Discrete Approximation of Impulsive Differential Inclusions *

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Dedicated to Frank Lempio on the occasion of his 65th birthday.

Abstract

The paper deals with the approximation of the solution set and the reachable sets of an impulsive differential inclusion with variable times of impulses. It is strongly connected to [11] and is its continuation. We achieve order of convergence 1 for the Euler approximation under Lipschitz assumptions on the set-valued right-hand side and on the functions describing the jump surfaces and jumps themselves. Another criterion prevents the beating phenomena and generalizes available conditions. Several test examples illustrate the conditions and the practical evaluation of the jump conditions.

1 Preliminaries

We study discrete approximations of finite dimensional autonomous impulsive systems, having the form:

$$\dot{x}(t) \in F(x(t)),\ x(0) = x_0 \ \text{ a.e. } t \in I = [0,1],\ t \neq \tau_i(x(t)), \tag{1}$$

$$\Delta x|_{t=\tau_i(x(t))} = S_i(x(t)),\ i = 1, \dots, p. \tag{2}$$

Here $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction with nonempty compact and convex values, $S_i : \mathbb{R}^n \to \mathbb{R}^n$ yields the magnitude of the impulses (jumps) and $\tau_i : \mathbb{R}^n \to \mathbb{R}$ are the impulsive surfaces. If a solution hits the impulsive surface, it has to jump (time of impulse). Sometimes we will simply write τ_i or $\tau_i(x)$ (not $\tau_i(x(t))$). We assume that $t = 0$ is not a time of impulse, otherwise we simply replace the initial condition x_0 by $x_0 + S_i(x_0)$.

Recall (compare [23, p. 34]) that the piecewise absolutely continuous function $x(\cdot)$ is said to be a solution of (1)–(2), if:

a) $x(\cdot)$ is left continuous, i.e. $x(t) = x(t-0)$, and satisfies (1) for almost all $t \in I$, $t \neq \tau_i(x(t))$ and (2).

b) It has jumps on $t = \tau_i(x(t))$ (discontinuities of the first kind), defined by: $\Delta x\big|_{t=\tau_i(x(t))} = x(t+0) - x(t) = S_i(x(t))$.

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Given a solution $x(\cdot)$ of (1)–(2) we define a function, which zeros coincide with the jump times:

$$f_i^x(t) = \tau_i(x(t)) - t \tag{3}$$

The multiple hitting of one switching surface is called **beating phenomena**.

In the paper we study the approximation of the solution set and the reachable set of (1)–(2) with the help of the following approximation scheme:

Let $\Delta_k = \left\{ t_j = \frac{j}{k}\Big|_{j=0,1,\ldots,k} \right\}$ be a uniform grid of I (if we have to cover $I_\delta := [0, 1-\delta]$ with $\delta > 0$, we drop some grid points). To (1)–(2) we juxtapose the following discrete system below.

We let $x_0^k = x_0$ and for $j = 0, 1, \ldots, k-1$ we define $x_{j+1}^k = x(t_{j+1} + 0)$ inductively as follows:

$$x^k(t) = x_j^k + (t - t_j) f_j, \;\; f_j \in F(x_j^k), \tag{4}$$

$$x_{j+1}^k = \begin{cases} x^k(t_{j+1}) = \lim\limits_{t \uparrow t_{j+1}} x^k(t) & t \neq \tau_i(x^k(t)), \;\; \forall t \in (t_j, t_{j+1}], \\ x^k(t_{j+1}) + S_i(x^k(t_{j+1})) & \text{otherwise}. \end{cases} \tag{5}$$

The formula above means that the jump times are permitted to be only on the grid points, i.e. if some discrete solution $x^k(\cdot)$ must have a jump time $t_j < \tau_i(x(t)) = t < t_{j+1}$, then the jump is "postponed" to the time t_{j+1}. For simplicity we did not explicitly express the dependence of x and the superscript k for such solutions and jump times.

On every interval $(t_j, t_{j+1}]$ we check the signs of $f_i^x(\cdot)$. If $f_i^x(t_j) = 0$, then there will be no jump on $(t_j, t_{j+1}]$ (at least for step-sizes small enough). If $f_i^x(t_j) > 0$ ($f_i^x(t_j) < 0$) and $f_i^x(t_{j+1}) \leq 0$ ($f_i^x(t_{j+1}) \geq 0$), then $x(\cdot)$ has a jump in $(t_j, t_{j+1}]$ and we set $x_{j+1}^k = x^k(t_{j+1}) + S_i(x^k(t_{j+1}))$. Here, i is the minimal index for which the change of the inequality from "$>$" to "$\leq$" (or vice versa) occurs for $f_i^x(\cdot)$.

We prove in case of Lipschitzian $F(\cdot)$ under some natural assumption that the order of convergence is $\mathcal{O}(h)$ (with respect to a kind of distance $\rho(\cdot,\cdot)$ from Definition 1).

Our other aim is to test this scheme in some concrete examples, where we compare (whenever it is possible) the graph of the approximate and the graph of the exact systems.

We refer to [18, 24] for the theory of impulsive differential equations. Among others we note the interesting books [21] and [23] (the second one studies impulsive differential inclusions with averaging methods in the first chapter). Notice that the impulsive systems have connection with so-called hybrid systems (see [1] and [3, 15]) which are out of the scope of this paper. We do not study measure-driven systems, where intensive research is done and mention in this direction [5, 14].

