Nonlinear sampled-data redesign: analytical formulas and their practical implementation

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Abstract: We investigate the sampled-data redesign problem for nonlinear control affine multi-input systems and consider sampled-data feedback laws for which the trajectories of the sampled-data closed loop system converge to the continuous time trajectories with a prescribed rate of convergence as sampling time vanishes. We analyze geometric existence conditions and propose a line search method for the practical implementation.

1 Introduction

Feedback controllers are nowadays typically implemented using digital devices. In contrast to analog implementations, these devices are not able to evaluate the feedback law continuously in time but only at discrete sampling time instances. Thus, the controller must be designed as a sampled-data controller, whose simplest (and most widely used) implementation is a zero order hold, i.e., the feedback law is evaluated at each sampling time and the resulting control value is kept constant and applied on the sampling interval until the next sampling time.

A popular design method for sampled-data controllers is the design of a controller based on the continuous-time plant model, followed by a discretization of the controller. In other words, the continuous control function generated by the continuous-time controller is replaced by a piecewise constant control function generated by the sampled-data controller. If hardware or communication constraints prohibit the use of small sampling intervals, more sophisticated design techniques have to be used in order to guarantee satisfactory performence, see [7] for a nice introduction to the subject. An important class of such techniques is the *sampled-data redesign*, in which a sampled-data controller is constructed which inherits certain properties of a previously designed feedback law for the continuous-time system. For surveys of analytical redesign techniques for nonlinear single-input systems see, e.g., [1, 8].

In this paper we present analytical redesign techniques for multi-input control affine nonlinear systems extending the results from [3] and discuss practical implementation issues. Our analytically designed controllers solve the redesign problem by maximizing the order at which the difference between the trajectories of the continuous and the sampled-data system converge to zero as the sampling time tends to zero, i.e., by optimizing the asymptotic behavior. This amounts to investigating whether a sampled-data feedback law for a desired order exists and, in case the answer is positive, how it can be computed.

Concerning the conditions for the existence of higher order sampled-data feedback laws, it turns out that the answer lies in the geometry of the system, expressed via Lie brackets of the vector fields. Since for higher orders the existence conditions and formulas for the sampled-data feedback laws become fairly complicated, we restrict our analytical results to low orders in order to illustrate the geometric nature of the conditions and provide a Maple code for arbitrary orders. Since our analytical results are valid only asymptotically, i.e., for sufficiently small sampling periods, and since we will show that in general arbitrary high orders are not feasible, a direct implementation may not yield satisfactory results for large sampling periods. In order to overcome this limitation, we propose a numerically cheap line search for optimizing the gain of the analytically determined correction term. To illustrate our approach we consider the example of a combustion engine test bench from [6], where our proposed method significantly enhances the performance.

2 Problem formulation

We consider a nonlinear plant model

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1}$$

with vector field $f : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ which is continuous and locally Lipschitz in x, state $x(t) \in \mathbb{R}^n$ and control $u(t) \in \mathbb{U} \subset \mathbb{R}^m$. Throughout the paper we assume that a smooth static state feedback $u_0 : \mathbb{R}^n \to \mathbb{R}^m$ has been designed which solves some given control problem for the continuous-time closed-loop system

$$\dot{x}(t) = f(x(t), u_0(x(t))), \qquad x(0) = x_0. \tag{2.2}$$

Our goal is now to design $u_T(x)$ such that the corresponding sampled-data solution $\phi_T(t, x_0, u_T)$ of the closed-loop system using a sampler and zero order hold

$$\dot{x}(t) = f(x(t), u_T(x(kT))), \quad t \in [kT, (k+1)T), \tag{2.3}$$

$k = 0, 1, \ldots$, reproduces the behavior of the continuous-time system. Our approach uses an asymptotic analysis in order to study the difference between the continuous-time model (2.2) and the sampled-data model (2.3). To this end, for a function $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and a compact set $K \subset \mathbb{R}^n$ we write $a(T, x) = \mathcal{O}(T^q)$ *on* K, if there exists a constant $C > 0$ such that the inequality $a(T, x) \le CT^q$ holds for all elements $x \in K$.
In order to obtain asymptotic estimates, for an "output" function $h : \mathbb{R}^n \to \mathbb{R}$ we define the difference

