

Horizon Adaptation for Nonlinear Model Predictive Controllers with guaranteed Degree of Suboptimality

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Abstract

We propose *adaptation strategies* to modify the standard constrained model predictive controller scheme in order to guarantee a certain lower bound on the degree of suboptimality. Within this analysis, the length of the optimization horizon is the parameter we wish to adapt. We develop and prove several shortening and prolongation strategies which also allow for an effective implementation. Moreover, extensions of stability results and suboptimality estimates to model predictive controllers with varying optimization horizon are shown.

Keywords: nonlinear model predictive control, suboptimality, stability, adaptation strategies

1. Introduction

Nowadays, nonlinear model predictive controllers (NMPC), sometimes also called receding horizon controllers (RHC) are used in a variety of industrial applications, cf. [1]. As shown in [12, 17], theory for such controllers is also widely understood both for linear and nonlinear systems. The control method itself deals with the problem of approximately solving an infinite horizon optimal control problem which is computationally intractable in general. Reasons for its success are on the one hand its capability to directly incorporate constraints depending on the states and inputs of the underlying process. On the other hand, the fundamental steps of this method are very simple: First, a solution of a finite horizon optimal control problem is computed for a given initial value. In a second step, the

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first part of the resulting control is implemented at the plant and in the third and last step, the finite horizon is shifted forward in time which renders this method to be iteratively applicable. As a consequence, the control which is applied at the plant is a static state feedback.

Due to considering only finite horizons, the inherent stability property of the infinite horizon problem does in general not carry over to the NMPC problem. To cope with the stability issue, several solutions have been proposed in the past, i.e. by imposing endpoints constraints [10] or adding so called Lyapunov function type endpoint weights and a terminal region to the NMPC problem [2]. A third idea deals with the plain NMPC problem without the requirement of added constraints or a modified cost function. To show stability of the resulting closed loop, in [7, 9] a relaxed Lyapunov inequality is assumed.

In either case, the horizon needs to be chosen as a worst case scenario which is usually needed to cope with small regions of state space only. Our aim in this work is to develop online applicable adaptation strategies for the horizon length which guarantee stability of the closed loop. Here, we follow the third approach since the original intention of the infinite horizon cost stays untouched, and make use of the suboptimality estimates given in [7]. Based on the structure of these suboptimality estimates and on the structure of the NMPC problem itself, we propose several techniques to fit the horizon to the control task, the current state of the system and also to the internal information of the NMPC controller itself. Due to the change of the structure of the controller, however, known stability proofs and suboptimality results [6, 7, 8, 9] cannot be applied. To cover these issues, stability results with varying optimization horizons are presented.

To some extent adaptation strategies of the horizon are known in the literature, see, e.g., [5, 19]. In contrast to these pure heuristics, our approach can be proven rigorously and does not require any insight into the process under consideration. Moreover, a change of the cost functional is possible without modification of the adaptation law which allows for testing various settings of the controller. Last, the quality of the resulting closed loop is tunable by a single variable characterizing the allowable tradeoff compared to the infinite horizon optimal control law.

The paper is organized as follows: In Section 2 we describe the problem setup and state the a posteriori and a priori suboptimality estimates which will be the foundation of our analysis. In the following Section 3, we first show how known stability results and estimates can be extended to the case of varying optimization horizons. In Section 4, we develop various shortening and prolongation strategies based on the suboptimality estimates from [7]. To show applicability and effectiveness of the proposed methods we present numerical results in Section 5. Finally,

Section 6 concludes the paper and points out directions of future research.

2. Setup and Preliminaries

In this work we consider nonlinear discrete time systems of the form

$$x(n+1) = f(x(n), u(n)), \quad x(0) = x_0 \quad (1)$$

with $x(n) \in X$ and $u(n) \in U$ for $n \in \mathbb{N}_0$ where \mathbb{N}_0 denotes the natural numbers including zero. In this context, the state space X and the control value space U are arbitrary metric spaces. Therefore, all presented results also apply to the discrete time dynamics induced by a sampled infinite dimensional system. State and control constraints can be incorporated by replacing X and U by appropriate subsets $\mathbb{X} \subset X$ and $\mathbb{U} \subset U$. Here, we denote the space of control sequences $u : \mathbb{N}_0 \rightarrow \mathbb{U}$ by $\mathbb{U}^{\mathbb{N}_0}$ and the solution trajectory for given control $u \in \mathbb{U}^{\mathbb{N}_0}$ and initial value $x_0 \in \mathbb{X}$ by $x_u(\cdot, x_0)$.

The task which we pursue is to find a static state feedback $u = \mu(x) \in \mathbb{U}^{\mathbb{N}_0}$ for a given control system (1) which minimizes the *infinite horizon cost functional* $J_\infty(x_0, u) = \sum_{n=0}^{\infty} l(x_u(n, x_0), u(n))$ with stage cost $l : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}_0^+$ where \mathbb{R}_0^+ denotes the nonnegative real numbers. The *optimal value function* for this problem is denoted by $V_\infty(x_0) = \inf_{u \in \mathbb{U}^{\mathbb{N}_0}} J_\infty(x_0, u)$. Moreover, one can prove optimality of the *infinite horizon feedback law* $\mu(\cdot)$ given by

$$\mu(x(n)) = \operatorname{argmin}_{u \in \mathbb{U}} \{V_\infty(x_u(1, x(n))) + l(x(n), u)\} \quad (2)$$

using Bellman's optimality principle for a given optimal value function. Here we use the argmin operator in the following sense: for a map $a : \mathbb{U} \rightarrow \mathbb{R}$, a nonempty subset $\tilde{\mathbb{U}} \subseteq \mathbb{U}$ and a value $u^* \in \tilde{\mathbb{U}}$ we write

$$u^* = \operatorname{argmin}_{u \in \tilde{\mathbb{U}}} a(u) \quad (3)$$

if and only if $a(u^*) = \inf_{u \in \tilde{\mathbb{U}}} a(u)$ holds. Whenever (3) holds the existence of the minimum $\min_{u \in \tilde{\mathbb{U}}} a(u)$ follows. However, we do not require uniqueness of the minimizer u^* . In case of uniqueness equation (3) can be understood as an assignment, otherwise it is just a convenient way of writing “ u^* minimizes $a(u)$ ”. Here we assume that the minimum with respect to $u \in \mathbb{U}$ is attained.

Since the computation of the desired control law requires the solution of a Hamilton–Jacobi–Bellman equation, we use a model predictive control approach

in order to avoid the problem of solving an infinite horizon optimal control problem. The NMPC methodology is simple and consists in three steps which are repeated at every discrete time instant during the process run: Upon start of each iterate, an optimal control for the problem on a finite horizon is computed. Then, the first element of the control is implemented at the process and in the third step the entire optimal control problem considered in the first step is shifted forward in time by one discrete time instant, see, e.g., [12] for an overview of this method.

