Input-to-state stability, numerical dynamics and sampled-data control

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We investigate the relation between asymptotic stability for dynamical systems and families of approximations. Using suitably perturbed systems and the input-to-state stability property we develop a framework which yields necessary and sufficient conditions on the stability of the approximations ensuring stability of the approximated system. The results are formulated for numerical one step schemes for ordinary differential equations and for sampled-data systems.

1 Introduction

The analysis of stability properties is a central topic in mathematical control theory as well as in the theory of dynamical systems. When investigating the behavior of dynamical systems, the first (and often the only feasible) approach are numerical simulations. Beyond simulations, the progress in the development of fast and reliable algorithms, e.g., in the area of optimization, nowadays allows for numerical controller design methods even for complex nonlinear control systems. In the context of stability analysis, the key question in both simulation and design is whether stability properties — either observed in a numerical simulation or obtained for a numerical approximation by a suitable controller design method — carry over from the numerical approximation to the original approximated system. The answer to this question is nontrivial, because the usual convergence properties of numerical approximations hold on finite time intervals only, while stability properties are inevitably linked to the asymptotic behavior of the system, i.e., to infinite time intervals.

In this paper we will address this question for asymptotic stability properties for numerical simulations of ordinary differential equations as well as for for numerically designed controllers for sampled-data systems. We will treat these two system classes in a unified fashion by developing an abstract framework for perturbed systems which is applied to both settings by embedding the numerical approximation into the perturbed system. Proceeding this way, it is immediately clear that some kind of robustness of the asymptotic stability properties must be imposed such that they carry over to perturbed systems. In contrast to many papers (like, e.g., [16, 23, 24]) we will not use Lyapunov functions in order to obtain these robustness

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properties. Rather, as in [8], we work with a version of the input-to-state stability (ISS) property from nonlinear control theory and the fact that for continuous systems, bounded stability neighborhoods and sufficiently small perturbation range ISS is actually an inherent property of asymptotically stable sets, cf. Proposition 5.2. In principle, the approaches via Lyapunov functions and via ISS are equivalent, since ISS is equivalent to the existence of an appropriate ISS Lyapunov function, cf. also Remark 5.4, below. However, for two reasons we prefer an ISS based analysis: on the one hand, it leads to surprisingly elementary and straightforward proofs. On the other hand, throughout this paper we aim at formulating necessary *and* sufficient conditions for the approximating systems which are more easily obtained from ISS directly than from converse Lyapunov theorems, because estimates for attraction rates and robustness gains are directly available in the ISS formulation. This is crucial for our analysis because the uniformity of these rates and gains with respect to the approximation error turns out to be the right property in order to obtain necessary *and* sufficient conditions instead of the merely sufficient conditions often found in the literature. Specifically, for numerical approximation this allows us to combine the main results from [16] and [8] into one statement in Theorem 6.2 while for sampled-data systems we can show that appropriate versions of certain sufficient conditions derived in [24, 23] are in fact also necessary, cf. Theorems 7.1 and 7.2.

The aim of this paper is to highlight the similarities in the stability analysis of numerical approximations and sampled-data systems and to illustrate the contribution which the ISS property from nonlinear control theory can provide for this analysis. Thus, in order to avoid technicalities and keep the paper self contained we will not aim the the most detailed and general results possible. In particular, we restrict our attention to practical asymptotic stability of compact sets A, i.e., asymptotic stability up to a small neighborhood of A whose size depends on the numerical accuracy (or, equivalently, the existence of asymptotically stable sets contained in this neighborhood). Furthermore, we will always consider stability neighborhoods B which are arbitrarily large but bounded and do not depend on the approximation accuracy, and we only give asymptotic estimates with respect to the approximation error instead of trying to give bounds.

The organization of this paper is as follows: After fixing setup and notation in Section 2 we provide some background about asymptotically stable sets in 3. The class of perturbed systems and the concept of embedding are introduced in Section 4 while the ensuing Section 5 provides the necessary results on input-to-state stability. Our main results are then stated and proved in Section 6 for one step approximations to ordinary differential equations and in Section 7 for sampled-data systems. Finally, the Appendix contains a technical Lemma which is used in several proofs.

2 Setup and notation

In this paper we consider semi-dynamical systems, i.e., maps

$$\Phi : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n \tag{2.1}$$

satisfying $\Phi(t+s, x) = \Phi(t, \Phi(s, x))$ and $\Phi(0, x) = x$ for all $t, s \in \mathbb{T}$ and all $x \in \mathbb{R}^n$, on the time scale $\mathbb{T} = \mathbb{R}_0^+$ or $\mathbb{T} = h\mathbb{N}_0$ for some time step $h > 0$, where $\mathbb{N}_0$ denotes the natural numbers including 0. This ambiguous choice of the time scale $\mathbb{T}$ (whose precise definition will always be clear from the context) allows to treat continuous and discrete time systems in a unified way.

The goal of this paper is to investigate the relation between stability properties of a system of type (2.1) and a family of approximating systems. To this end, we fix a system

$$\Phi_0 : \mathbb{T}_0 \times \mathbb{R}^n \to \mathbb{R}^n, \tag{2.2}$$

with time scale $\mathbb{T}_0$ which — depending on the application — may either be $\mathbb{T}_0 = \mathbb{R}_0^+$ or $\mathbb{T}_0 = h_0\mathbb{N}_0$ for some time step $h_0 > 0$.

System (2.2) is our abstract model for an "exact" system which may not be exactly computable, e.g., when Φ_0 is the solution of an ordinary differential equation. Therefore, we assume that we have a sequence of approximating dynamical systems for $k \in \mathbb{N}$ given by

$$\widetilde{\Phi}_k : \mathbb{T}_k \times \mathbb{R}^n \to \mathbb{R}^n, \tag{2.3}$$

again of type (2.1). These approximate systems are always discrete time systems, more precisely we choose the time scales $\mathbb{T}_k = h_k\mathbb{N}_0$ for a sequence of time steps $h_k \in (0, h_{max}]$ for some $h_{max} > 0$, which are chosen such that $\mathbb{T}_k \subseteq \mathbb{T}_0$ for all $k \in \mathbb{N}$. Note that if $\mathbb{T}_0 = h_0\mathbb{N}_0$, then $\mathbb{T}_k \subseteq \mathbb{T}_0$ is equivalent to h_k being an integer multiple of h_0.

Using this formalism, we say that the family of approximate systems is *consistent* if there exists a sequence $\varepsilon_k \to 0$ such that

$$\|\Phi_0(h_k, x) - \widetilde{\Phi}_k(h_k, x)\| \le h_k\varepsilon_k \tag{2.4}$$

and we say that the approximate system is *convergent* if there exists a continuous function $C : \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$\|\Phi_{h_k}(t, x) - \widetilde{\Phi}_k(t, x)\| \le C(t)\varepsilon_k \tag{2.5}$$

holds for all $t \in \mathbb{T}_k$. Here and in what follows we assume for simplicity of exposition that all inequalities we impose hold globally, i.e., for all $x \in \mathbb{R}^n$, cf. Remark 2.2, below.

Example 2.1 A typical example which meets this setup is when Φ_0 is generated by the solution $\varphi(t, x)$ of the initial value problem

$$\dot{\varphi}(t, x) = f(\varphi(t, x)), \quad \varphi(0, x) = x \tag{2.6}$$

with vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\|f(x) - f(y)\| \le L\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n. \tag{2.7}$$

The time scale $\mathbb{T}_0$ for Φ_0 can then be chosen as $\mathbb{R}_0^+$ or, if we prefer to consider the time–h map of the solution for some $h > 0$, as $\mathbb{T}_0 = h\mathbb{N}_0$. In this case, $\widetilde{\Phi}_k$ can be generated by any consistent and stable (and thus convergent) numerical one step method, e.g., by a Runge-Kutta scheme. If we use the step sizes $\Delta t_k = h_k$ for a sequence $h_k \to 0$, we obtain $\varepsilon_k = Ch_k^p$ for constants $p, C > 0$, where $p > 0$ is the consistency order of the Runge-Kutta scheme, cf., e.g., the textbooks [2, 14, 32, 33]. Alternatively, we may fix $h_k \equiv h > 0$ and choose the step sizes $\Delta t_k = h/k$ for some $k \in \mathbb{N}$, in which case we get $\varepsilon_k = Ck^{-p}$. Schemes with adaptive step size control do not directly fit into our setting due to the time varying nature of the step size, and must be treated with more sophisticated formalisms, like, e.g., in [17, 20, 28]. If however, we fix $h_k \equiv h > 0$, then we can compute $\widetilde{\Phi}_k$ using a scheme with adaptive step size control on the interval $[0, h]$, in which case ε_k depends on the error tolerances realized by the scheme.