$$\Delta h(T, x_0, u_T) := |h(\phi(T, x_0)) - h(\phi_T(T, x_0, u_T))|, \tag{2.4}$$

where $\phi(T, x_0)$ denotes the solution of the continuous-time system (2.2). Note that h here is not a physical output of the system but rather a scalar auxiliary function which can be chosen arbitrarily. In particular, we will use $h_i(x) = x_i$, $i = 1, \ldots, n$, in order to establish $\Delta h_i(T, x_0, u_T) = \mathcal{O}(T^q)$ which then implies

$$\Delta\phi(T, x_0, u_T) := \|\phi(T, x_0) - \phi_T(T, x_0, u_T)\|_\infty = \mathcal{O}(T^q) \tag{2.5}$$

measured in the maximum norm $\|\cdot\|_\infty$. From this estimate it follows by a standard induction argument that on each compact interval $[0, t^*]$ we obtain $\Delta\phi(t, x_0, u_T) \le \mathcal{O}(T^{q-1})$ for all times $t = kT$, $k \in \mathbb{N}$ with $t \in [0, t^*]$ which in particular allows to carry over stability properties from ϕ to ϕ_T, see [10]. An alternative application of (2.4) is obtained if $h = V$ is a Lyapunov function, see [11] for details.
In order to facilitate this analysis we restrict ourselves to control affine systems where (2.1)–(2.3) take the form

$$\dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_{0,i}(x(t)) = g_0(x) + G(x)u_0(x) \quad \text{with} \quad G(x) = \begin{pmatrix} g_{1,1}(x) & \cdots & g_{m,1}(x) \\ \vdots & \ddots & \vdots \\ g_{1,n}(x) & \cdots & g_{m,n}(x) \end{pmatrix}, \tag{2.6}$$

with smooth vector fields $g_0, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}^n$ and feedback laws $u_0 = (u_{0,1}(x), \ldots, u_{0,m}(x))^T$ with $u_{0,1}, \ldots, u_{0,m} : \mathbb{R}^n \to \mathbb{R}$. We look at sampled-data feedback laws meeting the following definition.

Definition 2.1. *An admissible sampled-data feedback law u_T is a family of maps $u_T : \mathbb{R}^n \to \mathbb{R}^m$, parameterized by the sampling period $T \in (0, T^*]$ for some maximal sampling period T^*, such that for each compact set $K \subset \mathbb{R}^n$ the inequality* $\sup_{x \in K, T \in (0, T^*]} \| u_T(x) \|_\infty < \infty$ *holds.*

Note that for existence and uniqueness of the solutions of (2.3), we do not need any continuity assumptions on u_T. Local boundedness is imposed, because unbounded feedback laws are physically meaningless and may lead to closed-loop systems which are very sensitive to modelling or approximation errors, cf., e.g., the examples in [10, 12].
We use the following notation: for subsets $D \subset \mathbb{R}^n$ we write $\operatorname{cl} D$, $\operatorname{int} D$ for the closure and the interior of D. The notation $|\cdot|$ stands for the Euclidean norm while $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$ denotes the maximum norm in $\mathbb{R}^n$. Furthermore, we denote by $L_g h(x) := \frac{d}{dx} h(x) \cdot g(x)$ and $[g_i, g_j] := \frac{d}{dx} g_j \cdot g_i - \frac{d}{dx} g_i \cdot g_j$ the directional derivative of a function $h : \mathbb{R}^n \to \mathbb{R}$ in the direction of $g : \mathbb{R}^n \to \mathbb{R}^n$ and the Lie bracket of vector fields g_i, $g_j : \mathbb{R}^n \to \mathbb{R}^n$.
In the sequel, we will see that the resulting sampled-data feedback (if existing) will be of the form

$$u_T(x) = \sum_{j=0}^{M} T^j u_j(x), \tag{2.7}$$

where the $u_j(x)$ are vectors $(u_{j,1}(x), \ldots, u_{j,m}(x))^T$ with $u_{0,1}(x), \ldots, u_{0,m}(x)$ from (2.2) and $u_{1,1}(x), \ldots, u_{M,m}(x) : \mathbb{R}^n \to \mathbb{R}$ being locally bounded functions. This structure appears to be rather natural and was also obtained as the outcome of the design procedure in several other papers, e.g. in [13]. Hence we introduce the following theorem which is based on the Fliess series expansion, see [4, Theorem 3.1.5] and generalizes [3, Theorem 2.3]. It provides the basis for the redesign of u_T. In order to formulate this result we define multinomial coefficients $\binom{n}{n_0 \ \ldots \ n_M} := \frac{n!}{n_0! n_1! \ldots n_M!}$ as well as multi-indices $\nu := (n_0, n_1, \ldots, n_M)$ and use the notations $|\nu| := n_0 + n_1 + \ldots + n_M$ and $||\nu|| = \sum_{i=0}^{M} i\, n_i$.