Concerning the computing step of the control law, we consider a finite horizon optimal control problem, that is we minimize the truncated cost functional

$$J_N(x_0, u) = \sum_{k=0}^{N-1} l(x_u(k, x_0), u(k)). \quad (4)$$

For reasons of clarity, we denote the closed loop solution at time instant n by $x(n)$ throughout this work while $x_u(\cdot, x_0)$ denotes the open loop trajectory of the prediction. Moreover, we use the abbreviation

$$u_N(\cdot, x_0) = \underset{u \in \mathbb{U}^N}{\operatorname{argmin}} J_N(x_0, u) \quad \text{and} \quad u_N(x_0) = u_N(0, x_0) \quad (5)$$

for the minimizing *open loop control* sequence of the truncated cost functional and its first element respectively. Moreover, we denote the optimal value function of the finite cost functional (4) by $V_N(x_0) = \min_{u \in \mathbb{U}^N} J_N(x_0, u)$.

Given the initial value $x_{u_N}(0, x_0) = x_0$, the open loop control (5) induces the *open loop solution*

$$x_{u_N}(k+1, x_0) = f(x_{u_N}(k, x_0), u_N(k, x_0)), \quad \forall k \in \{0, \dots, N-1\}. \quad (6)$$

Via the implementation and shift steps of the NMPC controller described earlier, we obtain a feedback control $\mu_N(\cdot)$ which can be defined via Bellman's principle of optimality

$$\mu_N(x(n)) = \underset{u \in \mathbb{U}}{\operatorname{argmin}} \{V_{N-1}(x_u(1, x(n))) + l(x(n), u)\}. \quad (7)$$

Using the feedback $\mu_N(\cdot)$, the *closed loop system* is given by

$$x(n+1) = f(x(n), \mu_N(x(n))), \quad x(0) = x_0, \quad n \in \mathbb{N}_0. \quad (8)$$

In the following, we are interested in the stability and suboptimality properties of the closed loop solution (7), (8). Note that due to the truncation of the infinite

horizon cost functional, stability and optimality properties induced by the infinite horizon optimal control (2) are not preserved in general. Here, we focus on the NMPC implementation without additional stabilizing endpoint constraints or a Lyapunov function type endpoint costs and a terminal region which are outlined in, e.g., [10] and [2] respectively.

Our aim in this work is to show that the requirement of considering the worst case optimization horizon N for all initial values $x \in \mathbb{X}$ can be weakened without losing stability of the closed loop (7), (8). Additionally, we show that the resulting closed loop trajectory satisfies locally a predefined degree of suboptimality compared to the infinite horizon solution (1), (2) with $u(n) = \mu(x(n))$. To this end, we compare the infinite horizon cost induced by the NMPC control law $\mu_N(\cdot)$, that is $V_\infty^{\mu_N}(x_0) := \sum_{n=0}^{\infty} l(x(n), \mu_N(x(n)))$, and the finite horizon cost $V_N(\cdot)$ or the infinite horizon optimal value function $V_\infty(\cdot)$. In particular, the latter gives us estimates about the degree of suboptimality of the controller $\mu_N(\cdot)$ of the NMPC process.

Note that since we do not assume terminal constraints to be imposed, feasibility of the NMPC scheme is an issue that cannot be neglected. In particular, without these constraints the closed loop trajectory might run into a dead end. To exclude such a scenario, we assume the following viability condition to hold. We like to note that in case of stabilizing endpoint constraints [10] or a terminal region [2] this assumption holds implicitly.

Assumption 1. For each $x \in \mathbb{X}$ there exists a control $u \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}$.

In order to derive adaptation strategies for the horizon length in this setting, we make extensive use of the suboptimality estimates derived in [7]. Methods to evaluate these estimates rely on a rather straightforward and easily proved “relaxed” version of the dynamic programming principle, see also [9, 11]. For proofs of the following estimate see [7, Proposition 3].

Proposition 2. Consider a feedback law $\mu_N : \mathbb{X} \rightarrow \mathbb{U}$ and its associated trajectory $x(\cdot)$ according to (8) with initial value $x(0) = x_0 \in \mathbb{X}$. If there exists a function $V_N : \mathbb{X} \rightarrow \mathbb{R}_0^+$ satisfying

$$V_N(x(n)) \geq V_N(x(n+1)) + \alpha l(x(n), \mu_N(x(n))) \quad (9)$$

for some $\alpha \in [0, 1]$ and all $n \in \mathbb{N}_0$ then

$$\alpha V_\infty(x(n)) \leq \alpha V_\infty^{\mu_N}(x(n)) \leq V_N(x(n)) \leq V_\infty(x(n)) \quad (10)$$

holds for all $n \in \mathbb{N}_0$.

Note that Proposition (2) is an *a posteriori* estimate since $V_N(x(n+1))$ is not available at time n . A more conservative *a priori* estimate is given in [7, Theorem 7] using the following assumptions:

Assumption 3. For given $N, \hat{N} \in \mathbb{N}, N \geq \hat{N} \geq 2$, there exists a constant $\gamma > 0$ such that for the open loop solution $x_{u_N}(k, x(n))$ given by (6) the inequalities

$$\begin{aligned} \frac{V_{\hat{N}}(x_{u_N}(N - \hat{N}, x(n)))}{\gamma + 1} &\leq \max_{j=2, \dots, \hat{N}} l(x_{u_N}(N - j, x(n)), \mu_{j-1}(x_{u_N}(N - j, x(n)))) \\ \frac{V_k(x_{u_N}(N - k, x(n)))}{\gamma + 1} &\leq l(x_{u_N}(N - k, x(n)), \mu_k(x_{u_N}(N - k, x(n)))) \end{aligned}$$

hold for all $k \in \{\hat{N} + 1, \dots, N\}$ and all $n \in \mathbb{N}_0$.

Theorem 4. Consider $\gamma > 0$ and $N, \hat{N} \in \mathbb{N}, N \geq \hat{N}$ such that $(\gamma + 1)^{N - \hat{N}} > \gamma^{N - \hat{N} + 2}$ holds. If Assumption 3 is fulfilled for these γ, N and \hat{N} , then (10) holds for all $n \in \mathbb{N}_0$ where

$$\alpha := \frac{(\gamma + 1)^{N - \hat{N}} - \gamma^{N - \hat{N} + 2}}{(\gamma + 1)^{N - \hat{N}}}. \quad (11)$$

Comparing the estimates from [7, 9, 18], we call the maximal value of α satisfying (9) *local suboptimality degree* if $x(n) \in \mathbb{X}$ is fixed. For a given closed loop trajectory $x(\cdot)$ we call $\alpha := \max\{\alpha \mid (9) \text{ holds } \forall n \in \mathbb{N}_0\}$ the *closed loop suboptimality degree* and for a given set \mathbb{X} we call $\alpha := \max\{\alpha \mid (9) \text{ with } x(n) = x \text{ holds } \forall x \in \mathbb{X}\}$ the *global suboptimality degree*.