In case that Φ_0 is induced by (2.6) satisfying (2.7) and $\widetilde{\Phi}_k$ is a stable numerical one step scheme, one obtains Lipschitz continuity of both Φ_0 and $\widetilde{\Phi}_k$. More precisely, for both $\Phi = \Phi_0$ and $\Phi = \widetilde{\Phi}_k$ we obtain Lipschitz estimates of the type

$$\|\Phi(t,x) - \Phi(t,y)\| \le e^{Lt}\|x-y\| \tag{2.8}$$

for all x, $y \in \mathbb{R}^n$, all $t \in \mathbb{T}$ with $\mathbb{T} = \mathbb{T}_0$ or $\mathbb{T} \in \mathbb{T}_k$, respectively, and some suitable $L > 0$ which in the case of Φ_0 coincides with the Lipschitz constant for f in (2.7). We will use these estimates in some of our results. As we will see in Section 7 there are, however, other applications where this Lipschitz property cannot be expected and even convergence (2.5) may fail. Hence we will also formulate general results which hold with merely imposing consistency (2.4).

Remark 2.2 The global estimates in (2.4), (2.5), (2.7) and (2.8) are in general quite restrictive. However, since in what follows we will be interested in the behavior on bounded subsets of the state space, one can always obtain these properties from the respective local inequalities by applying standard cutoff techniques. Proceeding this way one should, however, always keep in mind that the larger the regions of interest become, the larger the respective values ε_k and L become.

3 Asymptotically stable sets

In this paper we are interested in the asymptotic stability of compact sets $A \subset \mathbb{R}^n$. In order to characterize this property, we make use of the comparison functions classes $\mathcal{K}_\infty$ and $\mathcal{KL}$ introduced by Hahn [13] to stability analysis problems. As usual we call a continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of class $\mathcal{K}_\infty$ if it is zero at zero and strictly increasing to ∞ and we call a continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of class $\mathcal{KL}$ if it is of class $\mathcal{K}_\infty$ in the first and strictly decreasing to zero in the second argument.

In order to measure distances between points and sets, for $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ we define

$$\|x\|_A := \inf_{y\in A} \|x-y\|$$

with $\|\cdot\|$ denoting the Euclidean metric in $\mathbb{R}^n$. For two sets $A, \widetilde{A} \subseteq \mathbb{R}^n$ we define the *nonsymmetric Hausdorff distance* dist and the *Hausdorff metric* d_H, respectively, by

$$\operatorname{dist}(A,\widetilde{A}) := \sup_{x\in A} \|x\|_{\widetilde{A}} \quad \text{and} \quad d_H(A,\widetilde{A}) := \max\{\operatorname{dist}(A,\widetilde{A}),\, \operatorname{dist}(\widetilde{A},A)\}.$$

Figure 1 illustrates this distance concept. In this figure we have $d_H(A,\widetilde{A}) = \operatorname{dist}(A,\widetilde{A})$.

Defining the η–ball around a set by $B_\eta(A) := \{x \in \mathbb{R}^n \mid \|x\|_A \le \eta\}$ and denoting its closure by $\operatorname{cl} B_\eta(A)$, for compact sets A and $\widetilde{A}$ one easily proves the equivalences $\operatorname{dist}(A,\widetilde{A}) \le \eta \Leftrightarrow A \subseteq \operatorname{cl} B_\eta(\widetilde{A})$. Furthermore, for compact sets A and $\widetilde{A}$ the equivalence $d_H(A,\widetilde{A}) \le 0 \Leftrightarrow A = \widetilde{A}$ holds and d_H is indeed a metric on the space of compact subsets of $\mathbb{R}^n$.

Using these distance concepts we say that a compact set $A \subset \mathbb{R}^n$ is *asymptotically stable* for a system Φ of type (2.1) with open *stability neighborhood* $B \subseteq \mathbb{R}^n$ with $A \subset B$, if there exists $\beta \in \mathcal{KL}$ such that the inequality

$$\|\Phi(t,x)\|_A \le \beta(\|x\|_A, t)$$

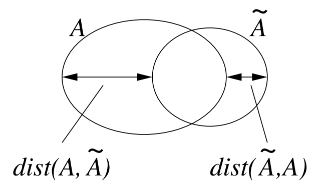

Fig. 1 Illustration of $\operatorname{dist}(A, \widetilde{A})$ and $\operatorname{dist}(\widetilde{A}, A)$

holds for all $x \in B$ and all $t \in \mathbb{T}$. Here the function β is called *attraction rate*.

As an example, if we consider $A = \{0\}$ then the $\mathcal{KL}$–function $\beta(r,t) = e^{-t}r$ will be an attraction rate for the dynamical system induced by the differential equation $\dot{x}(t) = -x(t)$ while $\beta(r,t) = r/\sqrt{2tr^2+1}$ will be an attraction rate for $\dot{x}(t) = -x(t)^3$. For higher dimensional examples we will typically have $\beta(r,0) > r$, i.e., the convergence $\|\Phi(t,x)\|_A \to 0$ as $t \to \infty$ induced by this inequality need not be monotone, which is in contrast to the concept of Lyapunov functions, which are always assumed to decay monotonically.

For systems Φ which are continuous in x, asymptotic stability is equivalent to the properties of *forward invariance* and *attraction*, i.e.,

$$\Phi(h,A) \subseteq A \quad \text{and} \quad \lim_{t\to\infty} \operatorname{dist}(\Phi(t,B),A) = 0, \tag{3.1}$$

cf. [8, Lemma 2.7].

Our goal in this paper is to analyze the relation between asymptotically stable sets of Φ_0 and $\widetilde{\Phi}_k$. More precisely, we will investigate necessary and sufficient conditions under which asymptotic stability of a family of sets A_k for $\widetilde{\Phi}_k$ carries over to Φ_0 for the limiting set A satisfying $\lim_{k\to\infty} d_H(A_k, A) = 0$.

Taking into account the interpretation of $\widetilde{\Phi}_k$ as a numerical approximation of Φ_0 from Example 2.1, the usefulness of such results for the interpretation of numerical simulations is obvious. However, as we will see in Section 7 the results also have applications in the stability analysis of sampled-data systems.

Given the mismatch between the exact system Φ_0 and its approximations $\widetilde{\Phi}_k$, it is usually not reasonable to expect that asymptotic stability carries over from one to another. In this situation, the following weaker notion, which resembles dissipativity conditions in dynamical systems theory, is useful:

For a given $\eta > 0$, we say that a compact set $A \subset \mathbb{R}^n$ is η*–practically asymptotically stable* for a system Φ of type (2.1) with open *stability neighborhood* $B \subseteq \mathbb{R}^n$ with $A \subset B$, if there exists $\beta \in \mathcal{KL}$ such that the inequality

$$\|\Phi(t,x)\|_A \le \beta(\|x\|_A, t) + \eta$$

holds for all $x \in B$ and all $t \in \mathbb{T}$.

For a sequence of systems Φ_k, $k \in \mathbb{N}$, of type (2.1) we say that A is *practically asymptotically stable* with stability neighborhood B, if for each $k \in \mathbb{N}$ it is η_k–practically asymptotically stable for Φ_k with stability neighborhood B for a sequence $\eta_k \to 0$ as $k \to \infty$.

The following lemma shows that for systems continuous in x η–practical asymptotic stability of a set A implies the existence of a "real" asymptotically stable set in the closed η–ball around A.