There are a lot of papers devoted to approximation of solutions and of the reachable set of systems without impulses. We mention only [12, 16], survey papers [13, 19] and references therein. Numerical realizations of set-valued Euler's method and careful estimates of the space discretization of reachable sets are contained in [7, 6]. For an Euler-type approximation of impulsive systems with linear dependence on the measure (state-independent jump times), the graph convergence is shown for nonlinear systems with measure decomposition ([26]) as well as convergence order 1 for linear systems with polyhedral approximations in [17].

This paper studies a similar subject to [10, 11]. In the first paper, some qualitative properties play a major role, in the second one discrete approximations of impulsive differential inclusions are considered, however, there are some difficulties to approximate numerically such systems. The main goal here is the numerical justification of the approximation of those nonlinear systems.

The multifunction $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with nonempty compact values is said to be upper semi continuous (USC) at x_0, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $G(x_0 + \delta\mathbb{B}) \subset G(x_0) + \varepsilon\mathbb{B}$, where $\mathbb{B}$ is the closed unit ball. We call a map $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ almost USC, if for any $\varepsilon > 0$ there exists a compact $I_\varepsilon \subset I$ with $\text{meas}(I \setminus I_\varepsilon) < \varepsilon$ such that $G(\cdot, \cdot)$ is USC on $I_\varepsilon \times \mathbb{R}^n$. The multifunction $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be Lipschitzian with a constant L, if for every $x, y \in \mathbb{R}^n$ one has $D_H(F(x), F(y)) \leq L|x - y|$. Recall that $D_H(A, B) = \max\{\text{dist}(A, B), \text{dist}(B, A)\} = \max\left\{\sup_{a\in A} \inf_{b\in B} |a - b|, \sup_{b\in B} \inf_{a\in A} |b - a|\right\}$ is the Hausdorff distance between the bounded sets A and B. For compact sets A we let $\sigma(l, A) = \max_{a\in A}\langle l, a\rangle$ – the support function.

We refer to [2] or [9] for all the concepts used but not explicitly given here.

Standing hypotheses.

F. $F(\cdot)$ is Lipschitz (with a constant L) or it is locally Lipschitz, satisfying the linear growth condition $|F(x)| \leq \beta(1 + |x|)$ for some positive constant β.

A1. $\tau_i(x) \neq \tau_j(x + S_i(x))$ for every x and every $j \neq i$. Each $\tau_i(\cdot)$ is Lipschitz with a constant N and Clarke's subdifferential $\partial\tau_i(\cdot)$.

A2. The functions $S_i : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz with a constant μ.

A3. Just one of the following assumptions holds for every $x \in \mathbb{R}^n$:

a) $\tau_i(x) \geq \tau_i(x + S_i(x))$, $\tau_{i+1}(x) > \tau_i(x)$ for $i = 1, \dots, p - 1$, $\tau_1(x_0) > 0$ and $\exists\ \alpha < 1$ such that $\sigma(\nabla\tau_i(x), F(x)) \leq \alpha$ (whenever the gradient $\nabla\tau_i(x)$ exists).

b) $\tau_i(x) \leq \tau_i(x + S_i(x))$, $\tau_{i+1}(x) < \tau_i(x)$ for every $i = 1, \dots, p - 1$, $\tau_1(x_0) < 0$ and $\exists\ \kappa > 1$ such that $\sigma(-\nabla\tau_i(x), F(x)) \leq -\kappa$ (whenever $\nabla\tau_i(x)$ exists).

Assumptions **A3 a)** resp. **b)** will guarantee strong monotonicity of $f_i^x(\cdot)$ such that even in A3 b) with the condition $\tau_{i+1}(x) < \tau_i(x)$ we have that the jump times $\tau_i^\star < \tau_{i+1}^\star$ for a given function $x(\cdot)$ are ordered in the right way.

Lemma 1. *Under the standing hypotheses there exist $\alpha' < 1$ and $\kappa' > 1$ such that it follows for every $i = 1, \dots, p$:*
a) *$H_F(l, x) := \sigma(l, F(x)) \leq \alpha'\ \forall\ l \in \partial_x\tau_i(x)$ in case of **A3 a)**,*
b) *$h_F(l, x) := -\sigma(-l, F(x)) \geq \kappa'\ \forall\ l \in \partial_x\tau_i(x)$ in case of **A3 b)**.*

Proof. Let $G_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be the minimal USC map with convex compact values, which graph contains the graph of $x \to \nabla\tau_i(x)$. From [8, Section 1.2, (4)]) we know that $G_i(x) = \partial\tau_i(\cdot)$. Since the support function $\sigma(l, F(\cdot))$ is Lipschitz and $\sigma(\cdot, F(x))$ is convex, the proof is complete. □

Lemma 2. *Under condition **A3 a)**, the functions $f_i^x(t) = \tau_i(x(t)) - t$ in* (3) *are strongly montone decreasing with $f_i^x(\cdot) < f_{i+1}^x(\cdot)$ and $\tau_i^\star < \tau_{i+1}^\star$.*