Unfortunately, we cannot expect the relaxed Lyapunov inequality (9) or Assumption 3 to hold in practice for the following reason: In many cases the discrete time system (1) is obtained from a discretization of a continuous time system, e.g. sampling with zero order hold, see [13, 14]. Hence, even if, e.g., a continuous time system is stabilizable to a setpoint x^* and no numerical errors occur during optimization and integration, the corresponding sampled-data system is most likely practically stabilizable at x^* only.

For this reason, the *a posteriori* and *a priori* estimates from Proposition 2 and Theorem 4 have been extended to cover the case of practical stability as well, see also [7, Proposition 14 and Theorem 20]. Since these practical suboptimality estimates can be used in a similar manner as in the non-practical case, we show stability of the horizon adaptation technique and adaptation strategies for the non-practical case only. Corresponding results can be found in [16, Chapter 3 and 4].

3. Adapting the NMPC Scheme

Since the suboptimality estimates from Proposition 2 and Theorem 4 are computable online, we may utilize them to repeatedly adapt the optimization horizon. To this end, we extend our notation of the horizon length from N to N_n , of the corresponding local suboptimality degree from α to $\alpha(N_n)$ and of the closed loop control law from $\mu_N(\cdot)$ to $\mu_{(N_n)}(\cdot)$ where n indicates the discrete time instant. Defining a fixed suboptimality bound $\bar{\alpha} \in (0, 1)$, we propose the following algorithm to guarantee local suboptimality degree $\bar{\alpha}$:

Algorithm 5. Set $n := 0$ and choose $\bar{\alpha} \in (0, 1)$ and $N_n \in \mathbb{N}$.

1. Obtain new measurements $x(n)$.
2. Set $\tilde{\alpha} = 0$. While $\tilde{\alpha} \leq \bar{\alpha}$ do
 - (a) Compute the open loop optimal control sequence $u_N(\cdot, x(n))$ from (5)
 - (b) Compute suboptimality degree $\tilde{\alpha} := \alpha(N_n)$ from, e.g., Proposition 2 or Theorem 4
 - (c) If $\tilde{\alpha} \geq \bar{\alpha}$: Call shortening strategy for N_n
Else: Call prolongation strategy for N_n
3. Implement $\mu_{(N_n)}(x(n)) := u(0, x(n))$, set $n := n + 1$ and goto Step 1.

The problem which we are facing for such an adaptive MPC algorithm is the fact that none of the existing stability proofs, see, e.g., [6, 7, 8], can be applied in this context since these results assume N to be constant while here the optimization horizon N_n may change in every step of the MPC algorithm. The major obstacle to apply the idea of Proposition 2 in the context of varying optimization horizons N_n is the lack of a common Lyapunov function along the closed loop. To compensate for this deficiency, we assume that if for a horizon length N_n we have $\alpha(N_n) \geq \bar{\alpha}$, then the controller shows a bounded guaranteed performance if N_n is increased. For ease of notation, we give this assumption in a set valued manner, however, within the following stability proof it is only required to hold along the closed loop.

Assumption 6. Given an initial value $x \in \mathbb{X}$ and a horizon length $N < \infty$ such that $\mu_N(\cdot)$ guarantees local suboptimality degree $\alpha(N) \geq \bar{\alpha}$, $\bar{\alpha} \in (0, 1)$, we assume that for $\tilde{N} \geq N$, $\tilde{N} < \infty$, there exist constants $C_l, C_\alpha > 0$ such that the inequalities

$$l(x, \mu_N(x)) \leq C_l l(x, \mu_{\tilde{N}}(x)) \frac{V_{\tilde{N}}(x) - V_{\tilde{N}}(f(x, \mu_N(x)))}{V_{\tilde{N}}(x) - V_{\tilde{N}}(f(x, \mu_{\tilde{N}}(x)))} \quad (12)$$

$$\alpha(N) \leq \frac{1}{C_\alpha} \alpha(\tilde{N}) \quad (13)$$

hold where $\alpha(\widetilde{N})$ is the local suboptimality degree of the controller $\mu_{\widetilde{N}}(\cdot)$ corresponding to the horizon length \widetilde{N} .

Note that Assumption 6 is indeed not very restrictive since we allow for non-monotone developments of the suboptimality degree $\alpha(\cdot)$ if the horizon length is increased which may occur as shown in [3]. Moreover, we only make sure that if a certain suboptimality degree $\bar{\alpha} \in (0, 1)$ holds for a horizon length N , then the estimate $\alpha(\widetilde{N})$ does not drop below zero if the horizon length \widetilde{N} is increased.

Using Assumption 6 to hold along the closed loop, we obtain stability and a performance estimate of the closed loop for changing horizon lengths:

Theorem 7. *Consider $\bar{\alpha} \in (0, 1)$ and a sequence $(N_n)_{n \in \mathbb{N}_0}$, $N_i \in \mathbb{N}$, where $N^* = \max\{N_n \mid n \in \mathbb{N}_0\}$, such that the NMPC feedback law $\mu_{(N_n)}$ defining the closed loop solution (8) guarantees*

$$V_{N_n}(x(n)) \geq V_{N_n}(x(n+1)) + \bar{\alpha}l(x(n), \mu_{N_n}(x(n))) \quad (14)$$

for all $n \in \mathbb{N}_0$. Moreover suppose Assumption 6 to hold for all pairs $(x(n), N_n)$, $n \in \mathbb{N}_0$. Then we obtain

$$\alpha_C V_\infty(x(n)) \leq \alpha_C V_\infty^{\mu_{(N_n)}}(x(n)) \leq V_{N^*}(x(n)) \leq V_\infty(x(n)) \quad (15)$$

to hold for all $n \in \mathbb{N}_0$ with $\alpha_C := \min_{j \in \mathbb{N}_0, j \geq n} \frac{C_\alpha(j)}{C_l(j)} \bar{\alpha}$ and $C_\alpha(j), C_l(j)$ from (12), (13) for $x = x(j)$, $j \geq n \in \mathbb{N}_0$.