Lemma 3.1 *Consider a system Φ of type* (2.1) *which is continuous in x and possesses an η–practically asymptotically stable set A with stability neighborhood B with attraction rate $\beta \in \mathcal{KL}$ satisfying $\operatorname{cl} B_\eta(A) \subset B$. Then there exists an asymptotically stable set A_η with stability neighborhood B with $d_H(A_\eta, A) \le \eta$ and attraction rate $\beta_\eta(r,t) \le \beta(r+\eta,t)+\eta$.*

P r o o f. Checking the properties of forward invariance and attraction in (3.1), one sees that the set

$$D := \bigcap_{T\ge 0} \operatorname{cl} \bigcup_{t\in\mathbb{T},\, t\ge T} \Phi(t,B)$$

is an asymptotically stable set (even an attractor, see [33, Theorem 2.7.2] for details). Furthermore, the η–practical asymptotic stability yields $\operatorname{dist}(D,A) \le \eta$. Defining

$$E := \operatorname{cl} \bigcup_{t\in\mathbb{T}} \Phi(t,A) \quad \text{and} \quad A_\eta := D \cup E$$

one sees that the definition of E implies forward invariance of E and $A \subseteq E$, thus $\operatorname{dist}(A,E) = 0$. Furthermore, the practical asymptotic stability implies $E \subset \operatorname{cl} B_\eta(A)$. Thus, for A_η we obtain

- $d_H(A_\eta, A) \le \eta$ because
  $\operatorname{dist}(A_\eta, A) \le \max\{\operatorname{dist}(D,A), \operatorname{dist}(E,A)\} \le \eta$ and
  $\operatorname{dist}(A, A_\eta) \le \operatorname{dist}(A,E) = 0$
- A_η is forward invariant as a union of the forward invariant sets D and E
- $\operatorname{dist}(\Phi(t,B), A_\eta) \le \operatorname{dist}(\Phi(t,B), D) \to 0$
- B is a neighborhood of A_η, because $D,\ E \subseteq \operatorname{cl} B_\eta(A) \subset B$

This shows the claim using the alternative asymptotic stability characterization (3.1). The bound on the attraction rate follows from the inequality

$$\|\Phi(t,x)\|_{A_\eta} \le \|\Phi(t,x)\|_A \le \beta(\|x\|_A, t) + \eta \le \beta(\|x\|_{A_\eta} + \eta, t) + \eta$$

where in the first inequality we used $A \subseteq A_\eta$. Thus, if $\tilde\beta \in \mathcal{KL}$ is any attraction rate of A_η, we can conclude that

$$\beta_\eta(r,t) = \min\{\tilde\beta(r,t), \beta(r+\eta,t)\} + \eta$$

is again an attraction rate and satisfies the assertion. □

4 Perturbed systems and embedding

In order to obtain a relation between Φ_0 and $\widetilde{\Phi}_k$ which can be treated with control theoretic methods, we need to define suitable perturbed systems.

To this end, in continuous time for time scale $\mathbb{T} = \mathbb{R}_0^+$ we consider the solutions $\Psi(t, x, w)$ of a perturbed ordinary differential equation

$$\dot{x}(t) = f_p(x(t), w(t)) \tag{4.1}$$

with vector field $f_p : \mathbb{R}^n \times W \to \mathbb{R}^n$, *perturbation functions* $w \in \mathcal{W} = L_\infty(\mathbb{R}_0^+, W)$ and some *perturbation set* $W \subseteq \mathbb{R}^m$. For a perturbation function $w \in \mathcal{W}$ we use the usual L_∞–norm $\|w\|_\infty = \text{ess sup}_{t \geq 0} \|w(t)\|$.

In discrete time for time scale $\mathbb{T} = h\mathbb{N}_0$ we consider a map

$$P : \mathbb{R}^n \times W \to \mathbb{R}^n,$$

again with *perturbation set* $W \subseteq \mathbb{R}^m$, and the discrete time perturbed system $\Psi : \mathbb{T} \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$ with $\mathcal{W} := l_\infty(\mathbb{T}, W)$ induced by P via

$$\Psi(0, x, w) = x, \quad \Psi(t + h, x, w) = P(\Psi(t, x, w), w(t)), \;\; t = 0, h, 2h, \ldots \tag{4.2}$$

In the discrete time case, for $w \in \mathcal{W}$ we use the l_∞ norm $\|w\|_\infty = \sup_{t \in \mathbb{T}} \|w(t)\|$.

We do now want to relate two different perturbed systems Ψ and $\widetilde{\Psi}$ of type (4.1) or (4.2) with time scales $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ and perturbation sets W and $\widetilde{W}$, respectively. For this purpose we define the concept of embedding:

Let $\delta, \rho > 0$. The system $\widetilde{\Psi}$ is called (δ, ρ)–*embedded* in the system Ψ on a time scale $\mathbb{T}^e$, if $\mathbb{T}^e \subseteq \mathbb{T}$, $\mathbb{T}^e \subseteq \widetilde{\mathbb{T}}$ and for each $x \in \mathbb{R}^n$ and each $\tilde{w} \in \widetilde{W}$ there exists $w \in W$ satisfying

$$\|w\|_\infty \leq \delta + \rho \|\tilde{w}\|_\infty \;\text{ and }\; \Psi(t, x, w) = \widetilde{\Psi}(t, x, \tilde{w})$$

for all $t \in \mathbb{T}^e$.

In order to establish an embedding property for the perturbed systems induced by Φ_0 and $\widetilde{\Phi}_k$, we consider a simple class of perturbed systems, the so called *inflated systems*, cf. [15], by specifying f_p in (4.1) and P in (4.2) to

$$f_p(x, w) := f(x) + w \quad \text{and} \quad P(x, w) := \Phi(h, x) + hw, \tag{4.3}$$

respectively, for $w \in W \subseteq \mathbb{R}^n$.

The following proposition establishes an embedding property for these perturbed systems.

Proposition 4.1 *Consider the systems* (2.2) *and* (2.3) *satisfying* (2.4) *and the corresponding perturbed systems* Ψ_0 *and* $\widetilde{\Psi}_k$ *constructed via* (4.1), (4.3) *or* (4.2), (4.3) *with perturbation sets* $W = \operatorname{cl} B_\alpha(0) \subset \mathbb{R}^n$ *and* $\widetilde{W} = \operatorname{cl} B_{\tilde{\alpha}}(0) \subset \mathbb{R}^n$.

Then the following embedding properties hold on $T^e = \widetilde{\mathbb{T}}_k$.

(i) *If* (2.2) *is a discrete time system, then* $\widetilde{\Psi}_k$ *is* $(\varepsilon_k, 1)$–*embedded in* Ψ_0 *if* $\alpha \geq \tilde{\alpha} + \varepsilon_k$, *and* Ψ_0 *is* $(\varepsilon_k, 1)$–*embedded in* $\widetilde{\Psi}_k$ *if* $\tilde{\alpha} \geq \alpha + \varepsilon_k$.

(ii) *If* (2.2) *is a continuous time system induced by the ordinary differential equation* (2.6) *satisfying* (2.7)*, then* $\widetilde{\Psi}_k$ *is* $((1+Lh_k)\varepsilon_k, (1+Lh_k))$*–embedded in* Ψ_0 *if* $\alpha \geq (1+Lh_k)(\tilde{\alpha}+\varepsilon_k)$*, and* Ψ_0 *is* $(\varepsilon_k, e^{Lh_k})$*–embedded in* $\widetilde{\Psi}_k$ *if* $\tilde{\alpha} \geq e^{Lh_k}\alpha + \varepsilon_k$*.*

P r o o f. (i) We only show the embedding $\widetilde{\Psi}_k$ in Ψ_0 since the converse embedding follows by symmetry. For this it is sufficient to show that for any $x \in \mathbb{R}^n$ and any $\tilde{w} \in \widetilde{W}$ there exists $w \in W$ such that

$$\|w\| \leq \varepsilon_k + \|\tilde{w}\| \text{ and } \Psi_0(h_k, x, w) = \widetilde{\Psi}_k(h_k, x, \tilde{w}) \tag{4.4}$$

holds for P and $\widetilde{P}_k$ from (4.3) and $h = h_k$, because then the stated embedding property follows by a straightforward induction over jh_k, $j = 1, 2, \ldots$.

Now from (4.3) it follows that the perturbation value

$$w = \frac{\widetilde{\Psi}_k(h_k, x, \tilde{w}) - \Phi_0(h_k, x)}{h_k}$$

satisfies the second property in (4.4). Using (2.4) and the triangle inequality it is immediate that this w also satisfies the first property in (4.4). From this, the definition of W yields $w \in W$ which finishes the proof.