Under condition ***A3 b)****, the functions* $f_i^x(\cdot)$ *in* (3) *are strongly montone increasing with* $f_i^x(\cdot) > f_{i+1}^x(\cdot)$ *and* $\tau_i^\star < \tau_{i+1}^\star$*.*

Proof. If $\tau^\star$ is a jump time of $x(\cdot)$, then $f_i^x(\tau^\star + 0) \leq f_i^x(\tau^\star)$ in case of **A3 a)**, since

$$\begin{aligned} f_i^x(\tau^\star + 0) &= \tau_i(x(\tau^\star + 0)) - (\tau^\star + 0) = \tau_i(x(\tau^\star) + S_i(x(\tau^\star))) - \tau^\star \\ &\leq \tau_i(x(\tau^\star)) - \tau^\star = f_i^x(\tau^\star) = 0, \end{aligned}$$

Let (s, t) be the interval of continuity of $x(\cdot)$. Applying the chain rule for Clarke's subdifferential yields

$$\begin{aligned} f_i^x(t) - f_i^x(s) &= s - t + \tau_i(x(t)) - \tau_i(x(s)) \\ &\leq s - t + \int_s^t \sigma(\dot{x}(r), \partial_x(\tau_i(x(t)))) \leq (t - s)(\alpha' - 1) < 0. \end{aligned}$$

The inequalities $f_i^x(\cdot) < f_{i+1}^x(\cdot)$ and $\tau_i^\star < \tau_{i+1}^\star$ follow immediately from $\tau_i(x) < \tau_{i+1}(x)$ and $\tau_1(x_0) > 0$.

Similarly, in case of **A3 b)** we have $f_i^x(\tau^\star + 0) \geq f_i^x(\tau^\star)$, $\tau_i^\star < \tau_{i+1}^\star$ and

$$f_i^x(t) - f_i^x(s) > 0.$$

Hence, the functions $f_i^x(\cdot)$ are strongly monotone except in the jump times for every i and hence $f_i^x(\tau) = 0$ has at most one solution, which is a jump time. □

Remark 1. *If* $\tau_i(\cdot)$ *is differentiable, then A3 a) is a nonsmooth and set-valued generalization of the corresponding pointwise condition* $\frac{\partial}{\partial x}\tau_i(x) \cdot f(x) < 1$ *as e.g. in [18, Theorem 1.3.5]. Condition A3 a) replaces the condition A3 in [10, 11], i.e.*

$$|F(x)| \leq C \text{ for each } x \in \mathbb{R}^n \text{ and } NC < 1. \tag{6}$$

Since $\tau_i(\cdot)$ *is Lipschitz,* $|\partial\tau_i(x)| \leq N$*. For every subgradient* l *we have* $\sigma(l, F(x)) \leq |l| \cdot |F(x)| \leq NC < 1$*, i.e. A3 a) holds. Condition A3 b) is a new condition and helps to deal with situations in which A3 a) is not fulfilled, cf. Example 1, case 2).*

Let us provide a simple illustrative example.

Example 1. *Consider the following one dimensional differential inclusion:*

$$\dot{x}(t) \in F(x),\ x(0) = \frac{1}{3},\ \tau(x) = x - 1. \tag{7}$$

Here, the impulsive surface is the (red) straight line $x = t + 1$ *in Figure 1 and the right-hand side is*

$$F(x) := \begin{cases} [1, 3x] & x \geq \frac{1}{3}\,, \\ [3x, 1] & x \leq \frac{1}{3}\,. \end{cases}$$

Case 1) *We let* $S(x) = -\dfrac{x}{2}$*.*

It is easy to see that the system (7) satisfies ***A1*** *and since the impulsive surface is just one, also* ***F*** *and* ***A2****. But neither* a) *nor* b) *of* ***A3*** *is true.*

There are "good" solutions as $x(t) = \dfrac{1}{3} + t$ with no jumps. However, if we study the solution $x^*(\cdot)$ with $\dot{x}^*(t) = 3x^*(t)$, we see after some standard calculations that this solution hits several times the impulsive surface (beating phenomena), see Figure 1.

Denote by $\tau_i^\star$ the i-th time of impulse for $x^*(\cdot)$, one can show that $\tau_{i+1}^\star - \tau_i^\star > \frac{1}{3} \cdot ln(2) \approx 0.23105$.

The distances of jump times for $x^*(\cdot)$ shrink from 0.2899 ($i = 1$) to 0.2467 ($i = 15$), if the zeros of the function are calculated numerically.

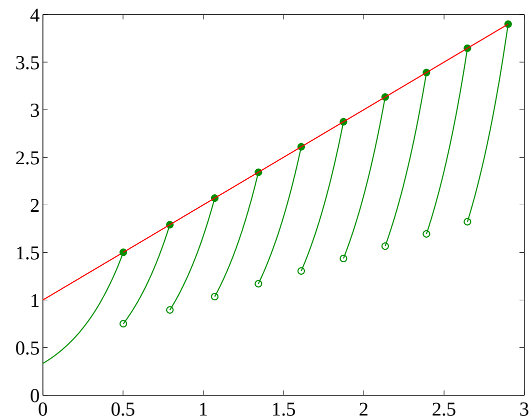

Figure 1: solution for Example 1, case 1) with multiple jumps

Case 2) *We let $S(x) = \dfrac{x}{2}$.*

Here, no multiple hitting of the jumping surface is possible, because if a solution intersects the impulsive surface $t = \tau(x(t))$, then $x(t+0) > t+1$. Since the derivative $\dot{x}(t) \geq 1$, $x(t) > t+1$ is valid for all times after the only jump time. In this case b) *of* ***A3*** *holds.*

In case 2) of Example 1 the functions $f_i^x(\cdot)$ are strongly increasing except in its jump times.