Proof. Given a pair $(x(n), N_n)$, Assumption 6 guarantees $\alpha(N_n) \leq \alpha(\widetilde{N})/C_\alpha^{(n)}$ for $\widetilde{N} \geq N_n$. Now we choose $\widetilde{N} = N^*$ within this local suboptimality estimate. Hence, we obtain $\bar{\alpha} \leq \alpha(N_n) \leq \alpha(N^*)/C_\alpha(n)$ using the relaxed Lyapunov inequality (14). Multiplying by $l(x(n), \mu_{N_n}(x(n)))$ and using (12), we can conclude $\bar{\alpha}l(x(n), \mu_{N_n}(x(n))) \leq \frac{C_l(n)}{C_\alpha(n)} (V_{N^*}(x(n)) - V_{N^*}(x(n+1)))$. Since the latter condition relates the closed loop varying optimization horizon to a fixed one, it allows us to use an identical telescope sum argument as in the proof of [7, Proposition 3]. Hence, summing the running costs along the closed loop trajectory reveals $\alpha_C \sum_{j=n}^K l(x(j), \mu_{N_j}(x(j))) \leq V_{N^*}(x(n)) - V_{N^*}(x(K+1))$ where we defined $\alpha_C := \min_{j \in \{n, \dots, K\}} \frac{C_\alpha(j)}{C_l(j)} \bar{\alpha}$ with constants $C_\alpha(j)$ and $C_l(j)$ from (12) and (13) for $x = x(j)$ and $j \in \{n, \dots, K\}$. Since $V_{N^*}(x(K+1)) \geq 0$ holds, taking K to infinity reveals

$$\alpha_C V_\infty^{\mu_{(N_n)}}(x(n)) = \alpha_C \lim_{K \rightarrow \infty} \sum_{j=n}^K l(x(j), \mu_{N_j}(x(j))) \leq V_{N^*}(x(n)).$$

Since the first and the last inequality of (15) hold by the principle of optimality, the assertion follows. \square

Similar to Proposition 2, Theorem 7 can be extended to the practical case, cf. [16, Chapter 4]. We like to point out that Theorem 7 is a generalization of Proposition 2 which is reobtained if $N_n = N$ for all $n \in \mathbb{N}_0$.

Note that the closed loop estimate α_C in (15) may be smaller than the local suboptimality bound $\bar{\alpha}$. In particular, since $l(x, \mu_{\bar{N}}(x))$ may tend to zero if \bar{N} is increased, we obtain that C_l in (12) is in general unbounded. The special case $l(x, \mu_{\bar{N}}(x)) = 0$, however, states that the equilibrium of our problem has been reached and can be neglected in this context, i.e. outside the equilibrium $\alpha_C > 0$ is always retained. Yet, α_C may become very small depending on C_α and C_l from Assumption 6. During our numerical experiments, however, no such case occurred, see also Section 5.

4. Adaptation Strategies

As we have seen in Theorem 7, the methods from Proposition 2 and Theorem 4 can be applied to compute a suboptimality estimate for a given pair $(x(n), N_n)$. Yet, these local estimates have to be reinterpreted along the closed loop. In particular, we require the existence of a finite horizon length N_n guaranteeing stability with suboptimality degree greater than $\bar{\alpha}$ in order to conclude finite termination of the algorithm proposed in Section 3.

Assumption 8. Given $\bar{\alpha} \in (0, 1)$, for all $x_0 \in \mathbb{X}$ there exists a finite horizon length $\bar{N} = N(x_0) \in \mathbb{N}$ such that the relaxed Lyapunov inequality (9) holds with $\alpha(N) \geq \bar{\alpha}$ for all horizon lengths $N \geq \bar{N}$.

Note that Assumption 8 is satisfied if $\bar{\alpha}$ is small enough and \bar{N} is large enough, see, e.g., [6].

4.1. Simple Adaptation Strategies

A basic adaptation technique for the horizon length can be obtained using Proposition 2. In particular, we can repeatedly shorten the horizon length and check Assumption (9) as the solution evolves:

Theorem 9. Consider an optimal control problem (5), (6) with initial value $x_0 = x(n)$, $N_n \in \mathbb{N}$ and $\bar{\alpha} \in (0, 1)$ to be fixed and denote the optimal control sequence by u^* . Suppose there exists an integer $\bar{k} \in \mathbb{N}_0$, $0 \leq \bar{k} < N_n$ such that

$$V_{N_n - k}(x_{u^*}(k, x_0)) \geq V_{N_n - k}(x_{u^*}(k + 1, x_0)) + \bar{\alpha}l(x_{u^*}(k, x_0), \mu_{N_n - k}(x_{u^*}(k, x_0))) \quad (16)$$

holds true for all $0 \leq k \leq \bar{k}$. Then, setting $N_{n+k} = N_n - k$ and $\mu_{N_{n+k}}(x(n+k)) = u^*(k)$ for $0 \leq k \leq \bar{k}$, inequality (14) holds for $k = n, \dots, n + \bar{k}$ with $\alpha = \bar{\alpha}$.

Proof. The assertions follows directly from the fact that for $\mu_{N_{n+k}}(x(n+k)) = u^*(k)$ the closed loop satisfies $x(n+k) = x_{u^*}(k, x(n))$. Hence, (14) follows from (16). \square

Note that the result of Theorem 9 can be extended to consider an m -step feedback as defined in [8] by supposing $\bar{k} \geq m - 1$. With the choice $N_{n+k} = N_n - k$, due to the principle of optimality we obtain that the optimal control problems within the next \bar{k} NMPC iterations are already solved since $\mu_{N_{n+k}}(x(n+k))$ can be obtained from the optimal control sequence $u^*(\cdot)$ computed at time n . This implies that the most efficient way for the reducing strategy is not to reduce N_n itself but rather to reduce the horizons N_{n+k} by k for the subsequent sampling instants $n + 1, \dots, n + \bar{k}$, i.e., we choose the initial guess of the horizon $N_{n+1} = N_n - 1$. Still, if the a posteriori estimate is used, the evaluation of (16) requires the solution of an additional optimal control problem in each step.

In order to use the *a priori* estimate given by Theorem 4 the following result can be used as a shortening strategy:

Theorem 10. Consider an optimal control problem (5), (6) with initial value $x_0 = x(n)$ and $N_n, \hat{N} \in \mathbb{N}$, $N_n \geq \hat{N} \geq 2$ and denote the optimal control sequence by u^* . Moreover, $\bar{\alpha} \in (0, 1)$ is supposed to be fixed inducing some $\bar{\gamma}(\cdot)$ via (11). If there exists an integer $\bar{k} \in \mathbb{N}_0$, $0 \leq \bar{k} < N_n - \hat{N} - 1$ such that for all $0 \leq k \leq \bar{k}$ there exist $\gamma_n > 0$, $\gamma_n < \bar{\gamma}(N_n - k)$ satisfying

$$V_{\hat{N}}(x_{u^*}(N_n - \hat{N}, x_0)) \leq (\gamma_n + 1) \max_{j=2, \dots, \hat{N}} l(x_{u^*}(N_n - j, x_0), \mu_{j-1}(x_{u^*}(N_n - j, x_0))) \quad (17)$$

$$V_{j_k}(x_{u^*}(N_n - j_k, x_0)) \leq (\gamma_n + 1) l(x_{u^*}(N_n - j_k, x_0), \mu_{j_k}(x_{u^*}(N_n - j_k, x_0))) \quad (18)$$

for all $j_k \in \{\hat{N} + 1, \dots, N_n - k\}$. Then, setting $N_{n+k} = N_n - k$ and $\mu_{N_{n+k}}(x(n+k)) = u^*(k)$ for $0 \leq k \leq \bar{k}$, inequality (14) holds for $k = n, \dots, n + \bar{k}$ with $\alpha = \bar{\alpha}$.