(ii) We first show the embedding of $\widetilde{\Psi}_k$ in Ψ_0. Again, it is sufficient to show the embedding for $t = h_k$, i.e, to construct a suitable perturbation function w on $[0, h_k]$. To this end we set

$$\Delta x := \frac{\widetilde{\Psi}_k(h_k, x, \tilde{w}) - \Phi_0(h_k, x)}{h_k}, \quad y(t) := \Phi_0(t, x) + t\Delta x$$

and define the perturbation function

$$w(t) = f(\Phi_0(t, x)) - f(y(t)) + \Delta x.$$

This yields $\dot{y}(t) = f(y(t)) + w(t)$ and thus by uniqueness of the solution $\Psi_0(h_k, x, w) = y(h_k) = \Psi_k(h_k, x, \tilde{w})$, i.e., the second property in (4.4). Furthermore, from (2.4) and (4.2) we obtain the estimate $\|\Delta x\| \leq \varepsilon_k + \|\tilde{w}\|_\infty$ and thus for $t \in [0, h_k]$

$$\|w(t)\| \leq L\|\Phi_0(t, x) - y(t)\| + \|\Delta x\| \leq (1 + Lh_k)\|\Delta x\| \leq (1 + Lh_k)(\varepsilon_k + \|\tilde{w}\|_\infty).$$

This shows the first property in (4.4).

For the converse embedding, from the estimate

$$\|\Psi_0(h_k, x, w) - \Phi_0(h_k, x)\| \leq \int_0^{h_k} \|f(\Psi_0(t, x, w)) - f(\Phi_0(h_k, x))\| + \|w(t)\| dt$$

using the Lipschitz continuity of f and Gronwall's Lemma we obtain

$$\|\Psi_0(h_k, x, w) - \Phi_0(h_k, x)\| \leq e^{Lh_k} h_k \|w\|_\infty. \tag{4.5}$$

Now, setting

$$\tilde{w} = \frac{\Psi_0(h_k, x, w) - \widetilde{\Phi}_k(h_k, x)}{h_k}$$

we obtain the second property in (4.4) and consistency (2.4) and the above estimate (4.5) yield

$$\|\tilde{w}\| \leq \frac{\|\Psi_0(h_k,x,w) - \Phi_0(h_k,x)\| + \|\Phi_0(h_k,x) - \widetilde{\Phi}_k(h_k,x)\|}{h_k} \leq e^{Lh_k}\|w\|_\infty + \varepsilon_k$$

which shows the first property in (4.4). □

5 Input-to-state stability

The input-to-state stability (ISS) property was introduced by Sontag [29] as a robust stability property for nonlinear perturbed systems and — along with its various variants — has found a vast amount of applications, see, e.g., the survey papers [30, 31]. Here, we use a version of this property which applies to arbitrary compact sets and is restricted to a certain stability neighborhood B:

We say that the system Ψ is *input-to-state stable* (ISS) with respect to a compact set $A \subset \mathbb{R}^n$, a *stability neighborhood* B and a perturbation set W, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the inequality

$$\|\Psi(t,x,w)\|_A \leq \beta(\|x\|_A, t) + \gamma(\|w\|_\infty)$$

holds for all $x \in B$ and all $t \in h\mathbb{N}_0$. Here the function β is called *(ISS) attraction rate* and the function γ is called *robustness gain*.

Analogous to the η–practical asymptotic stability for some $\eta > 0$ we say that the system Ψ is *η–practically input-to-state stable* (η–ISS) with respect to a compact set $A \subset \mathbb{R}^n$, a *stability neighborhood* B and a perturbation set W, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the inequality

$$\|\Psi(t,x,w)\|_A \leq \beta(\|x\|_A, t) + \gamma(\|w\|_\infty) + \eta$$

holds for all $x \in B$ and all $t \in h\mathbb{N}_0$.

For a sequence of systems Φ_k, $k \in \mathbb{N}$, of type (2.1) we say that A is *practically input-to-state stable* (practically ISS) with stability neighborhood B, if for each $k \in \mathbb{N}$ it is η_k–ISS for Φ_k with stability neighborhood B for a sequence $\eta_k \to 0$ as $k \to \infty$.

It is easily seen that ISS (η–ISS, practical ISS) implies asymptotic stability (η–practical asymptotic stability, practical asymptotic stability, respectively) of A for the unperturbed system $\Psi(t,x,0)$.

The following proposition will be crucial for the analysis in this paper. It shows that ISS implies η–ISS for an (δ,ρ)–embedded system.

Proposition 5.1 *Let $\delta, \rho > 0$ and consider two perturbed systems Ψ and $\widetilde{\Psi}$ such that $\widetilde{\Psi}$ is (δ,ρ)–embedded in the system Ψ. Assume that Ψ is η–ISS with respect to a compact set $A \subset \mathbb{R}^n$ and a stability neighborhood B with $\eta > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$.*

Then the system $\widetilde{\Psi}$ is $\tilde{\eta}$–ISS with respect to A and the stability neighborhood B with $\tilde{\eta} = \eta + \gamma(\delta)$, the same $\beta \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ given by $\tilde{\gamma}(r) = \gamma(\rho r + \delta) - \gamma(\delta)$.

P r o o f. From the embedding property one obtains

$$\begin{aligned} \|\widetilde{\Psi}(t,x,\tilde{w})\|_A = \|\Psi(t,x,w)\|_A &\leq \beta(\|x\|_A, t) + \gamma(\|w\|_\infty) + \eta \\ &\leq \beta(\|x\|_A, t) + \gamma(\rho\|\tilde{w}\|_\infty + \delta) + \eta \\ &= \beta(\|x\|_A, t) + \tilde{\gamma}(\|\tilde{w}\|_\infty) + \tilde{\eta} \end{aligned}$$

which shows the claim. □

For inflated systems (4.3) based on unperturbed systems satisfying (2.7) or (2.8), ISS is an inherent property of asymptotically stable sets, if the perturbation set W is chosen sufficiently small. Even more, the following proposition shows that the ISS attraction rate β coincides with the attraction rate of the unperturbed system and that there exists a robustness gain γ, which only depends on the attraction rate β, the Lipschitz constant L in (2.7) or (2.8) and the size of the stability neighborhood B, measured via $d_H(B, A)$ and

$$d_{min}(B, A) = \sup\{\varepsilon > 0 \mid B_\varepsilon(A) \subset B\}.$$

Note that the strict inequality $d_{min}(B, A) < \operatorname{dist}(B, A)$ holds if B is not a ball around A. Furthermore we have $d_{min}(B, A) = 0$ if B is not a neighborhood of A.

Proposition 5.2 *There exist maps*

$$\gamma : \mathcal{KL} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathcal{K}_\infty \quad \textit{and} \quad \alpha : \mathcal{KL} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$$

such that for each system (2.1) *satisfying* (2.6), (2.7) *if* $\mathbb{T} = \mathbb{R}$ *and* (2.8) *if* $\mathbb{T} = h\mathbb{N}_0$, *and each asymptotically stable set* $A \subset \mathbb{R}^d$ *with attraction rate* $\beta \in \mathcal{KL}$ *and bounded attracted neighborhood* B*, the inflated perturbed system* (4.2), (4.3) *with perturbation set* $W = B_{\alpha(\beta, d_H(B,A), d_{min}(B,A), L)}(0)$ *is ISS with respect to* A *with stability neighborhood* B*, same attraction rate* β *and robustness gain* $\gamma(\beta, d_H(B, A), L)$.

P r o o f. First observe that from Gronwall's Lemma and (2.7) in the continuous time case or by induction and (2.8) in the discrete time case we obtain

$$\|\Psi(t, x, w) - \Phi(t, x)\| \le e^{Lt}\|w\|_\infty. \tag{5.1}$$

Abbreviate $d_{max} = d_H(B, A)$ and $d_{min} = d_{min}(B, A)$. We pick $\varepsilon \in (0, d_{min}/2]$ maximal such that $\beta(\varepsilon, 0) \le d_{min}/2$ holds, $T > 0$ minimal with $\beta(d_{max}, T) \le \varepsilon/2$ and set

$$\alpha = \alpha(\beta, d_H(B, A), d_{min}(B, A), L) := \frac{\varepsilon}{2e^{LT}}.$$

Using (5.1) and induction over $i \in \mathbb{N}$ this yields $\|\Psi(iT, x, w)\|_A \le \varepsilon$ for all perturbation functions w with $w(t) \in W = B_\alpha(0)$. For $t \in [iT, (i + 1)T)$ we can conclude

$$\|\Psi(t, x, w)\|_A \le \beta(\|\Psi(iT, x, w)\|_A, t - iT) + e^{Lt}\|w\| \le \beta(\varepsilon, 0) + e^{Lt}\|w\| < d_{min}$$

which implies $\Psi(t, x, w) \in B_{d_{min}}(A) \subset B$ for all $x \in B$, all $t \ge T$ and all perturbation functions w with $w(t) \in W$.