Given $\varepsilon > 0$ and motivated by USC case (see [11]) we study inner perturbations:

$$(8) \qquad \begin{cases} \dot{x}(t) \in F(x(t) + \varepsilon \mathbf{B}), \ x(0) = x_0 \ \text{ a.e. } t \in I = [0,1], \ t \neq \tau_i(x(t)), \\ \Delta x|_{t=\tau_i(x(t))} = S_i(x(t)), \ i = 1, \dots, p. \end{cases}$$

First we prevent the beating phenomena:

Lemma 3. *Under* **A1** – **A3** *for every solution of (1)–(2) (if it exists) and $i = 1, \dots, p$ there exists at most one $t \in I$ with $t = \tau_i(x(t))$. Furthermore, the solution set is nonempty and bounded.*

Proof. Let $x(\cdot)$ be a solution of (1)–(2). Due to Lemma 2, the functions $f_i^x(\cdot)$ are strongly monotone for every i except in the jump times and hence $f_i^x(\tau) = 0$ has at most one solution, which is a jump time. The solution exists on each interval $(s,t]$ of continuity due to A1) and the reachable set is bounded. If the solution has a jump at τ due to the i-th jump surface, then the new value $x(\tau + 0) =$

$x(\tau)+S_i(x(\tau))$ will form a new starting value for the same differential inclusion (8). Since A2) holds, we can estimate

$$|x(\tau+0)-x(\tau)| \leq |S_i(x(\tau))| \leq |S_i(x_0)| + \mu|x(\tau)-x_0|.$$

Since all values $x(\tau)$ of each solution before the jump are uniformly bounded, the values after the jump are also uniformly bounded which results in the precompactness of the reachable set due to the maximal number p of the possible jumps.

The existence is already proved in [10] in a more general form. $\square$

For Lemma 3, we may allow $\alpha \leq 1$ and $\kappa \geq 1$ in A3 a) resp. A3 b) as long as we have a strict inequality between $\tau_i(x)$ and $\tau_i(x+S_i(x))$.

Using the following standard arguments one can also show the same existence result for problem (8), i.e. for small $\varepsilon > 0$ there exist constants C, M, $\bar{\alpha}'$ resp. $\bar{\kappa}'$ such that $|y(t)| \leq M$ for solutions $y(\cdot)$ of (1)–(2), if $F(y)$ is replaced by $\overline{co}\ F(y+\varepsilon\mathbb{B})$, $|F((M+\varepsilon)\mathbb{B})| \leq C$ and $H_F(l, x+\varepsilon\mathbb{B}) \leq \bar{\alpha}' < 1\ \forall\ l \in \partial\tau_i(x)$ (resp. $h_F(l, x+\varepsilon\mathbb{B})$ $\geq \bar{\kappa}' > 1\ \forall\ l \in \partial\tau_i(x)$). In this case we will replace α' by $\bar{\alpha}'$ and κ' by $\bar{\kappa}'$.

Remark 2. *Calculating the solution set under the scheme (4)–(5) (or some other one), the problem arises to verify, if the approximate solution say $\eta(\cdot) = x^k(\cdot)$ has a jump time between two grid points $t_j^k < t_{j+1}^k$. The criterion is very easy: the jump occurs if and only if f_i^η changes its sign.*

Corollary 1. *Assume that the conditions **A1, A2, A3** hold, then there exists a constant $\nu > 0$ such that for every solution $y(\cdot)$ of (8) all of the absolute values*

$$\begin{aligned} &|\tau_{i+1}(y(\cdot)) - \tau_i(y(\cdot))|, \quad |\tau_{i+1}(y(\cdot+0)) - \tau_i(y(\cdot))|, \\ &|\tau_{i+1}(y(\cdot)) - \tau_i(y(\cdot+0))|, \quad |\tau_{i+1}(y(\cdot+0)) - \tau_i(y(\cdot+0))| \end{aligned}$$

are greater or equal to ν for $i = 1, 2, \ldots, p-1$.

The proof for A3 a) and the inequality for the first term is given in [11], however, we include here the full proof for the reader's convenience.

Proof. Since $\tau_i(\cdot)$ are Lipschitz for $i = 1, \ldots, p$, one has that on every compact set $K \subset \mathbb{R}^n$ the sets $\mathcal{A}_i := \left\{ \bigcup_{x\in K} \{(x, \tau_i(x))\right\}$ and $\mathcal{A}_{j,i} := \left\{ \bigcup_{x\in K} (x, \tau_j(x+S_i(x)))\}\right\}$ are compacts. Due to **A1, A3** $\mathcal{A}_i \bigcap \mathcal{A}_j = \emptyset$ and $\mathcal{A}_i \bigcap \mathcal{A}_{j,i} = \emptyset$ for any $i \neq j$. Consequently the first estimation follows from $\min_{i\neq j} D_H(\mathcal{A}_i, \mathcal{A}_j) = \nu > 0$, similar arguments help for the estimation of the other terms. $\square$

The above corollary shows that the jump surfaces have a minimum distance of $\nu > 0$, if both surfaces are evaluated at the values of a solution $y(\cdot)$. The next corollary shows that the jump times for a solution also have a minimal distance to each other.