Proof. Since (17), (18) hold for $k = 0$ with $\gamma_n > 0$ and $\gamma_n < \bar{\gamma}(N_n)$, Theorem 4 guarantees that the local suboptimality degree is at least as large as $\bar{\alpha}$. If $\bar{k} > 0$ holds, we can make use of the fact that for $\mu_{N_{n+k}}(x(n+k)) = u^*(k)$ the closed loop satisfies $x(n+k) = x_{u^*}(k, x(n))$. By (17), (18), we obtain Assumption 6 to hold along the closed loop. Accordingly, the assertion follows from Theorem 4 which concludes the proof. \square

Similar to the a posteriori case, the shortening strategy given by Theorem 10 can be extended to consider an m -step feedback as defined in [8] by supposing $\bar{k} \geq m - 1$. Note that while the a priori estimate from Theorem 4 is slightly more conservative than the result from Proposition 2, it is also computationally less demanding if the value \hat{N} is small.

The shortening strategies induced by Theorems 9 and 10 can be extended to cover the case of practical stability. To this end, the inequalities (16), (17) and (18) have to be replaced by their practical equivalents given in [7, Proposition 14] and [7, Theorem 21].

In contrast to these efficient and simple shortening strategies it is quite difficult to obtain efficient methods for prolongating the optimization horizon N_n . In order to provide a simple prolongating strategy, we invert the approach of Theorem 9:

Theorem 11. *Consider an optimal control problem (5), (6) with initial value $x_0 = x(n)$ and $N_n \in \mathbb{N}$. Moreover, for fixed $\bar{\alpha} \in (0, 1)$ supposed Assumption 8 to hold. Then, any algorithm which iteratively increases the optimization horizon N_n terminates in finite time and computes a horizon length N_n such that (14) holds with local suboptimality degree $\bar{\alpha}$.*

Proof. Follows directly from Assumption 8. □

Note that the prolongation strategy described in Theorem 11 only requires Assumption 8 to hold. This allows us to use any of the suboptimality estimates stated in Section 2. Unfortunately, if (14) does not hold, it is in general difficult to assess by how much N_n should be increased such that (14) holds for the increased N_n . The most simple strategy of increasing N_n by one in each iteration shows satisfactory results in practice, however, when starting the iteration with N_n , in the worst case (14) has to be checked $\bar{N} - N_n + 1$ times at each sampling instant. In contrast to the shortening strategy, the principle of optimality cannot be used here to establish a relation between the optimal control problems for different N_n and, moreover, these problems may exhibit different solution structures which makes it a hard task to provide a suitable initial guess for the optimization algorithm.

4.2. Advanced Adaptation Strategies

Since the shortening strategies based on both the a posteriori and the a priori estimates can be implemented with negligible additional computational effort, we focus on the prolongation of the horizon. To reduce the additional effort, we

analyze the relationship of $\alpha(N_n)$ and $\gamma(N_n)$ given by Theorem 4. If we consider $\bar{\alpha} \in (0, 1)$, we obtain a lower bound for N_n from (11) by

$$N_n \geq \left\lceil \hat{N} + \frac{2 \ln(\gamma(N_n)) - \ln(1 - \bar{\alpha})}{\ln(\gamma(N_n) + 1) - \ln(\gamma(N_n))} \right\rceil =: \Phi(N_n) \quad (19)$$

for fixed $x(n)$, \hat{N} and $\bar{\alpha}$. Since we want to guarantee local suboptimality degree $\bar{\alpha}$ and N_n to be as small as possible, we seek a horizon length N_n satisfying $N_n = \Phi(N_n)$, i.e. a fixed point of the function $\Phi(\cdot)$.

Theorem 12. Consider $N_n, \hat{N} \in \mathbb{N}$, $N_n \geq \hat{N} \geq 2$, and $\bar{\alpha} \in (0, 1)$ to be fixed and $\gamma(N_n)$ to minimally satisfy Assumption 3. If for a given $n \in \mathbb{N}_0$ there exists a constant $\theta \in [0, 1)$ such that the function $\Phi(\cdot)$ defined in (19) satisfies

$$|\Phi(\Phi(N_n)) - \Phi(N_n)| \leq \theta |\Phi(N_n) - N_n| \quad \theta \in [0, 1) \quad \forall N_n \geq \hat{N}. \quad (20)$$

and $\Phi^k(N_n) \geq \hat{N}$ for all $k \in \mathbb{N}$, then there exists $N_n^* \in \mathbb{N}$ with $N_n^* = \Phi(N_n^*)$ and $\Phi^k(N_n) \rightarrow N_n^*$, $k \rightarrow \infty$. If we use $N_n = N_n^*$ in Algorithm 5 for all $n \in \mathbb{N}_0$ and Assumption 6 holds for $x = x(n)$ and all $n \in \mathbb{N}_0$, then the closed loop solution (8) is asymptotically stable and exhibits local suboptimality degree $\alpha(N_n^*) \geq \bar{\alpha}$.

Proof. Since $\gamma(N_n)$ satisfies all requirements of Theorem 4, we can obtain an estimate $\alpha(N_n)$ via (11). In order to guarantee a certain degree of suboptimality $\bar{\alpha}$ we have to show $\bar{\alpha} \leq \alpha(N_n) = \frac{(\gamma+1)^{N_n-\hat{N}} - \gamma^{N_n-\hat{N}+2}}{(\gamma+1)^{N_n-\hat{N}}}$. This can be solved for N_n giving $N_n \geq \Phi(N_n)$ with $\Phi(\cdot)$ from (19). Due to (20) we have

$$|\Phi^k(N_n) - \Phi^{k-1}(N_n)| \leq \theta^{k-1} |\Phi(N_n) - N_n|. \quad (21)$$

Since $\theta \in [0, 1)$ the right hand side of (21) tends to zero. Hence, there exists an index $\bar{k} \in \mathbb{N}$ such that $\theta^{\bar{k}-1} |\Phi(N_n) - N_n| < 1$. Defining the sequence of optimization horizons via $(N_n^{(i)})_{i \in \mathbb{N}_0} := (\Phi^i(N_n))_{i \in \mathbb{N}_0}$ we obtain $N_n^{(j)} = N_n^{(k)} \geq \hat{N}$ for all $j, k \geq \bar{k}$. Hence, $(N_n^{(i)})_{i \in \mathbb{N}_0}$ is converging and $N_n^* = \Phi(N_n^*)$ holds for $N_n^* = N_n^{(\bar{k})}$. Choosing $N_n = N_n^*$, the local suboptimality degree satisfies $\alpha(N_n) \geq \bar{\alpha}$ by construction of $\Phi(\cdot)$. Hence, a new initial value can be obtained by implementing the controller in a receding horizon fashion. Since we can apply this procedure along the resulting trajectory, i.e. for all $n \in \mathbb{N}_0$, asymptotic stability of the closed loop solution (8) follows by Assumption 6 and Theorem 7 and $\alpha(N_n^*) \geq \bar{\alpha}$ follows directly from Theorem 4. \square

