Digital redesign of nonlinear multi-input systems

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(joint work with Dragan Nesić, Karl Worthmann)

1. Introduction

At the Oberwolfach Control Theory Meeting 2005 I presented the following open problem:

Consider a single input control affine closed loop system

\begin{equation}
\dot{x}(t) = g_0(x(t)) + g_1(x(t))u(x(t))
\end{equation}

with \( x \in \mathbb{R}^n \) and a smooth feedback controller \( u : \mathbb{R}^n \to \mathbb{R} \) and the corresponding sampled-data system

\begin{equation}
\dot{x}_T(t) = g_0(x_T(t)) + g_1(x_T(t))u_T(x_T(iT)), \quad t \in [iT, (i + 1)T), \ i = 0, 1, \ldots
\end{equation}

with a family of sampled-data controllers \( u_T : \mathbb{R}^n \to \mathbb{R} \) parameterized with the (sufficiently small) sampling rate \( T > 0 \) which are locally bounded uniformly in \( T \) but not necessarily continuous. We consider the mismatch after one time step given by

\( \Delta_T(x_0) := \|x(T, x_0, u) - x_T(T, x_0, u_T)\| \),

with \( x(t, x_0, u) \) and \( x_T(t, x_0, u_T) \) denoting the solutions of (1) and (2), respectively, with initial value \( x_0 \) at time \( t = 0 \).

It is easy to prove that for \( u_T \equiv u \) we obtain \( \Delta_T = O(T^2) \) while for

\begin{equation}
u_T(x) = u(x) + \frac{T}{2} \frac{\partial u(x)}{\partial x} [g_0(x) + g_1(x)u(x)]
\end{equation}

we obtain \( \Delta_T = O(T^3) \) (this follows from [4, Theorem 4.11] setting \( V(x) = x_i \) observing that positive definiteness of \( V \) is not needed). Remark 4.12 in [4] suggests that higher order cannot be obtained in general.

**Problem:** Find conditions on \( g_0, g_1, u \) under which \( \Delta_T \leq O(T^4) \) can be achieved.

In this report a solution to the problem and an extension to multi-input systems will be presented. In the talk, we will in addition discuss performance issues and present a novel numerical optimization approach based on these results.

2. Single-Input systems

We use the following notation: for two vector fields \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) we define the usual Lie bracket by \([f, g] = \frac{d}{dx} g \cdot f - \frac{d}{dx} f \cdot g\). Furthermore, for \( k \in \mathbb{N} \) we define

\begin{equation}
u^k(x_0) := \left. \frac{d^k}{dt^k} \right|_{t=0} u(x(t, x_0, u)).
\end{equation}

\(1\Delta_T = O(T^m) \) means: for each compact \( K \subset \mathbb{R}^n \) there is \( C > 0 \) with \( \sup_{x \in K} \Delta_T(x) \leq CT^m \)
Note that with this notation (3) can be written as

\[ u_T(x) = u(x) + \frac{T}{2} u^1(x). \]

**Theorem 2.1:** A feedback law \( u_T \) with \( \Delta_T = O(T^4) \) exists if and only if there exists a bounded function \( \alpha : \mathbb{R}^n \to \mathbb{R} \) satisfying

\[ [g_0, g_1](x)u^1(x) = \alpha(x)g_1(x). \]

If this condition holds, then the feedback laws \( u_T \) are given by

\[ u_T(x) = u(x) + \frac{T}{2} u^1(x) + \frac{T^2}{6} u^2(x) + \frac{T^2}{12} \alpha(x) \]

and these \( u_T \) are uniquely determined up to terms of order \( O(T^3) \) for all \( x \) with \( g_1(x) \neq 0 \).

The proof of this theorem relies on comparing the Taylor expansion of \( x(T, x_0, u) \) with the Fliess expansion of \( x_T(T, x_0, u_T) \) in \( T = 0 \), see [1, Theorem 3.6] for details.

**Remark 2.2:**

(i) Conditions for higher order \( \Delta_T \leq O(T^5) \) can be stated similarly but become more and more involved. However, computer mathematics systems like, e.g., MAPLE can be used to check the conditions recursively and compute the corresponding \( u_T \).