Corollary 2. *Under the conditions of Lemma 3 there exists a constant $\lambda > 0$ such that for every solution $y(\cdot)$ of (8) and every jump time $\tau_i^\star$, $i = 1, \ldots, p$ from $y(\cdot)$ the estimate $\tau_{i+1}^\star - \tau_i^\star \geq \frac{\lambda}{1-\kappa}$ holds in the case of A3 a). Hereby, $\kappa \leq \min_{l\in(\partial\tau_i)(x(\tau))} H_F(l, x(\tau))$, $i = 1, \ldots, p$. If A3 b) holds, the estimate $\tau_{i+1}^\star - \tau_i^\star \geq \frac{\lambda}{\alpha-1}$ holds with $\alpha \geq \max_{l\in(\partial\tau_i)(x(\tau))} H_F(l, x(\tau))$, $i = 1, \ldots, p$.*

Proof. By Corollary 1 there exists a lower bound $\lambda > 0$ of $\tau_{i+1}(y(\tau^\star_{i+1}))-\tau_i(y(\tau^\star_{i+1}))$ for the case A3 a). We have $\kappa \le \alpha' < 1$, $\tau_j(y(\tau^\star_j)) = \tau^\star_j$, $j = 1, \dots, p$ and

$$\begin{aligned} &\tau_i(y(\tau^\star_{i+1})) - \tau_i(y(\tau^\star_i)) \ge \kappa \cdot (\tau^\star_{i+1} - \tau^\star_i), \\ &\quad \tau^\star_{i+1} = \tau_{i+1}(y(\tau^\star_{i+1})) \ge \lambda + \tau_i(y(\tau^\star_{i+1})) \ge \lambda + \tau_i(y(\tau^\star_i)) + \kappa \cdot (\tau^\star_{i+1} - \tau^\star_i), \\ &\quad (1-\kappa) \cdot (\tau^\star_{i+1} - \tau^\star_i) \ge \lambda. \end{aligned}$$

By Corollary 1 there exists $\lambda > 0$ with the bound on $\tau_i(y(\tau^\star_{i+1}))-\tau_{i+1}(y(\tau^\star_{i+1})) \ge \lambda$ for the case A3 b). We have $\alpha \ge \kappa' > 1$ and

$$\begin{aligned} \tau_i(y(\tau^\star_{i+1})) - \tau_i(y(\tau^\star_i)) &\le \alpha \cdot (\tau^\star_{i+1} - \tau^\star_i), \\ \tau^\star_{i+1} + \lambda = \tau_{i+1}(y(\tau^\star_{i+1})) + \lambda &\le \tau_i(y(\tau^\star_{i+1})) \le \tau_i(y(\tau^\star_i)) + \alpha \cdot (\tau^\star_{i+1} - \tau^\star_i) \\ = \tau^\star_i + \alpha \cdot (\tau^\star_{i+1} - \tau^\star_i), &\qquad \text{i.e. } \lambda \le (\alpha - 1) \cdot (\tau^\star_{i+1} - \tau^\star_i). \end{aligned}$$

□

One can take the number k of subintervals so big that the step-size is less than λ. Due to Corollary 2 it is impossible for an approximate solution (say $x^k(\cdot)$) to have more that one jump in $[t_j, t_{j+1}]$.

2 Approximation of the solution set

The numerical approximation of the Lipschitzian case has been studied among others in [12, 13, 16, 19]. Euler's method in the presence of state constraints is realized in [7].

Definition 1. *Let $x(\cdot)$ and $y(\cdot)$ be solutions of (8) we say that $\rho(x(\cdot), y(\cdot)) \le \varepsilon$ if they intersect successively the impulsive surfaces, i.e. $\tau_i(x) \ge \tau_j(y)$ is impossible for $i < j$ and vice versa. Moreover $\sum_{i=1}^{p} \left(\tau_i^+ - \tau_i^-\right) \le \varepsilon$ and $|x(t) - y(t)| \le \varepsilon$ for every $t \in I \setminus \left(\bigcup_{i=1}^{p} [\tau_i^-, \tau_i^+]\right)$. Here $\tau_i^- = \min\{\tau_i(x), \tau_i(y)\}$ and τ_i^+ is the corresponding maximum.*

To study the approximation of the solution set of (1)–(2) in $C(I, \mathbb{R}^n)$ with discrete trajectories (the solution set of (4)–(5)), we will use the Filippov-Plis lemma. To prove this lemma, we first mention the following known lemma (Lemma 2 in [22]), a discrete version of Gronwall inequality.

Lemma 4. *Let $a_1, a_2, b, d \ge 0$ and let $\delta_0^+ = \delta_0^- = \delta_0$. If for $i = 1, 2, \dots, p$*

$$\delta_i^+ \le a_1 \delta_i^- + d, \quad \delta_i^- \le a_2 \delta_{i-1}^+ + b$$

then $\delta_i^- \le (a_2 d + b) \sum_{j=0}^{i-1} (a_1 a_2)^j + \delta_0 (a_1 a_2)^i$, where $\delta_0 \ge 0$.