Note that, in general, we cannot a priori check whether $\Phi(\cdot)$ satisfies (20). Moreover, an algorithm derived from Theorem 12 may cause overshoots. Numerical experience has shown that $\sigma = 5$ is a suitable choice to bound the change of the the horizon length, yet, this variable should be chosen with respect to the considered problem. Additionally, numerical simulations indicate that the “best” choice of σ depends on the occurring horizon lengths N_n , i.e. larger horizons allow for larger choices of σ .

Theorem 12 can also be utilized to shorten the horizon. However, the computation of N^* requires nonnegligible effort. Hence, this strategy should only be considered when $\alpha(N) < \bar{\alpha}$. Yet, $N_{i+1} := \Phi(N_i)$ may be a suitable choice for the optimization horizon in the subsequent optimal control problem.

Different from the fixed point idea, a map $\Psi(\cdot)$ can be designed which generates a sequence of horizons $(N_n^{(i)})$ via $N_n^{(i+1)} := \Psi(N_n^{(i)})$ such that the suboptimality estimate $\alpha(N_n^{(i)})$ is monotonely increasing:

Lemma 13. *Suppose $N_n, \hat{N} \in \mathbb{N}$, $N_n \geq \hat{N} \geq 2$, $\bar{\alpha} \in (0, 1)$ and $0 \leq \delta < 1 - \alpha(N_n)$ are given and Assumption 3 holds. Suppose there exists a constant $\vartheta > 0$ such that $\gamma(\tilde{N}) \leq \vartheta\gamma(N_n)$ holds for*

$$\tilde{N} \geq \left\lceil \hat{N} + \left(\frac{\ln \left(\left(\frac{\gamma(N_n)}{\gamma(N_n)+1} \right)^{N_n - \hat{N}} - \frac{\delta}{\gamma(N_n)^2} \right) - 2 \ln(\vartheta)}{\ln(\vartheta\gamma(N_n)) - \ln(\vartheta\gamma(N_n) + 1)} \right) \right\rceil =: \Psi(N_n), \quad (22)$$

then $\alpha(\tilde{N})$ as defined in (11) with $\gamma = \gamma(\tilde{N})$ from Assumption 3 satisfies

$$\alpha(\tilde{N}) \geq \alpha(N_n) + \delta \quad (23)$$

Proof. In order to show $\alpha(\tilde{N}) \geq \alpha(N) + \delta$ we use (11) in (23) which gives us

$$\gamma(\tilde{N})^2 \left(\frac{\gamma(\tilde{N})}{\gamma(\tilde{N}) + 1} \right)^{\tilde{N} - \hat{N}} \leq \gamma(N)^2 \left(\frac{\gamma(N)}{\gamma(N) + 1} \right)^{N - \hat{N}} - \delta.$$

Overestimating the left hand side using $\gamma(\tilde{N}) \leq \vartheta\gamma(N)$ this leaves us to show

$$\vartheta^2 \gamma(N)^2 \left(\frac{\vartheta\gamma(N)}{\vartheta\gamma(N) + 1} \right)^{\tilde{N} - \hat{N}} \leq \gamma(N)^2 \left(\frac{\gamma(N)}{\gamma(N) + 1} \right)^{N - \hat{N}} - \delta$$

to guarantee (23). Since $\vartheta > 0$, this inequality is equivalent to

$$(\tilde{N} - \hat{N}) \left[\ln \left(\frac{\vartheta\gamma(N)}{\vartheta\gamma(N) + 1} \right) \right] \leq \ln \left(\left(\frac{\gamma(N)}{\gamma(N) + 1} \right)^{N - \hat{N}} - \frac{\delta}{\gamma(N)^2} \right) - 2 \ln(\vartheta)$$

Using negative definiteness of $\ln \left(\frac{\vartheta\gamma(N)}{\vartheta\gamma(N) + 1} \right)$ and (22) the assertion follows. \square

Similar to $\Phi(\cdot)$ from Theorem 12 the map $\Psi(\cdot)$ may be used to shorten the horizon, a feature which can be avoided easily:

Lemma 14. *If $\vartheta \geq 1$ and $\delta \geq 0$ hold, then we have $\Psi(N) > N$ for $\Psi(\cdot)$ from (22).*

Proof. In the special case $\delta = 0$, (22) simplifies to

$$\Psi(N) := \left[\hat{N} + \frac{(N - \hat{N})(\ln(\gamma(N) + 1) - \ln(\gamma(N))) + 2 \ln(\vartheta)}{\ln(\vartheta\gamma(N) + 1) - \ln(\vartheta\gamma(N))} \right].$$

Since $\frac{\ln(x+1) - \ln(x)}{\ln(\vartheta(x+1)) - \ln(\vartheta(x))} > 1$ and $\vartheta \geq 1$ hold, we obtain $\Psi(N) > N$. Moreover, increasing δ results in an enlarged value $\Psi(N)$ showing the assertion. \square

Obtaining a suitable approximation of ϑ is the most crucial part of the monotone prolongation method. In order to avoid computing the open loop optimal control (5) for all initial values $x \in \mathbb{X}$ and all $N \geq \hat{N}$ and deriving the corresponding values $\gamma(\cdot)$ to obtain ϑ , one can iteratively update the value of ϑ by setting $\vartheta := \max\{\vartheta, \gamma(N_n^{(i+1)})/\gamma(N_n^{(i)})\}$. This method is not only computationally cheap and gives us a lower bound for ϑ , it also moderates a possible overshoot. Note that this approximation has to be restarted for each $n \in \mathbb{N}_0$.