Theorem 2.2. *Consider the control affine system* (2.6)*, a smooth function $h : \mathbb{R}^n \to \mathbb{R}$, the continuous-time closed-loop system* (2.2) *and the sampled-data closed-loop system* (2.3) *with controller u_T given by* (2.7)*. Then, for sufficiently small T, we can write:*

$$h(\phi_T(T, x, u_T)) = h(x) + \sum_{s=0}^{M} T^{s+1} \left[\sum_{i=1}^{m} L_{g_i} h(x) u_{s,i} + p_s(x, u_0, \ldots, u_{s-1}) \right] + \mathcal{O}(T^{M+2}) \tag{2.8}$$

where $p_0(x) = L_{g_0}h(x)$ *and* $p_s(x, u_0, \dots, u_{s-1})$, $s = 1, \dots, M$, *is given by*

$$\sum_{k=1}^{s} \sum_{i_0=0,\dots,i_k=0}^{m} \frac{L_{g_{i0}} \cdots L_{g_{ik}} h(x)}{(k+1)!} \cdot \sum_{\substack{v \in \mathbb{N}_0^m: \\ \sum_{i=1}^m v_i = s-k}} \prod_{j=1}^{m} \left(\sum_{\substack{|\nu_j| = c_j \\ \|\nu_j\| = v_j}} \binom{c_j}{n_{0,j}\, n_{1,j} \cdots n_{M,j}} \prod_{l=0}^{M} u_{l,j}^{n_{l,j}} \right) \tag{2.9}$$

with $u_i = (u_{i,1}, \dots, u_{i,m})^T$. *Here* c_j *denotes* $\#\{i_l \mid l = 1, \dots, k : i_l = j\}$.

For a proof as well as for explicit expressions for $p_1(x, u_0)$ and $p_2(x, u_0, u_1)$ see [2]. Computer algebra systems, such as MAPLE, can be used to compute expansions of the difference (2.4) for particular examples, cf. Remark 3.4.

3 Necessary and sufficient conditions

In this section we investigate conditions for the existence of an admissible feedback law u_T which achieves

$$\Delta h(T, x, u_T) = \mathcal{O}(T^q) \qquad \text{or} \qquad \Delta\phi(T, x, u_T) = \mathcal{O}(T^q) \tag{3.1}$$

and provide formulas for these feedback laws. Since the computations with respect to the sufficient condition turn out to be fairly involved we restrict our analytical computations to the case $q \leq 4$ and provide a Maple procedure for the general case. As we will see, $q = 4$ is the first nontrivial case in the sense that (3.1) for $q \leq 3$ can always be satisfied without any further conditions. For the formulation of the theorems we use the notation

$$u^{(i)}(x) = \frac{1}{(i+1)!} \left. \frac{d^i u_0(\phi(t, x))}{dt^i} \right|_{t=0}. \tag{3.2}$$

Note that this definition coincides with the continuous-time controller for $i = 0$. The next theorems are consequences from Theorem 2.2 based on a careful evaluation of the p_i-terms. They generalize corresponding results for the single input-case $m = 1$ [11, Theorem 4.11] and [3, Theorems 3.1 and 3.6]. To prove these results we first show the assertions for $\Delta h(T, x, u_T)$ and deduce the claims for the full state trajectory by choosing $h_i(x) = x_i$, $i = 1, \dots, n$.

Theorem 3.1. *Consider the vector field* (2.6)*, the continuous-time closed-loop system* (2.2)*, the sampled-data closed-loop system* (2.3)*, a smooth function* $h : \mathbb{R}^n \to \mathbb{R}$ *and a compact set* $K \subset \mathbb{R}^n$*. Then the following assertions hold for* $u^{(i)}$ *from* (3.2)*:*

(i) $\Delta h(T, x_0, u_T) = \mathcal{O}(T^2)$ *and* $\Delta\phi(T, x_0, u_T) = \mathcal{O}(T^2)$ *hold on* K *for* $u_T(x) = u^{(0)}(x)$.