We need a variant of Filippov–Plis lemma, which is proved in [11] for F being OSL and USC under slightly stronger assumptions on the jumps, but not for the assumption A3 b). Here, we study the Lipschitzian case.

Theorem 1. *(Lemma of Filippov–Plis) Under the* ***standing hypotheses*** *there exists a constant K such that if $\varepsilon > 0$ is sufficiently small, then for any solution $y(\cdot)$ of (8) there exists a solution $x(\cdot)$ of (1)–(2) with $\rho(x(\cdot), y(\cdot)) \le K\varepsilon$ and vice versa.*

Proof. Let $y(\cdot)$ be a solution of (8). It is easy to see that $\dot{y}(t) \in F(y(t) + L\varepsilon\mathbb{B})$ (however, $|\dot{y}(t)| \le C$). We are looking for a solution $x(\cdot)$ of (1)–(2), satisfying the condition of the theorem.

On the interval $[0, 1]$ define

$$\Gamma_\varepsilon(t, z) := \left\{u \in F(z) : \langle y(t) - z, \dot{y}(t) - u\rangle \le L\left(|y(t) - z|^2 + \varepsilon|y(t) - x(t)|\right)\right\}.$$

It is easy to see that $\Gamma(\cdot, \cdot)$ is almost USC with nonempty convex and compact values. Let $x(\cdot)$ be a piecewise defined solution of

$$\dot{x}(t) \in \Gamma_\varepsilon(t, x(t)) \tag{9}$$

with starting value $x(0) = y(0)$, later with starting value $x(\tau_i^+ + 0) = x(\tau_i^+) + S_i(x(\tau_i^+))$ resp. $x(\tau_i^- + 0) = x(\tau_i^-) + S_i(x(\tau_i^-))$.
It follows that

$$\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 \le L\left(|x(t) - y(t)|^2 + \varepsilon|x(t) - y(t)|\right).$$

Since $|x(\cdot) - y(\cdot)|$ is an AC function, it is a.e. differentiable and the integrable derivative could be estimated by $\frac{d}{dt}|x(t) - y(t)| \le L\left(|x(t) - y(t)| + \varepsilon\right)$. If $x(\cdot)$ and $y(\cdot)$ have no impulses on (r, s), then

$$|x(s) - y(s)| \le e^{L(s-r)}\left(|x(r+0) - y(r+0)| + L\varepsilon\int_r^s e^{-L(\tau - r)}d\tau\right). \tag{10}$$

By A1) the following holds:

$$|f_i^x(t) - f_i^y(t)| = |\tau_i(x(t)) - \tau_i(y(t))| \le N \cdot |x(t) - y(t)|.$$

Since $x(\cdot)$ and $y(\cdot)$ are continuous in 0, $x(0) = y(0) = 0$ and we iteratively show that $|x(t) - y(t)| = \mathcal{O}(\varepsilon)$, we can assume that $x(\cdot)$ and $y(\cdot)$ intersect the impulse surfaces successively, i.e. $x(\cdot)$ does not intersect the $(i+1)$-th surface before $y(\cdot)$ to intersect the i-th one and vice versa. The minimal length of the interval $[\tau_i(x(t)), \tau_{i+1}(x(t))]$ is $\lambda > 0$ by Corollary 1. Hence, it is possible to choose $\varepsilon > 0$ so small that $\tau_i(y(t)) \in (\tau_i(x(t)), \tau_{i+1}(x(t)))$ and vice versa. Assume that $x(\cdot)$ intersects the first impulsive surface before $y(\cdot)$.

Denote $a_2 = e^L$. We let $b := (e^L - 1)\varepsilon$ and $\delta_i^- = |y(\tau_i^-) - x(\tau_i^-)|$ and $\delta_i^+ = |y(\tau_i^+ + 0) - x(\tau_i^+ + 0)|$.

Modifying the proof of Theorem 2 of [22] and applying the Gronwall inequality for $|y(t) - x(t)|$ of (10) we will derive the following estimates:

$$\delta_i^+ \le a_1\delta_i^-, \quad \text{where } a_1 \text{ will be defined later in (16),} \tag{11}$$

$$\delta_{i+1}^- \le a_2\delta_i^+ + b, \tag{12}$$

$$|\tau_i^+ - \tau_i^-| \le \frac{N\delta_i^-}{1 - \alpha'} \quad \text{by } \mathbf{A3\ a)}, \tag{13}$$

$$|\tau_i^+ - \tau_i^-| \le \frac{N\delta_i^-}{\kappa' - 1} \quad \text{by } \mathbf{A3\ b)}. \tag{14}$$

Inequality (12) holds, since there is no jump in (τ_i^+, τ_{i+1}^-) and

$$\delta_{i+1}^- = |y(\tau_{i+1}^-) - x(\tau_{i+1}^-)| \le a_2 \delta_i^+ + b$$

follows due to (10).