Now for each $\eta \in (0, \alpha]$ we pick $t_\eta \ge T$ maximal with $e^{Lt_\eta} \le 1/\sqrt{\eta}$, which implies $t_\eta \to \infty$ as $\eta \to 0$. With this choice of t_η from (5.1) for each $x \in B$ and each w with $w(t) \in W$ we obtain

$$\|\Psi(t, x, w)\|_A \le \beta(\|x\|_A, t) + \sqrt{\|w\|_\infty}$$

for $t \in [0, t_{\|w\|_\infty}]$. For $t \ge t_{\|w\|_\infty}$ we proceed inductively over $i \in \mathbb{N}$ in order to obtain $\|\Psi(it_{\|w\|_\infty}, x, w\| \le \beta(d_{max}, t_{\|w\|_\infty}) + \sqrt{\|w\|_\infty}$. For $t \ge t_{\|w\|_\infty}$ this implies

$$\|\Psi(t, x, w)\|_A \le \beta(\beta(d_{\max}, t_{\|w\|_\infty}) + \sqrt{\|w\|_\infty}, 0) + \sqrt{\|w\|_\infty}.$$

This yields

$$\|\Psi(t,x,w)\|_A \le \beta(\|x\|_A,t) + \gamma(\|w\|_\infty)$$

for $\gamma \in \mathcal{K}_\infty$ given by $\gamma(\eta) := \beta(\beta(d_{\max}, t_\eta) + \sqrt{\eta}, 0) + \sqrt{\eta}$. □

Remark 5.3 (i) Alternative versions of this result can be found in [8, Theorem 4.1 and Proposition 4.5] or [6, Proposition 3.4.1 and Proposition 3.4.4]. In these references the ISS attraction rate β is not determined, however, a tighter estimate for the robustness gain γ is given. In particular, if A attracts exponentially, i.e., $\beta(r,t) = \rho e^{-\lambda t} r$, then this tighter estimate yields that $\gamma(\beta, d_H(B,A), L)$ is a linear function, cf. [8, Example 4.2] or [6, Example 4.2.2].
(ii) An inspection of the proof reveals that we can replace the Lipschitz assumptions (2.7) or (2.8) by the inequality

$$\|\Psi(t,x,0) - \Psi(t,x,w)\| \le C(t)\nu(\|w\|)$$

for all $t \in \mathbb{T}$, $x \in B$ and $w \in \mathcal{W}$, a continuous function $C : \mathbb{R}_0^+ \to \mathbb{R}$ and a function $\nu \in \mathcal{K}_\infty$.
(iii) The proof technique of establishing ISS or similar properties via finite time trajectory based estimates has been used by a couple of authors, see, e.g., [1] or [22] for similar results.

Remark 5.4 In what follows we will work directly with the ISS estimate. Alternatively, as, e.g., in [16, 23], one may work with suitable Lyapunov functions which ensure a robust stability property. These two approaches are essentially equivalent, because for a variant of the ISS property, the so called input-to-state dynamical stability (ISDS), one can show that the attraction rate β and the robustness gain γ can be "encoded" in a suitable Lyapunov function, such that both approaches yield the same estimates, cf. [6, 7]. This, however, requires the use of technically complicated rate preserving converse ISDS Lyapunov function theorems, which we avoid here by working directly with the ISS inequality.

6 Numerical Dynamics

In this section we investigate the relation between stability properties of solutions of ordinary equations Φ_0 and numerical one step approximations $\widetilde{\Phi}_k$ as in the setting of Example 2.1. For this setting we make the following assumptions.

Assumption 6.1 (i) The exact system Φ_0 in (2.2) is induced by a differential equation (2.6) satisfying (2.7) and is thus defined for $\mathbb{T} = \mathbb{R}_0^+$.

(ii) The approximate systems $\widetilde{\Phi}_k$ in (2.3) are defined by stable and consistent (2.4) numerical one step schemes according to Example 2.1 and are thus convergent (2.5) and satisfy the Lipschitz condition (2.8). The corresponding time steps $h_k \in (0, h_{max}]$ are either constant, i.e., $h_k \equiv h > 0$, or decreasing to 0, i.e., $\lim_{k\to\infty} h_k = 0$.

Under this assumption, the following theorem shows the precise relation between asymptotically stable sets of Φ_0 and asymptotically stable and practically asymptotically stable sets, respectively, for $\widetilde{\Phi}_k$.

Theorem 6.2 *Consider exact and approximate systems Φ_0 from* (2.2) *and $\widetilde{\Phi}_k$ from* (2.3) *meeting Assumption 6.1. Let $\mathbb{T}_0 = h_k\mathbb{N}_0$ if the sequence h_k in Assumption 6.1(ii) is constant and let $\mathbb{T} = \mathbb{R}_0^+$ if $h_k \to 0$. Let $A \subset \mathbb{R}^n$ be a compact set with open and bounded neighborhood B. Then the following statements are equivalent.*

(i) *A is asymptotically stable for Φ_0 with stability neighborhood B.*

(ii) *For all sufficiently large $k \in \mathbb{N}$ the set A is practically asymptotically stable for $\widetilde{\Phi}_k$ with stability neighborhood B and attraction rate $\beta \in \mathcal{KL}$ independent of k.*

(iii) *For all sufficiently large $k \in \mathbb{N}$ there exist compact sets $A_k \subset B$ with $d_H(A_k, A) \to 0$ as $k \to \infty$ which are asymptotically stable for $\widetilde{\Phi}_k$ with stability neighborhood B and attraction rate $\tilde{\beta} \in \mathcal{KL}$ independent of k.*

Moreover, if these statements hold, then the functions β and $\tilde{\beta}$ from (ii) and (iii) are attraction rates for A and Φ_0.

Proof. "(i) $\Rightarrow$ (ii)": Consider the inflated systems Ψ_0 constructed according to (4.1), (4.3) from Φ_0. From Proposition 5.2 we obtain that this system is ISS with respect to A for perturbation set $W = \mathrm{cl}\, B_\alpha(0)$ for some $\alpha > 0$ and $\gamma \in \mathcal{K}_\infty$ and ISS attraction rate β equal to the unperturbed attraction rate β.

Now pick some $\varepsilon \in (0, \alpha/(1+Lh_{max}))$. Then for sufficiently large $k \in \mathbb{N}$, i.e., sufficiently small ε_k, we have that $(1 + Lh_k)(\varepsilon_k + \varepsilon) < \alpha$. Thus Proposition 4.1(ii) yields that the perturbed system $\widetilde{\Psi}_k$ constructed from $\widetilde{\Phi}_k$ via (4.2), (4.3) with perturbation set $\widetilde{W} = \mathrm{cl}\, B_\varepsilon(0)$ is $((1 + Lh_k)\varepsilon_k, (1 + Lh_k))$–embedded in Ψ_{h_k}.

From this embedding property Proposition 5.1 applied with $\eta = 0$ yields that A is η_k–ISS for $\widetilde{\Psi}_k$ for $\eta_k = \gamma((1 + Lh_k)\varepsilon_k)$, implying that A is η_k–practically asymptotically stable for $\widetilde{\Phi}_k$ with attraction rate β.

"(ii) $\Rightarrow$ (iii)": Let η_k be the constants in the practical asymptotic stability property. Then Lemma 3.1 implies the existence of the asymptotically stable sets A_k satisfying $d_H(A_k, A) \leq \eta_k \to 0$ as $k \to \infty$.

In order to show the existence of the uniform attraction rate $\tilde{\beta}$, observe that Lemma 3.1 yields an attraction rate β_k satisfying $\beta_k(r, t) \leq \tilde{\beta}(r+\eta_k, t)+\eta_k$ for A_k. From this inequality Lemma A.1 applied with $d_k = \eta_k$ yields the existence of $\tilde{\beta}$.