(ii) $\Delta h(T, x_0, u_T) = \mathcal{O}(T^3)$ *and* $\Delta\phi(T, x_0, u_T) = \mathcal{O}(T^3)$ *hold on* K *for* $u_T(x) = u^{(0)}(x) + Tu^{(1)}(x)$.

Sketch of proof. The assertion for Δh follows from comparing the Taylor series expansion of $h(\phi(T, x_0))$ in T with the Fliess series expansion provided in Theorem 2.2, for more details see [2, Proof of Theorem 4.1].
For $\Delta\phi$ we use that $\Delta\phi(T, x, u_T) = \mathcal{O}(T^q)$ is equivalent to $\Delta h(T, x, u_T) = \mathcal{O}(T^q)$ for $h_i(x) = x_i$, $i = 1, \dots, n$. Hence, assertions (i) and (ii) for $\Delta\phi(T, x, u_T) = \mathcal{O}(T^q)$ follow immediately by applying the corresponding assertions for $\Delta h(T, x, u_T)$ to $h_i(x) = x_i$, $i = 1, \dots, n$. □

For the case $\mathcal{O}(T^4)$ it turns out that in general a suitable sampled data feedback law u_T does not exist. Instead, we obtain a rather restrictive necessary condition as formulated in the following theorem.

Theorem 3.2. *Consider the vector field* (2.6)*, the continuous-time closed-loop system* (2.2)*, the sampled-data closed-loop system* (2.3) *and a compact set* $K \subset \mathbb{R}^n$*. Then the following assertions hold for* $u^{(i)}$ *from* (3.2)*:*

(iii a) If there exists for a smooth function $h : \mathbb{R}^n \to \mathbb{R}$*, a bounded function* $\alpha_h : K \to \mathbb{R}^m$ *satisfying*

$$\sum_{i=1}^{m} L_{g_i} h(x) \alpha_{h,i}(x) = \sum_{i=1}^{m} \Big[L_{[g_0, g_i]} h(x) + \sum_{\substack{j=1 \\ j \neq i}}^{m} L_{[g_j, g_i]} h(x) u_{0,j}(x) \Big] \dot{u}_i(x), \tag{3.3}$$

then there exists u_T *such that* $\Delta h(T, x_0, u_T) = \mathcal{O}(T^4)$ *holds on* K *with*

$$u_T(x) = \begin{cases} u^{(0)}(x) + Tu^{(1)}(x) + T^2 u^{(2)}(x) + \frac{T^2}{12} \alpha_h(x), & x \in \operatorname{cl} \widetilde{K} \\ u^{(0)}(x) + Tu^{(1)}(x), & x \notin \operatorname{cl} \widetilde{K}, \end{cases} \tag{3.4}$$

where $\widetilde{K} := \{x \in K \mid \exists i : L_{g_i} h(x) \neq 0\}$.
Conversely, if an admissible sampled-data feedback law $\tilde{u}_T = u_T + \mathcal{O}(T^3)$ *for* u_T *from* (3.4) *satisfies* $\Delta h(T, x_0, u_T) = \mathcal{O}(T^4)$ *on a set* $\widehat{K} \subseteq \widetilde{K}$*, then there exists a bounded function* α *satisfying* (3.3) *on* $\operatorname{cl} \widehat{K}$.

(iii b) If there exists a bounded function $\alpha : K \to \mathbb{R}^m$ satisfying

$$\sum_{i=1}^{m} \left[[g_0, g_i](x) + \sum_{\substack{j=1 \\ j \neq i}}^{m} [g_j, g_i](x) u_{0,j}(x) \right] \dot{u}_i(x) = \sum_{i=1}^{m} \alpha_i(x) g_i(x) \tag{3.5}$$

and K satisfies $K = \operatorname{cl} \operatorname{int} K$ then $\Delta\phi(T, x_0, u_T) = \mathcal{O}(T^4)$ holds on K for $u_T(x)$ from (3.4) with $\widetilde{K} := \{x \in K \,|\, \exists i : g_i(x) \neq 0\}$ and α instead of α_h. Indeed, $u_T(x)$ can be chosen arbitrarily for $x \notin \operatorname{cl} \widetilde{K}$. Furthermore, on $K^* = \{x \in K \,|\, G(x)$ from (2.6) has full column rank$\}$, any feedback law $\tilde{u}_T$ satisfying $\Delta\phi(T, x_0, \tilde{u}_T) = \mathcal{O}(T^4)$, is of the form $\tilde{u}_T(x) = u_T(x) + \mathcal{O}(T^{q-1})$ for u_T and the function α in (3.5) is unique if it exists. On $\operatorname{cl} K^*$ the sufficient condition (3.5) is also necessary for the existence of u_T in (iii).