Since $\tau_i^- = \tau_i(x)$, $\tau_i^+ = \tau_i(y)$ holds and we can use the C-Lipschitz continuity of $y(\cdot)$ on $[\tau_i^-, \tau_i^+]$ as well as Lemma 1 a) and A3 a) to show that

$$\tau_i^+ - \tau_i^- \le |\tau_i(x(\tau_i^-)) - \tau_i(y(\tau_i^-))| + \tau_i(y(\tau_i^+)) - \tau_i(y(\tau_i^-)) \le N\delta_i^- + \alpha'(\tau_i^+ - \tau_i^-).$$

Therefore $\tau_i^+ - \tau_i^- \le \dfrac{N\delta_i^-}{1-\alpha'}$, which proves (13) in case of **A3 a)**.

In case of **A3 b)** one uses Lemma 1 b) to estimate

$$\tau_i^+ - \tau_i^- \ge \tau_i(y(\tau_i^+)) - \tau_i(y(\tau_i^-)) - |\tau_i(x(\tau_i^-)) - \tau_i(y(\tau_i^-))| \ge \kappa'(\tau_i^+ - \tau_i^-) - N\delta_i^-,$$

i.e. $\tau_i^+ - \tau_i^- \le \dfrac{N\delta_i^-}{\kappa' - 1}$. Furthermore,

$$|x(\tau_i^-) - y(\tau_i^+)| \le |x(\tau_i^-) - y(\tau_i^-)| + |y(\tau_i^-) - y(\tau_i^+)| \le \delta_i^- + C|\tau_i^+ - \tau_i^-|. \tag{15}$$

Hence, we have in case of **A3 a)** by (15) and **A2**

$$\begin{aligned}
\delta_i^+ &= |x(\tau_i^+) - y(\tau_i^+ + 0)| \le |x(\tau_i^-) - y(\tau_i^-)| + \int_{\tau_i^-}^{\tau_i^+} |\dot{x}(t) - \dot{y}(t)|\, dt \\
&\quad + |S_i(x(\tau_i^-)) - S_i(y(\tau_i^+))| \le \delta_i^- + 2C(\tau_i^+ - \tau_i^-) + \mu|x(\tau_i^-) - y(\tau_i^+)| \\
&\le \delta_i^- + 2C(\tau_i^+ - \tau_i^-) + \mu[\delta_i^- + C(\tau_i^+ - \tau_i^-)] \\
&\le (1+\mu)\delta_i^- + \frac{N(2+\mu)C}{1-\alpha'}\delta_i^- = a_1\delta_i^-
\end{aligned}$$

which is just (11). In this case

$$a_1 = \frac{(1+\mu)(1-\alpha') + NC(2+\mu)}{1-\alpha'}. \tag{16}$$

In case of **A3 b)** one has to replace $1-\alpha'$ by $\kappa' - 1$.

If $\tau_i(x(t)) > \tau_i(y(t))$, the proof is very similar.

Now Lemma 4 applies. Due to **A1, A2** there exists a constant $\lambda > 0$ such that $\tau_{i+1}^- - \tau_i^- \ge \lambda$, thanks to Corollary 2. It follows from (13) and (14) that $\tau_i^+ - \tau_i^- < K\varepsilon$. Therefore there exists a (generic) constant $\widetilde{K} > 0$ such that:

$$\rho(x(\cdot), y(\cdot)) \le \widetilde{K}\varepsilon.$$

□

Now we study the discrete approximations. Notice that it is difficult to estimate the distance between the solution set of (1)–(2) and the piecewise linear approximations (4)–(5). So we successively approximate the solution set of (1)–(2) with other solution sets for which such an estimate is simpler.

First we prove a lemma for the approximation of solutions of (1)–(2) with the solutions of

$$\dot{x}^k(t) = f_j(t) \in F(x_j^k),\ x_0^k = x_0,\ x^k(t) = x_j^k + \int_{t_j}^t f_j(s)ds, \tag{17}$$

where $f_j(\cdot)$ is an integrable selection of the constant set-valued map $F(x_j^k)$.
If $\exists\, t' \in (t_j, t_{j+1})$ with $t' = \tau_i(x^k(t'))$, then we redefine the solution $x^k(\cdot)$ as $x^k(t) = x^k(t') + S_i(x^k(t')) + \int_{t'}^t \hat{f}_j(s)ds$ on $(t', t_{j+1}]$, where $\hat{f}_j(t) \in F(x^k(t') + S_i(x^k(t')))$.

Recall that we use the same partition Δ_k as in (4)–(5).

Lemma 5. *Let $\varepsilon > 0$ be fixed and k big enough. For every solution $y(\cdot)$ of (1)–(2) there exists a solution $x^k(\cdot)$ of (17) such that $\rho(x^k(\cdot), y(\cdot)) \le \varepsilon$ and vice versa. One can take $\varepsilon = O\left(\frac{1}{k}\right)$.*

Proof. **(Vice versa)** Let $x^k(\cdot)$ be a solution of (17). Then,

$$\begin{aligned}(x^k)'(t) &= f_j(t) \in F(x_j^k),\\ x^k(t) &\in x_j^k + \|x^k(t) - x_j^k\| \cdot B_1(0),\\ \|x^k(t) - x_j^k\| &\le \|\int_{t_j}^t f_j(s)ds\| \le C(t - t_j) \le Ch\end{aligned}$$

on a subinterval $[t_j, t_{j+1}]$. Hence, $x^k(\cdot)$ is a solution of (8) with $\varepsilon = C \cdot h$.