Next, we use Lemma 13 to show that a prolongation strategy based on $\Psi(\cdot)$ in Step 2 of Algorithm 5 terminates in finite time:

Theorem 15. *Suppose Assumptions 3 and 8 hold and suppose $\vartheta \geq 1$. Then a finite number of iterations $N_n^{(i+1)} := \Psi(N_n^{(i)})$ with $\Psi(\cdot)$ according to Lemma 13 are required to obtain a horizon length N_n^* which guarantees local suboptimality degree $\alpha(N_n^*) \geq \bar{\alpha}$.*

Proof. Using the stopping criterion $\alpha(N_n) \geq \bar{\alpha}$ of Algorithm 5 to define $\delta = \bar{\alpha} - \alpha(N_n)$, we always have $\delta > 0$. Hence, by Lemma 14 we can conclude that the horizon length $N_n^{(i)}$ is increasing in every step of the iteration due to $\vartheta \geq 1$. Since Assumption 8 guarantees the existence of a finite horizon length $\bar{N} \in \mathbb{N}$, $\bar{N} < \infty$, such that $\alpha(N_n) \geq \bar{\alpha}$ holds for all $N_n \geq \bar{N}$, the iteration $N_n^{(i+1)} := \Psi(N_n^{(i)})$ terminates in finite time. Hence, choosing $N_n = N_n^*$ we have that $\alpha(N_n) \geq \bar{\alpha}$ is guaranteed by the stopping criterion. \square

Note that we do not assume $\gamma(\cdot)$ in (19) to be computed in a specific way but only to satisfy Assumption 3. Hence, using Theorems 12 and 15 a suitable horizon N_n can be obtained such that $\alpha(N_n) \geq \bar{\alpha}$ holds even if some $\tilde{\gamma}(\cdot) \geq \gamma(\cdot)$ are used. Therefore, also the a priori practical estimate [7, Theorem 21] or the global

estimate presented in [9, Theorem 4.5 and 5.8] can be applied. Additionally, (11) defines a bijective mapping relating $\alpha(N_n)$ and $\gamma(N_n)$ which allows us to use the a posteriori estimates of Proposition 2 and [7, Proposition 14]:

Lemma 16. *Given $N_n, \hat{N} \in \mathbb{N}$, $N_n, \mathbb{N}_0 \geq 2$, the mapping $\Gamma : [0, \infty) \rightarrow (-\infty, 1]$, $\Gamma(x) := 1 - \frac{x^{N_n - \hat{N} + 2}}{(x+1)^{N_n - \hat{N}}}$ is bijective.*

Proof. In order to show bijectivity, we use continuity of $\Gamma(\cdot)$ on $[0, \infty)$ and $\Gamma(0) = 1$ and $\lim_{x \rightarrow \infty} \Gamma(x) = -\infty$ to show surjectivity. In order to obtain injectivity, we show that $\Gamma(\cdot)$ is strictly monotone on $[0, \infty)$, i.e. for all $x \in [0, \infty)$ and $\varepsilon > 0$ we have $\Gamma(x) > \Gamma(x + \varepsilon)$. Using the definition of $\Gamma(\cdot)$ in this last inequality, we obtain $(1 + \varepsilon/(x + 1))^{N_n - \hat{N}} < (1 + \varepsilon/x)^{N_n - \hat{N} + 2}$ which holds true for $x > 0$ since $N_n - \hat{N} \geq 0$. Last, considering $x = 0$, we see that $\Gamma(\varepsilon) < 1$ for all $\varepsilon > 0$ and hence the assertion follows. \square

Now, Lemma 16 allows us to solve (11) for $\gamma(N_n)$. Since $\Gamma(\cdot)$ is twice continuously differentiable on $[0, \infty)$, this can be done effectively using Newton's method

$$\gamma^{(k+1)}(N_n) := \gamma^{(k)}(N_n) + \frac{1 - \alpha - \frac{\gamma^{(k)}(N_n)^{N_n - \hat{N} + 2}}{(\gamma^{(k)}(N_n) + 1)^{N_n - \hat{N}}}}{\left(\frac{\gamma^{(k)}(N_n)}{\gamma^{(k)}(N_n) + 1}\right)^{N_n - \hat{N} + 1} (N_n - \hat{N} + 2 + 2\gamma^{(k)}(N_n))} \quad (24)$$

where $\gamma^{(0)}(N_n) := 1$ can be used as the initial value due to strict monotonicity of $\Gamma(\cdot)$. However, since we do not expect $\gamma(N_n)$ to vary massively along the closed loop, the information from previous steps may be reused.

Using the a posteriori estimate, we obtain an additional degree of freedom since \hat{N} can be chosen arbitrarily within the set $\{2, \dots, N_n\}$. However, it is not clear which value is the best: For one, the smallest possible $\gamma(N_n) > 0$ is obtained by setting $\hat{N} := 2$. Formulas (19) and (22), however, logarithmically depend on $\gamma(N_n)$, i.e small values of $\gamma(N_n)$ might lead to overshoots.

5. Numerical Results

To illustrate the effectiveness of our adaptation strategies, we consider a digital redesign problem (cf. [15]) of an arm/rotor/platform (ARP) model stated in [4]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + x_6(t)x_3(t) \\ \dot{x}_2(t) &= -k_1 x_1(t)/M - b_1 x_2(t)/M + x_6(t)x_4(t) - m r b_1 x_6(t)/M^2 \\ \dot{x}_3(t) &= -x_6(t)x_1(t) + x_4(t) \end{aligned}$$

$$\begin{aligned}
\dot{x}_4(t) &= -x_6(t)x_2(t) - k_1x_3(t)/M - b_1x_4(t)/M + mrk_1/M^2 \\
\dot{x}_5(t) &= x_6(t) \\
\dot{x}_6(t) &= -a_1x_5(t) - a_2x_6(t) + a_1x_7(t) + a_3x_8(t) - p_1x_1(t) - p_2x_2(t) \\
\dot{x}_7(t) &= x_8(t) \\
\dot{x}_8(t) &= a_4x_5(t) + a_5x_6(t) - a_4x_7(t) - (a_5 + a_6)x_8(t) + u(t)/J
\end{aligned}$$

For the digital redesign a continuous time full-state feedback u_0 was developed via backstepping such that the output $\zeta(t) := x_5(t) - \frac{a_3}{a_1 - a_2 a_3} [x_6(t) - a_3 x_7(t)]$ is close to $x_5(t)$ and tracks a given reference signal $\zeta_{\text{ref}}(\cdot)$, see [4, Chapter 7.3.2] for details on the backstepping design and the specification of the model parameters. The resulting continuous time solution of $x_{5,\text{ref}}(\cdot)$ is then used as a reference to compute a sampled-data control with zero order hold, cf. [15]. To solve the sequence of optimal control problems we use a direct approach and employ an SQP method to solve the resulting optimization problem. Here, we consider the initial value $x_0 = (0, 0, 0, 0, 10, 0, 0, 0)^\top$, the cost functional $J_N(x_0, u) = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |x_5(t) - x_{5,\text{ref}}(t)| dt$ where the sampling instances are equidistantly fixed via $t_i = t_0 + iT$ with $T = 0.2$, set tolerance levels of both the minimizer and the differential equation solver to 10^{-6} and use reference function

$$\zeta_{\text{ref}}(t) = \begin{cases} 10, & t \in [0, 5) \cup [9, 10) \\ 0, & t \in [5, 9) \cup [10, 15) \end{cases},$$

cf. Figure 1(a). For this problem it is known from [7, Section 5] that the subop-