Sketch of proof. In order to prove (iii a) we consider the Taylor series expansion of $\phi(T, x_0)$ in $T = 0$ up to order four. Comparing the resulting terms with Theorem 2.2 for u_T from (3.4) and using Theorem 3.1 and the definition of α_h one sees that the choice (3.4) ensures a forth order approximation, for details see [2, Proof of Theorem 4.1]. Note that the function α_h is bounded on K by assumption, which in particular implies that the control law (3.4) is admissible in the sense of Definition 2.1 on $\widetilde{K}$. For the converse statement the proof is identical to the single-input case in [3].
For the proof of (iii b) one first shows that under condition (3.5) any feedback of the stated form satisfies the assertion. This part of the proof follows the single-input case in [3] with slight technical modifications. The form of the $\tilde{u}_T$ on K^* follows since from Theorem 2.2 for $M = 0$ we obtain that any $\tilde{u}_T$ satisfying $\Delta\phi(T, x_0, \tilde{u}_T) = \mathcal{O}(T^2)$ must fulfill

$$\sum_{i=1}^{m} L_{g_i} h_k(x) u_{0,i}(x) = \sum_{i=1}^{m} L_{g_i} h_k(x) \tilde{u}_{T,i}(x) + \mathcal{O}(T)$$

for $k = 1, \ldots, n$ in order to satisfy part (i) of Theorem 3.1 for $h(x) = h_k(x) = x_k$, $i = k, \ldots, n$. Using $L_{g_i} h_k(x) = g_{i,k}(x)$ this is equivalent to $G(x)u_0(x) = G(x)\tilde{u}_T(x) + \mathcal{O}(T)$ and since $G(x)$ has full column rank this implies $\tilde{u}_T(x) = u_0 + \mathcal{O}(T)$. The higher order terms in T are obtained analogously using Theorem 3.1(ii) and part (iii a) of this theorem, respectively. The uniqueness of α on K^* follows because the right hand side of (3.5) equals $G(x)\alpha(x)$. Finally, from the uniqueness property of u_T the necessity of (3.5) on $\operatorname{cl} K^*$ follows from the converse statement in part (iii a) for $\widehat{K} = K^*$. $\square$

Remark 3.3. *The Theorems 3.1 and 3.2 have a nice geometric interpretation if we consider the possible directions of the system trajectories. To this end, consider the expansions*

$$\phi(T, x) = v_0 + Tv_1 + T^2 v_2 + T^3 v_3 + \ldots \quad \text{and} \quad \phi_T(T, x, u_T) = w_0 + Tw_1 + T^2 w_2 + T^3 w_3 + \ldots$$

While the control value in ϕ may vary in time, the control value in ϕ_T is constant on the sampling interval $[0, T)$. Thus, for each $i = 0, 1, \ldots$ the set of possible directions v_i which can be generated by different choices of u_0 is larger or equal than the corresponding set of possible directions w_i generated by different u_T.
Theorem 3.1 now shows that the sets of possible directions v_i and w_i are indeed identical for $i = 0, 1$ and 2, because $\mathcal{O}(T^2)$ and $\mathcal{O}(T^3)$ are unconditionally feasible. Note that the T-dependence of u_T is crucial for achieving $w_2 = v_2$. This is no longer possible for the directions v_3 and w_3 which affect the trajectories with order $\mathcal{O}(T^3)$. Indeed, our analysis shows that the direction v_3 can be decomposed as $v_3 = v_3^1 + v_3^2$, such that $w_3 = v_3^1$ can always be achieved via the $u^{(2)}$ term in u_T while v_3^2 cannot in general be reproduced by w_3. This direction v_3^2 is exactly the expression appearing on the left hand side of (3.5). Condition (3.5) now demands that v_3^2 lies in span$\langle g_1, \ldots, g_m \rangle$ such that it can be compensated by the α-term of the sampled-data feedback law u_T.