"(iii) $\Rightarrow$ (i)": The convergence (2.5) of $\widetilde{\Phi}_k$ implies for each fixed $t \in \mathbb{T}_0$ and each $x \in B$

$$\begin{aligned}
\|\Phi(t,x)\|_A &\leq \|\Phi(t,x)\|_{A_k} + d_H(A_k, A) \\
&\leq \|\widetilde{\Phi}_k(t,x)\|_{A_k} + C(t)\varepsilon_k + d_H(A_k, A) \\
&\leq \beta(\|x\|_{A_k}, t) + C(t)\varepsilon_k + d_H(A_k, A) \\
&\leq \beta(\|x\|_A + d_H(A_k, A), t) + C(t)\varepsilon_k + d_H(A_k, A)
\end{aligned}$$

for each $k \in \mathbb{N}$. For $k \to \infty$ this yields the estimate

$$\|\Phi(t,x)\|_A \leq \beta(\|x\|_A, t)$$

which shows that A is asymptotically stable with attraction rate β. A similar inequality shows that $\tilde{\beta}$ from (ii) is also an attraction rate for A. □

Remark 6.3 (i) If we follow the proof from (i) to (iii), then we obtain the inequality

$$d_H(A_k, A) \leq \eta_k \leq \gamma((1 + Lh_k)\varepsilon_k),$$

for A_k in (iii) in where $\gamma \in \mathcal{K}_\infty$ is an ISS robustness gain of A in (i). In particular, if A in (i) attracts exponentially then using Remark 5.3 we can conclude that in (iii) the estimate

$$d_H(A_k, A) \leq C\varepsilon_k$$

holds for some constant $C > 0$ and all sufficiently large $k > 0$.

(ii) The implication "(i) ⇒ (iii)" is a variant of the Theorem by Kloeden and Lorenz [16], who were the first to study the relation between asymptotic stability of ordinary differential equations and their one step discretizations.

Example 6.4 We illustrate the theorem with a slight variation of a classical example from [3]. Consider the two-dimensional ordinary differential equation given by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x - \max\{\|x\| - 1, 0\}x$$

for $x = (x_1, x_2)^T \in \mathbb{R}^2$.

Figure 2 shows, from left to right, two solutions of the original system, of its (explicit) Euler discretization (with time step $h = 1/2$) and of its implicit Euler discretization (with time step $h = 1/2$), respectively. The initial values for these solutions are $x_0' = (0, 2)$ and $x_0'' = (0, 1/2)$ and the solutions are computed for $t \in [0, 20]$. In addition, in the first two figures the shaded regions show the minimal attracting sets A and $\widetilde{A}$.

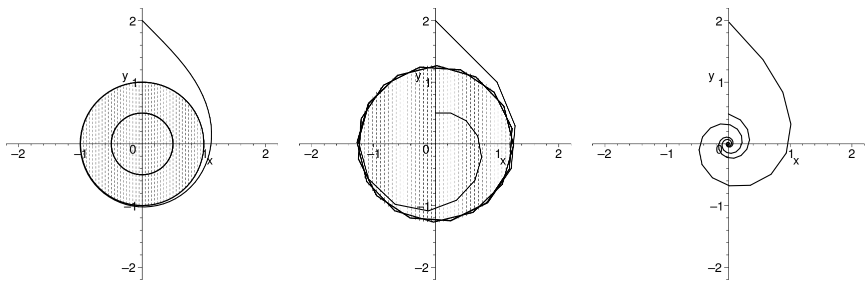

Fig. 2 Exact, explicit and implicit Euler solutions of Example 6.4

It is easily seen that for the original system each disc

$$D_a := \{x \in \mathbb{R}^2 \mid \|x\| \leq a\}$$

with $a \geq 1$ is an attracting set, while for the Euler discretization each disc D_a with $a \in [(1 + h - \sqrt{1 - h^2})/h, c(h)]$ is an attracting set, where $c(h) \to \infty$ as $h \to 0$. For the implicit Euler discretization it turns out that each disc D_a with $a \in [0, \tilde{c}(h)]$ is an attracting set, with $\tilde{c}(h) \to \infty$ as $h \to 0$. Hence, indeed, for both discretizations there exist attracting sets approaching D_a for each $a \geq 1$. According to Theorem 6.2, these sets must have an attraction rate independent from the numerical accuracy (which here is linked with the time step), which can be confirmed by numerical simulations.

In contrast to this, the asymptotically stable set $\widetilde{A} = \{0\}$ which exists for the implicit Euler scheme for each time step $h > 0$ cannot have attraction rate independent of h, because this set

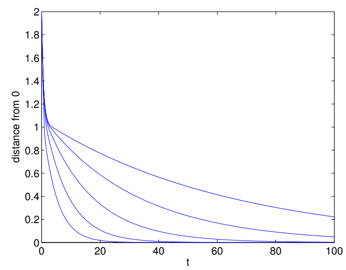

Fig. 3 Numerically measured attraction rate for the asymptotically stable set $\widetilde{A} = \{0\}$ in the implicit Euler discretization with time steps $h_k = 2^{-k}$, $k = 1, \ldots, 5$ (bottom to top)

is not asymptotically stable for the exact system. Figure 3 shows the attraction rates for time steps $h_k = 2^{-k}$, $i = k, \ldots, 5$ (bottom to top) and confirms this theoretical result numerically: obviously the attraction rate depends on the numerical accuracy.

7 Sampled-data control

In this section we consider control systems induced by ordinary differential equations

$$\dot{x} = g(x, u) \tag{7.1}$$

with vector field $g : \mathbb{R}^n \times U \to \mathbb{R}^n$ and *control value set* $U \subset \mathbb{R}^m$ which, again to keep the presentation simple, we assume to be compact. We want to control the system by a static state feedback law $u : \mathbb{R}^n \to U$. In continuous time the implementation of such a feedback law leads to the *continuous time closed loop system*

$$\dot{x}(t) = g(x(t), u(x(t))), \tag{7.2}$$

whose solutions we denote by $\Phi_0^u(t, x)$.

In practice, feedback laws are often implemented using a digital device which implies that a continuous evaluation of $u(x(t))$ for all $t \in \mathbb{R}_0^+$ is not possible. Instead, $u(x(t))$ can only be evaluated at discrete times $t_0, t_1, \ldots$, the *sampling times*, which here for simplicity of exposition we chose as $t_i = hi$ for $i \in \mathbb{N}_0$ and the *sampling period* $h > 0$. Assuming that the control value is held constant in each sampling interval, i.e., that the digital-analog converter is a zero order hold device, this leads to the *sampled-data closed loop system*

$$\dot{x}(t) = g(x(t), u(x(t_i))), \quad t \in [t_i, t_{i+1}), \;\; i \in \mathbb{N}_0 \tag{7.3}$$

For a given feedback $u : \mathbb{R}^n \to U$ by $\Phi_h^u(t, x) : \mathbb{T}_h \times \mathbb{R}^n \to \mathbb{R}^n$ we denote the discrete time system on $\mathbb{T}_h = h\mathbb{N}_0$ obtained from the solution of (7.3). Note that this is still a stylized model, because it does not incorporate additional properties of digital control schemes like, e.g, delays in computing $u(x(t_i))$ from the measurement $x(t_i)$. Still, it captures an essential feature of sampled-data control systems, namely that the control value cannot be continuously adapted to the current state. It is important to keep in mind that the sampling period h does not merely determine the time scale $\mathbb{T}_h$ on which Φ_h^u is defined. Rather, it has a physical meaning because it determines the intervals on which $u(x(t_i))$ is held constant in (7.3).

Our goal in this section is to give conditions under which a compact set A is practically asymptotically stable for the sampled-data system (7.3). These results are similar in spirit to [24, 23] (for an introduction to the subject see [18]), with the difference that here we derive necessary and sufficient conditions instead of merely sufficient conditions.

Our first result uses the assumption that we have a feedback law u which renders a certain compact set A asymptotically stable for the continuous time closed loop system (7.2). In order to ensure consistency of (7.3) and (7.2) we assume that the right hand side of (7.2) satisfies a suitable global Lipschitz assumption, furthermore we impose a global bound on g.[1]

Theorem 7.1 *Consider the continuous and sampled-data closed loop systems Φ_0^u from* (7.2) *and $\Phi_{h_k}^u$ from* (7.3) *for a sequence of sampling periods $h_k \to 0$. Assume that the function g from* (7.2) *is globally Lipschitz in x uniformly in u with constant L and that $x \mapsto g(x, u(x))$ is globally Lipschitz, again with constant L, and bounded by some constant $M > 0$. Let $A \subset \mathbb{R}^n$ be a compact set with open and bounded neighborhood B. Then the following statements are equivalent.*

(i) *A is asymptotically stable for Φ_0^u with stability neighborhood B.*

(ii) *For all sufficiently large $k \in \mathbb{N}$ the set A is practically asymptotically stable for $\Phi_{h_k}^u$ with stability neighborhood B and attraction rate $\tilde{\beta} \in \mathcal{KL}$ independent of k.*

(iii) *For all sufficiently large $k \in \mathbb{N}$ there exist compact sets $A_k \subset B$ with $d_H(A_k, A) \to 0$ as $k \to \infty$ which are asymptotically stable for $\Phi_{h_k}^u$ with stability neighborhood B and attraction rate $\beta \in \mathcal{KL}$ independent of k.*

P r o o f. It is sufficient to show that Φ_0^u and $\Phi_{h_k}^u$ satisfy the consistency inequality (2.4) for a suitable sequence $\varepsilon_k \to 0$, because then Theorem 6.2 applied with $\Phi_0 = \Phi_0^u$ and $\widetilde{\Phi}_k = \Phi_{h_k}^u$ yields the assertion.