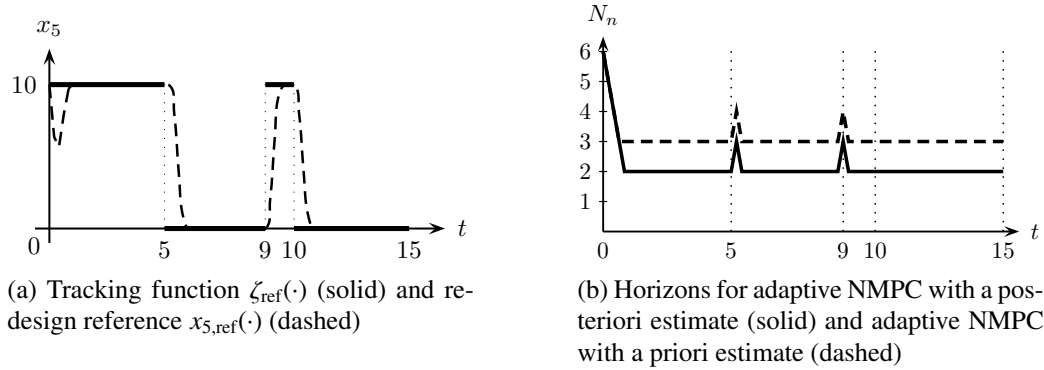


Figure 1: Tracking function and horizons of the adaptation scheme

tinality estimates of Proposition 2 and Theorem 4 show only poor performance.

For this reason we employ the practical variants [7, Proposition 14 and Theorem 21] with $\varepsilon = 10^{-5}$ and set the lower bound $\bar{\alpha} = 0.5$.

Using the a posteriori and a priori estimation techniques within Algorithm 5, we obtain the evolutions of horizons N_n along the closed loop for the suboptimality bound $\bar{\alpha} = 0.5$ as displayed in Figure 1(b). In particular, one observes that the less conservative a posteriori algorithms yields smaller optimization horizons which makes the resulting scheme computationally more efficient, see also Table 1. However, the evaluation of the a posteriori criterion itself is computationally more demanding, see also Figure 2, below.

Adaptive NMPC		Time in [$10^{-3}s$]			Horizon length		
Implementation	Estimate	max	min	\emptyset	max	min	\emptyset
Standard NMPC	—	86.50	3.23	23.14	6	6	6
Theorem 11	a posteriori	183.19	1.30	13.37	6	2	2.39
Theorem 12	a posteriori	184.79	1.49	11.55	6	2	2.25
Theorem 15	a posteriori	183.16	1.48	16.28	6	2	2.43
Theorem 11	a priori	226.10	2.12	17.46	6	3	3.21
Theorem 12	a priori	219.32	1.45	18.96	6	3	3.19
Theorem 15	a priori	219.44	1.47	15.60	6	3	3.13

Table 1: Comparison of NMPC results in the tracking type example

It is also interesting to compare these horizons to the standard NMPC Algorithm with fixed N which needs a horizon of $N = 6$ in order to guarantee $\alpha \geq \bar{\alpha}$ along the closed loop. Here, one observes that the required horizon N_n for the adaptive NMPC approach is typically smaller than $N = 6$ for both the a posteriori and the a priori estimate based variant. From Figure 1(b) one also observes that the horizon is increased at the jump points of the reference function (\cdot) , which is the behavior one would expect in a “critical” situation and nicely reflects the ability of the adaptive horizon algorithm to adapt to the new situation.

Although the algorithm chooses to modify the horizon length throughout the run of the closed loop, one can barely see a difference between the resulting $x_5(\cdot)$ trajectories and the (dashed) reference trajectory given in Figure 1(a). For this reason, we do not display the closed loop solutions. Instead, we additionally plotted the computing times of the two adaptive NMPC variants in Figure 2. Again, one can immediately see the spikes in the graph right at the points in which $\zeta_{\text{ref}}(\cdot)$ jumps. This figure also illustrates the disadvantage of the algorithm of having to solve multiple additional optimal control problems whenever N_n is increased,

which clearly shows up in the higher computation times at these points, in particular for the computationally more expensive a posteriori estimate.

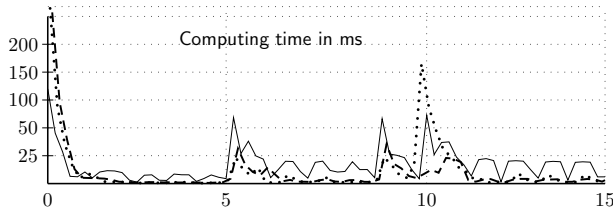


Figure 2: Computing times of the standard NMPC (solid), adaptive NMPC with a posteriori estimate (dotted) and adaptive NMPC with a priori estimate (dashed)

Last, we can use Theorem 7 to obtain the closed loop suboptimality degree α_C of the different implementations shown in Table 2. Note that we can restrict ourselves to those time instances where $l(\cdot, \cdot) > \varepsilon$ holds and that according to Theorem 7 α_C is given by combinations of $C_l(n)$ and $C_\alpha(n)$ while Table 2 shows minimal and maximal values of C_l and C_α .

Adaptive NMPC		C_l		C_α		α_C
Implementation	Estimate	min	max	min	max	
Theorem 11	a posteriori	0.9959	1.1282	1.0000	1.0506	0.4431
Theorem 12	a posteriori	1.1078	1.1078	1.0000	1.0000	0.4513
Theorem 15	a posteriori	0.9638	1.1760	1.0000	1.1088	0.4291
Theorem 11	a priori	0.8824	1.1460	1.0010	1.8321	0.4751
Theorem 12	a priori	0.9480	1.5334	0.9816	1.6457	0.3366
Theorem 15	a priori	0.8835	2.0362	0.9856	1.8360	0.2535

Table 2: Values of C_l , C_α and α_C of Theorem 7

From Table 2 we obtain that α_C may deteriorate if the a priori estimate is used while results based on the a posteriori estimate show that α_C is close to the minimal local suboptimality degree $\bar{\alpha} = 0.5$. In either case, the presented adaptation strategies guarantee stability of the closed loop and show a satisfactoring local and closed loop suboptimality degree.

6. Conclusion

We derived various adaptation strategies for the horizon length of an NMPC controller and showed stability and suboptimality of the resulting closed loop trajectory. Moreover, we have shown the practicability and effectiveness of these methods. Future work concerns many parts of this method. Probably the most important point is to improve the *a priori* estimates from Theorem 4 and [7, Theorem 20] by a more detailed analysis of the parameter \hat{N} and to develop other efficiently computable suboptimality estimates. Moreover, different feasibility conditions as well as development and investigation of alternatives to prolongate or shorten the optimization horizon will be an issue. In particular, combinations of iterates may allow for further insight of the process under control.

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