(ii) The condition (5) is rather restrictive. Hence, Theorem 2.1 shows that a mismatch \( \Delta_T \leq O(T^4) \) can hardly be expected in general, regardless of how \( u_T \) is chosen. In particular, the seemingly “natural” Taylor-like choice

\[ u_T(x) = u(x) + \frac{T}{2} u^1(x) + \frac{T^2}{6} u^2(x) \]

only works if \( \alpha \equiv 0 \). A sufficient condition for \( \alpha \equiv 0 \) is \( [g_0, g_1] \equiv 0 \), i.e., the vector fields commute.

(iii) A sufficient condition for (5) is \( [g_0, g_1] \in \text{span}(g_1) \). In [3] it was shown that this condition is necessary and sufficient for the fact that for each smooth controller \( u : \mathbb{R}^n \to \mathbb{R} \) there exists \( u_T \) satisfying \( \Delta_T \leq O(T^k) \) for arbitrary \( k \in \mathbb{N} \).

3. **Multi-Input systems**

We now extend our result to multi-input control affine systems of the form

\[ \dot{x}(t) = g_0(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(x(t)) \]

with vector fields \( g_i = (g_{i,1}, \ldots, g_{i,n})^T, i = 1, \ldots, m, m \in \mathbb{N}, m \leq n \), and controller \( u = (u_1, \ldots, u_m)^T \). We write the right hand side of the system briefly as

\[ g_0(x) + G(x)u(x) \quad \text{with} \quad G(x) = \begin{pmatrix} g_{1,1}(x) & \cdots & g_{m,1}(x) \\ \vdots & \ddots & \vdots \\ g_{1,n}(x) & \cdots & g_{m,n}(x) \end{pmatrix}, \]

and use definition (4) also for these vector valued feedback laws.
As in the single input case for \( u_T \equiv u \) we get \( \Delta_T = O(T^2) \) sets while for \( u_T(x) = u(x) + \frac{T}{2} u^1(x) \) we obtain \( \Delta_T = O(T^3) \), cf. [2, Theorem 4.1 (i)-(ii)]. For \( \Delta_T \leq O(T^4) \), Theorem 2.1 generalizes as follows, see [2, Theorem 4.1 (iii)]. Again, the proof relies on Taylor and Fliess expansions of the solution.

**Theorem 3.1:** For the multi-input system (6), a feedback law \( u_T \) with \( \Delta_T \leq O(T^4) \) exists if there exists a bounded function \( \alpha : \mathbb{R}^n \to \mathbb{R}^m \) satisfying

\[
\sum_{i=1}^{m} \left[ g_0, g_i \right](x) + \sum_{j=1}^{m} \sum_{j \neq i} g_j, g_i(x) u_{0,j}(x) \right] u_i^1(x) = \sum_{i=1}^{m} \alpha_i(x) g_i(x).
\]

If this condition holds, then the feedback laws \( u_T \) are given by

\[
u_T(x) = u(x) + T^2 \frac{T}{2} u^1(x) + T^2 \frac{T^2}{6} u^2(x) + \frac{T^2}{12} \alpha(x)
\]

and these \( u_T \) are uniquely determined up to terms of order \( O(T^3) \) for all \( x \) for which \( G(x) \) has full column rank. For these \( x \) condition (8) is also necessary.

As in the case of Theorem 2.1, the results can be extended to higher orders which is most conveniently done recursively using a computer mathematics system such as maple. This recursive design procedure leads to a feedback of the form

\[ u_T(x) = u(x) + T^2 \frac{T}{2} u^1(x) + \frac{T^2}{6} u^2(x) + \ldots \]

in which each \( \tilde{u}^k \) is the solution of a least squares problem of the form \( G(x) \tilde{u}^k(x) = b^k(x) \). If this problem is solvable with residual 0 for \( k = 1, \ldots, m \), then \( u_T \) is a sampled-data feedback yielding \( \Delta_T \leq O(T^{m+2}) \). In particular, this shows that

(i) the problem is solvable for arbitrary order \( O(T^k) \), \( k \in \mathbb{N} \), if \( G(x) \) is square and invertible for all \( x \in \mathbb{R}^n \)

(ii) the problem is in general not solvable for \( \Delta_T \leq O(T^3) \) if \( G(x) \) is not square, i.e., when \( m < n \).

**References**