In order to prove (2.4) observe that the boundedness assumption on g implies

$$\|\Phi_0^u(h_k, x) - x\| \le h_k M \quad \text{and} \quad \|\Phi_{h_k}^u(h_k, x) - x\| \le h_k M.$$

From this we obtain

$$\begin{aligned}
\|\Phi_0^u(h_k, x) - \Phi_{h_k}^u(h_k, x)\| &= \left\| \int_0^{h_k} g(\Phi_0^u(t,x), u(\Phi_0^u(t,x))) - g(x, u(x)) \right. \\
&\qquad \left. + \; g(x,u(x)) - g(\Phi_{h_k}^u(t,x), u(x)) dt \right\| \\
&\le \int_0^{h_k} L\|x - \Phi_0^u(t,x)\| + L\|x - \Phi_{h_k}^u(t,x)\| dt \\
&\le \int_0^{h_k} 2Lh_k M dt \;=\; 2LM{h_k}^2
\end{aligned}$$

which implies (2.4) with $\varepsilon_k = 2LMh_k$. □

[1] Again, these assumption are supposed to be global in order to simplify the presentation, cf. Remark 2.2

This theorem shows that a stabilizing feedback law u designed for the continuous time system (7.2) also works for the sampled-data system (7.3) if the sampling period $h > 0$ is sufficiently small. In practice, however, this way of designing u may perform poorly if the sampling period $h > 0$ cannot be chosen arbitrarily small, e.g., due to hardware limitations in the physical realization of the system or the controller. This is particularly important if we want our stability neighborhood B to become large, because the larger the region of interest becomes the larger the respective constants and robustness gains in our estimates may become, cf. Remark 2.2, and consequently the smaller the sampling period $h > 0$ must be.

It may therefore be desirable to design a feedback law u_h directly in discrete time for (7.3). When proceeding this way we face the problem that the discrete time system Φ_h^u induced by (7.3) is hardly ever known exactly, because it is the solution of a nonlinear ordinary differential equation. Therefore, typically one has to design the feedback controller for a family of approximations $\widetilde{\Phi}_k^u$, $k \in \mathbb{N}$, to Φ_h^u, see, e.g., [4, 5, 9, 10, 11, 12, 18, 21, 25, 27, 26] for examples of such design methods. Since a feedback law constructed on basis of an approximation $\widetilde{\Phi}_k^u$ will depend on k, we need to consider sequences of feedback laws u_k in the following definition of the approximating systems:

For a sequence of feedback laws $u_k : \mathbb{R}^n \to U$ and $k \in \mathbb{N}$ we consider the family of approximating systems

$$\widetilde{\Phi}_k^{u_k} : \mathbb{T}_k \times \mathbb{R}^n \to \mathbb{R}^n \tag{7.4}$$

on the time scale $\mathbb{T}_k := h_k\mathbb{N}_0$ for sampling periods $h_k \in (0, h_{max}]$. In what follows these sampling periods h_k may be chosen arbitrarily, in particular they may be constant $h_k \equiv h > 0$ or converging to 0 as $k \to \infty$.

In order to formalize the approximation property of the family $\widetilde{\Phi}_k^{u_k}$ we fix a sequence u_k of feedback laws and generalize the consistency and convergence conditions (2.4) and (2.5) to systems with feedback, an idea that apparently was first used in [24][2]:

We say that the family of approximating systems $\widetilde{\Phi}_k^{u_k}$ is *consistent*, if there exists a sequence $\varepsilon_k \to 0$ such that

$$\|\Phi_{h_k}^{u_k}(h_k, x) - \widetilde{\Phi}_k^{u_k}(h_k, x)\| \le h_k\varepsilon_k \tag{7.5}$$

holds, and we say that the approximate system is *convergent* if there exists a continuous function $C : \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$\|\Phi_{h_k}^{u_k}(t, x) - \widetilde{\Phi}_k^{u_k}(t, x)\| \le C(t)\varepsilon_k \tag{7.6}$$

holds for all $t \in \mathbb{T}_k$.

In practice, a *consistent* approximation $\widetilde{\Phi}_k^{u_k}(h_k, x)$ can be obtained as in Example 2.1, applying a numerical one step scheme with step size $\Delta t_k = h_k$ or $\Delta t_k = h/k$ to the vector field $f(x) = g(x, u)$ and evaluating the resulting map $\widetilde{\Phi}_k(h_k, x, u)$ in $(x, u_k(x))$. Alternatively, in the literature often Fliess series approximations to $\Phi_h^{u_k}$ are used as approximate models, e.g. in [11, 21, 25]. *Convergence* of these schemes follows if the systems $\widetilde{\Phi}_k(x, u)$ satisfy the Lipschitz condition (2.8) for a Lipschitz constant $L > 0$ independent of k, however, if the feedback laws u_k are discontinuous, then (2.8) cannot hold and convergence may fail. Of course, the

[2] In this reference convergence is called multi-step consistency.

regularity of the maps u_k strongly depends on the sampled data feedback design method employed. While for series based (re-)design methods as presented, e.g., in [11, 21, 25] or for the Euler backstepping method [27, 18] the Lipschitz property can be expected or even rigorously proved, optimal control based design methods as in [5, 9, 10, 12, 26] typically yield discontinuous feedback laws u_k, for which convergence (7.6) cannot be expected in general.

The question which naturally arises then is the following: Given a sequence of feedback laws u_k designed in order to (practically) stabilize a certain set A for the increasingly accurate models $\widetilde{\Phi}_k^{u_k}$, under which necessary and sufficient conditions will these feedbacks also render the set A practically asymptotically stable for the exact sampled-data model $\Phi_{h_k}^{u_k}$ for sufficiently large k?

In order to answer this question, Theorem 6.2 is not directly applicable, because in general we do not have a suitable limiting system for $k \to \infty$ playing the role of Φ_0 in statement (i) of Theorem 6.2. However, we can still formulate necessary and sufficient conditions using suitable variants of the statements (ii) or (iii) of Theorem 6.2 for both $\Phi_{h_k}^{u_k}$ and $\widetilde{\Phi}_k^{u_k}$. In the sequel we will demonstrate how to derive such variants based on statement (ii).

The precise formulation of these variants depends on the regularity and convergence properties of $\Phi_k^{u_k}$. Here, for the sake of brevity, we restrict ourselves to two cases, namely the case of $\widetilde{\Phi}_k^{u_k}$ satisfying (2.8) with L independent of k (and thus satisfying (7.6)) and the case of merely consistent but not necessarily convergent approximations $\widetilde{\Phi}_k^{u_k}$. Other cases can be treated as well, for instance, a sufficient condition for the case of possibly discontinuous but convergent approximations is given in [24, Theorem 1]. Since we aim at necessary and sufficient conditions, for the non-convergent case mere asymptotic stability of the respective sets is not sufficient. Instead, we need a robust form of asymptotic stability, which we formulate in terms of practical ISS for the respective inflated systems.