Remark 3.4. *From Theorem 2.2 we can obtain a simple recursive procedure for computing u_T for arbitrary orders: Assume that $u_0, \ldots, u_{M-1}$ in (2.7) are determined and realize the order $\mathcal{O}(T^{M+1})$. Then, comparing the summands for $s = M$ in (2.8) and in the Taylor expansion of $\phi(T, x)$ leads to a (in general overdetermined) linear system*

$$G(x) u_M(x) = b(x). \tag{3.6}$$

If (3.6) admits a solution, then this defines the M-th component of u_T in (2.7) which then realizes the order $\mathcal{O}(T^{M+2})$. If (3.6) does not admit a solution, then the order $\mathcal{O}(T^{M+2})$ cannot be achieved by a sampled data feedback law.
This procedure can be efficiently implemented in MAPLE[1] *using the least squares solver in order to solve (3.6) and checking the residual in order to decide whether (3.6) is solvable. Furthermore, this procedure shows that we can always achieve any desired order if the matrix G is square, i.e., $m = n$, and invertible.*

Remark 3.5. *In [9] it was shown for single-input systems, i.e., $m = 1$, that the condition $[g_0, g_1] \in$ span$\langle g_1 \rangle$ is necessary and sufficient for the existence of sampled-data feedback laws u_T realising $\Delta\phi(T, x) = \mathcal{O}(T^q)$ for all $q \geq 2$ and all continuous-time feedback laws u_0. We conjecture that the generalization of this condition to the multi-input case is $[g_i, g_j] \in$ span$\langle g_1, \ldots, g_m \rangle$ for all $i, j = 0, \ldots, m$. Note that the sampled-data feedback laws considered in [9] are not necessarily locally bounded and thus may not fulfill our Definition 2.1.*

[1]Our MAPLE implementation is available on `www.math.uni-bayreuth.de/~lgruene/publ/redesign_mtns.html`.

4 Practical implementation

For sufficiently small sampling times T, the application of Theorems 3.1 and 3.2 is straightforward: one can determine the highest possible order of approximation and directly implement the obtained feedback. Examples can be found in [3] for the single-input case and in [6] for the multi-input case for a practical example of a combustion engine test bench. Indeed, this works even for tracking type problems, see [5]. Unfortunately, for larger sampling periods the redesign procedure tends to overestimate the required correction terms in u_T. As a remedy, an approach for limiting the gain of the correction terms has been used in [11, section 5] for single-input systems. Here we generalize this procedure to the multi-input case for the controller u_T from Theorem 3.1 (ii).
For this purpose we have a closer look at the feedback law $u_T = u^{(0)} + Tu^{(1)}$ from Theorem 3.1 (ii). Numerical examinations revealed that typically the redesigned feedback u_T yields an adjustment in a suitable direction but that the gain of $Tu^{(1)}$ is too large for larger sampling intervals T, which in particular produces large overshoots. This leads to the idea of replacing the correction term $Tu^{(1)}$ by $\gamma Tu^{(1)}$ for some $\gamma \in [0, 1]$. As a first and simple approach, we choose $\gamma = \min\{1, |x|/|Tu^{(1)}|\}$. Note that it is important to include T in this computation in order to cope with larger sampling periods. In section 5 we will present results for this heuristic procedure under the name modified redesign.
A considerable improvement of this heuristic choice of γ can be obtained by minimizing the deviation from the continuous-time solution $|\Phi(T, x) - \Phi_T(T, x, u^{(0)} + \gamma Tu^{(1)})|$ after one sampling interval over $\gamma \in [0, 1]$, i.e. we carry out a line search between u_0 and u_T. The advantage in comparison with more sophisticated optimization based redesign methods, like, e.g., the Gauss-Newton or MPC based ones presented in [1, Section 4], relies on the analytically computed direction. Standard optimization algorithms typically consists of two steps: finding a suitable direction – which is the mathematically and computationally more demanding part – and then executing a line search along this direction. In our case we only have to handle the latter because our analytical results give us a good candidate for the search direction.

