Assumption 3.1 and the dynamic programming principle, we can prove that the values $\lambda_j = l(x^*(j), u^*(j))$ and $\nu = V_N(f(x^*(0), \mu_N(x^*(0)))) = V_N(x^*(1))$ satisfy the constraints in Problem 3.2. This observation can be used in order to conclude that (5) holds with α from Problem 3.2, which implies (i).
(ii) The proof of (ii) is obtained by an explicit construction of f and l.

Problem 3.2 is a small linear program which is fast to solve. Thus, it can be used in order to determine the minimal stabilizing horizon N for many parameter values C and σ in order to obtain detailed information about how N depends on C and σ. For a given control system (1), the information obtained this way can then be used in order to design running cost functions l which allow for MPC stabilization using small horizons N. Examples for such design procedures can be found in [5, 6].

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