Theorem 7.2 *Consider a sequence of feedback laws $u_k : \mathbb{R}^n \to U$ and the corresponding exact and approximate sampled data systems $\Phi_{h_k}^{u_k}$ from* (7.3) *and $\widetilde{\Phi}_k^{u_k}$ from* (7.4)*, which are supposed to be consistent in the sense of* (7.5)*. Let $A \subset \mathbb{R}^n$ be a compact set with open and bounded neighborhood B. Then the following assertions hold.*

1. (non-convergent case) *The following statements are equivalent:*

(i) *For all sufficiently large $k \in \mathbb{N}$ the set A is practically ISS with stability neighborhood B for the inflated system* (4.2), (4.3) *associated to $\Phi_{h_k}^{u_k}$ with robustness gains $\gamma_k \in \mathcal{K}_\infty$ satisfying $\lim_{k\to\infty} \gamma_k(C\varepsilon_k) \to 0$ for each $C > 0$.*

(ii) *For all sufficiently large $k \in \mathbb{N}$ the set A is practically ISS with stability neighborhood B for the inflated system* (4.2), (4.3) *associated to $\widetilde{\Phi}_k^{u_k}$ with robustness gains $\gamma_k \in \mathcal{K}_\infty$ satisfying $\lim_{k\to\infty} \gamma_k(C\varepsilon_k) \to 0$ for each $C > 0$.*

2. (Lipschitz and convergent case) *If the $\Phi_{h_k}^{u_k}$ and $\widetilde{\Phi}_k^{u_k}$ satisfy* (2.8) *with L independent of k and* (7.6)*, then the following statements are equivalent:*

(i) *For all sufficiently large $k \in \mathbb{N}$ the set A is practically asymptotically stable for $\Phi_{h_k}^{u_k}$ with stability neighborhood B and attraction rate $\beta \in \mathcal{KL}$ independent of $k \in \mathbb{N}$.*

(ii) *For all sufficiently large $k \in \mathbb{N}$ the set A is practically asymptotically stable for $\widetilde{\Phi}_k^{u_k}$ with stability neighborhood B and attraction rate $\tilde{\beta} \in \mathcal{KL}$ independent of k.*

P r o o f. First note that in both 1. and 2. the statements (i) and (ii) are symmetric. Thus, in both cases it is sufficient to show (i) $\Rightarrow$ (ii). For this purpose we denote the two inflated systems by Ψ_k and $\widetilde{\Psi}_k$, respectively. Then from (7.5) and Proposition 4.1(ii) we obtain that these systems are $(\varepsilon_k, 1)$–embedded in each other.

1. (i) From the practical ISS property of Ψ_k by Proposition 5.1 we can conclude practical ISS of A for $\widetilde{\Psi}_k$ with $\tilde{\gamma}_k(r) = \gamma_k(r + \varepsilon_k) - \gamma_k(\varepsilon_k)$. The asserted limit property follows from $\tilde{\gamma}_k(C\varepsilon_k) \leq \gamma_k((C+1)\varepsilon_k) \to 0$ for $k \to \infty$.

2. Assume that A is practically asymptotically stable for $\Phi_{h_k}^{u_k}$ with attraction rate β independent of k. Let $\eta_k \to 0$ denote the constants from the practical asymptotic stability property. Then Lemma 3.1 yields the existence of asymptotically stable sets A_k for $\Phi_{h_k}^{u_k}$ with $d_k := d_H(\widetilde{A}_k, A) \leq \eta_k$ and attraction rates $\beta_k(r,t) \leq \beta(r + \eta_k, t) + \eta_k$. From this, by Lemma A.1 we obtain that the sets A_k have an attraction rate $\tilde{\beta}$ independent of k.

By Theorem 5.2 the Lipschitz property (2.8) implies that these sets A_k are ISS with attraction rate $\tilde{\beta}$, robustness gain $\gamma \in \mathcal{K}_\infty$ and perturbation set $W = B_\alpha(0)$ independent of k for the inflated system Ψ_k. Thus, by Proposition 5.1, for sufficiently large k the sets A_k are practically ISS for the inflated system $\widetilde{\Psi}_k$ with attraction rate $\tilde{\beta}$ independent of k. Consequently, they are also practically asymptotically stable for $\widetilde{\Phi}_k^{u_k}$ with attraction rate $\tilde{\beta}$ independent of k. $\square$

Remark 7.3 (i) In the non-convergent case, it follows that the asserted ISS property of the controlled approximated systems (7.4) is sufficient for asymptotic stability of the exact sampled-data system (7.3). In fact, this robust stability property is at the heart of many stability proofs for sampled-data systems in the literature, even if rarely explicitly mentioned. The respective ISS property for the inflated system is often ensured via Lyapunov functions, as in [9], [25] or [23], or via optimal value functions as in [4]. Our result shows that these conditions are not only sufficient for practical asymptotic stability but in fact necessary and sufficient for the exact sampled-data system (7.3) being practically ISS w.r.t. inflation.
(ii) It is interesting to note that our theorem holds similarly for vanishing sampling period sequences $h_k \to 0$ and for constant sequences $h_k \equiv h$, while in the literature these cases are often treated separately, as in [9] or [23]. The reason for this is that when ensuring ISS via Lyapunov functions, for constant sampling rates equi-continuity of the corresponding Lyapunov functions V_k is sufficient while for vanishing sampling-rate stronger continuity properties (typically Lipschitz assumptions with constants independent of k) on the V_k are needed.
(iii) Our approach is tailored to treat asymptotic stability questions. However, it is reasonable to expect that similar necessary and sufficient uniformity statements could be made for other qualitative properties of sampled-data systems, in particular for properties which admit a dissipation-like characterization [19].

A Appendix: A technical result on $\mathcal{KL}$ functions

In this appendix we formulate and prove a useful property of a sequence of $\mathcal{KL}$–functions which is used in the proofs of Theorem 6.2 and Theorem 7.2.

Lemma A.1 *Let $\mu \in \mathcal{KL}$, $\beta_k \in \mathcal{KL}$ and $d_k \in \mathbb{R}_0^+$ for $k \in \mathbb{N}$ satisfying $d_k \to 0$ and $\beta_k(r,t) \leq \mu(r + d_k, t) + d_k$ for all $k \in \mathbb{N}$. Then there exists $\beta \in \mathcal{KL}$ with $\beta_k(r,t) \leq \beta(r,t)$ for all $k \in \mathbb{N}$.*

P r o o f. Defining $\beta_0(r,t) = 2\mu(r,t)$ and

$$\beta(r,t) := \sup_{k=0,1,2,\ldots} \beta_k(r,t),$$

this function β obviously satisfies $\beta_k \leq \beta$ for all $k \in \mathbb{N}$. It remains to show that $\beta \in \mathcal{KL}$.

To this end we first prove that for each compact set $D = [r_1, r_2] \times [0, T]$ for $r_2 > r_1 > 0$ and $T > 0$, there exists $k_0(D) > 0$ such that

$$\beta(r,t) = \max_{k=0,1,\ldots,k_0(D)} \beta_k(r,t) \tag{A.1}$$

holds for all $(r,t) \in D$.

In order to prove (A.1) we define $\varepsilon := \min_{(r,t)\in D} \mu(r,t) > 0$. Now μ is uniformly continuous on the compact set D. Thus, since $d_k \to 0$ we find $k_0 \in \mathbb{N}$ such that $\mu(r, d_k, t) + d_k \leq \varepsilon$ holds for all $(r,t) \in D$ and all $k > k_0(D)$. This choice of $k_0(D)$ yields

$$\beta_k(r,t) \leq \mu(r,t) + \varepsilon \leq 2\mu(r,t) = \beta_0(r,t)$$

for all $(r,t) \in D$ and all $k > k_0(D)$, which implies (A.1).

From (A.1) we immediately obtain that continuity and the strict monotonicity properties carry over from the β_k to β on D and thus on $(0,\infty) \times [0,\infty)$. Since $\beta(0,t) = 0$ is obvious for all $t \geq 0$ from the construction, it remains to show continuity at $\{0\} \times [0,\infty)$, i.e.,

$$\lim_{r\to 0} \sup_{t\geq 0} \beta(r,t) = 0 \tag{A.2}$$

and the limit property

$$\lim_{t\to\infty} \beta(r,t) = 0 \text{ for all } r \geq 0. \tag{A.3}$$

In order to prove (A.2) we show that for each $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $\beta(r_\varepsilon, 0) < \varepsilon$ holds. Then the monotonicity of β yields $\beta(r,t) < \varepsilon$ for all $r \in [0, r_\varepsilon]$ and all $t \geq 0$, which implies (A.2) since $\varepsilon > 0$ was arbitrary. In order to show the existence of this $r_\varepsilon > 0$, choose $k_\varepsilon > 0$ such that $\mu(d_k, 0) + d_k < \varepsilon$ holds for all $k > k_\varepsilon$. Then, for any $r > 0$ the inequality $\beta(r,0) \geq \varepsilon$ implies $\beta(r,0) \leq \max_{k=0,1,\ldots,k_\varepsilon} \beta_k(r,0)$. Since this expression tends to 0 as $r \to 0$, we obtain the existence of $r_\varepsilon > 0$ with $\beta(r_\varepsilon, 0) < \varepsilon$.

Inequality (A.3) is proved analogously. □

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