5 Example

We illustrate our results by means of a combustion engine test bench model given by (2.6) with $m = 2$:

$$f(x) = \begin{bmatrix} -(\tilde{c}_0 + \tilde{c}_1 x_3 + \tilde{c}_2 x_3^2) - \gamma_1 x_3 - \gamma_2 x_3^2 \\ \beta(x_3 - x_4) \\ \frac{1}{\theta_E}\left(\frac{c}{\beta}(x_1 - x_2) - d(x_3 - x_4)\right) \\ \frac{1}{\theta_D}\left(\frac{c}{\beta}x_2 + d((x_3 - x_4)\right) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} 2r_1k_1x_1 \\ r_2(2k_4x_4 + k_5x_2) \end{bmatrix}.$$

Here the continuous-time feedback was obtained via a control Lyapunov function, for further details see [6]. As initial value for the state we use $x_0 = [0.6\,0.4\,0\,0]^T$. For fast sampling the redesign improves the performance of the sampled-data system relative to the emulated solution and the modified redesign provides even better results, see Figure 1. Although the redesign solution exhibits very poor performance in case of the larger sampling period $T = 0.75$ sec.,

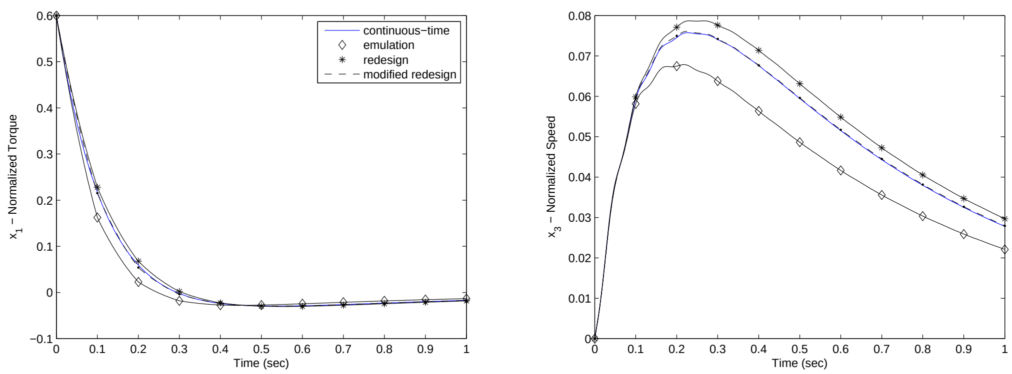

Figure 1: Response of the engine system with small sampling T=0.1 sec.

the modified redesign still outperforms the emulation. Using the line search method (applied here with at most 5 function evaluations) provides an even better approximation of the continuous-time solution. Basically these results are independent of the chosen initial conditions. For practical application we recommend the modified redesign because it shows to be even more robust in contrast to the redesigned controller and outperforms the emulation solution in the

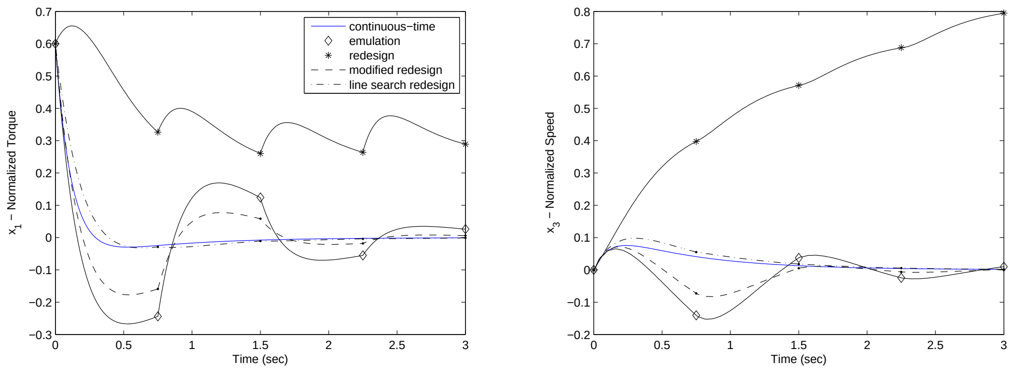

Figure 2: Response of the engine system with large sampling T=0.75 sec.

majority of cases. For large sampling periods the line search approach outperforms the other methods and exhibits generally very good results. The low numerical effort needed for this approch is remarkable, especially in comparison with more sophisticated optimization methods. Hence, the method is suitable for online optimization.

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