Now, the result holds due to Theorem 1.

(other assertion) Let $y(\cdot)$ be a solution of (1)–(2). We define $x^k(\cdot)$ successively on the intervals $[t_i^k, t_{i+1}^k]$. Let $x_i^k = x^k(t_i)$. Define the multifunction:

$$G_i(t) := \{u \in F(x_i^k) : |\dot{y}(t) - u| \le L|y(t) - x_i^k|\}. \tag{18}$$

It is easy to see that $G_i(\cdot)$ is measurable. Let $g_i(t) \in G_i(t)$ be a measurable selection. Define $x^k(t) = x_i^k + \int_{t_i}^t g_i(s)ds$. If there are no impulses in $[t_i^k, t_{i+1}^k]$, we define $x^k(\cdot)$ on the whole subinterval.

If there exists a jump $\tau_l(z(t^*)) = t^* \in (t_i^k, t_{i+1}^k)$, then we redefine $x^k(t)$ for $t = t^* + 0$ by setting $x^k(t^* + 0) = x^k(t^*) + S_l(x^k(t^*))$. Now we define $\widetilde{G}_i(t) := \{u \in F(x^k(t^* + 0)) : |\dot{y}(t) - u| \le L|y(t) - x^k(t^* + 0)|\}$ and extend $x^k(\cdot)$ on $(t^*, t_{i+1}^k]$ as $x^k(t) = x^k(t^* + 0) + \int_{t^*}^t \tilde{g}_i(s)ds$, where $\tilde{g}_i(s) \in \widetilde{G}_i(s)$ is a measurable selection. As in the proof of Theorem 1 we may assume only one jump in the subinterval and that $x^k(\cdot)$ and $y(\cdot)$ intersect the impulse surfaces successively. In this way, one can define $x^k(\cdot)$ on the whole interval I.

For $t \in [\tau_i^+, \tau_{i+1}^-]$

$$\operatorname{dist}(\dot{x}^k(t), F(x^k(t)) \le D_H(F(x_i^k), F(x^k(t))) \le L|x^k(t) - x_i^k| \le LC(t - t_i)$$

follows as well as

$$\langle y(t) - x^k(t), \dot{y}(t) - \dot{x}^k(t)\rangle \le L\left(|y(t) - x^k(t)|^2 + LC(t - t_i)|y(t) - x^k(t)|\right). \tag{19}$$

Thus, $\frac{d}{dt}|y(t) - x^k(t)| \leq L|y(t) - x^k(t)| + LC(t - t_i)$.

Denote $a_2 = \max_{t \in I} 2CLhe^{Lt} \int_0^t e^{-\tau} d\tau$ and δ_i^-, δ_i^- as in Theorem 1. If $b = \max\left\{a_2 \frac{1}{k}, \frac{2LC}{k}\right\}$, then $\delta_{i+1}^- = |y(t_{i+1}^-) - x(t_{i+1}^-)| \leq a_2\delta_i^+ + b$. Hence, (12) holds true.

Let **A3 a)** be valid. As in the proof of Theorem 1 we have

$$\tau_i^+ - \tau_i^- \leq N\delta_i^- + \alpha'(\tau_i^+ - \tau_i^-).$$

and hence, (13) is also true. In the case of **A3 b)**, one can also proceed in a similar way to show that (14) of Theorem 1 is fulfilled.

Due to Lemma 4 for sufficiently large k there exists a constant $M > 0$ (not depending on k) such that for every solution $y(\cdot)$ of (1)–(2) there exists an approximate solution $x^k(\cdot)$ (obtained by (18)) with $\rho(x^k(\cdot), y(\cdot)) \leq \frac{M}{k}$.

In case of **A3 b)** one obtains the same estimate with $\kappa' - 1$ replacing $1 - \alpha'$. $\square$

Consider now the new discrete system having the similar form as (17):

For $j = 0, 1, \ldots, k-1$ we let $x^k(t_{j+1}) = \lim_{t \uparrow t_{j+1}} x^k(t)$ and set for $t \in [t_j, t_{j+1})$:

$$x_0^k = x_0, \; x(t) = x_j^k + \int_{t_j}^t f_j(s)ds, \; f_j(\cdot) \text{ integrable with } f_j(s) \in F(x_j^k). \tag{20}$$

If $\exists\, t \in (t_j, t_{j+1})$ with $t = \tau_i(x(t))$, then we postpone the jump and proceed as in the scheme (4)–(5).

As a next step, we want to estimate the distance between the solution sets of (1)–(2) and (20). Similarly as in the proof of Lemma 5, one can prove as a corollary of Theorem 1:

Theorem 2. *Under the conditions of Theorem 1 we have (for k big enough):*

For every solution $x(\cdot)$ of (1)–(2) there exists a solution $x^k(\cdot)$ of (20) such that $\rho(x(\cdot), x^k(\cdot)) \leq O\left(\frac{1}{k}\right)$ and vice versa.

Proof. We only have to estimate the distance between the solution sets of (20) and (17) and afterwards to use Lemma 5. Let $\eta(\cdot)$ be a solution of (17). We look for a solution $x^k(\cdot)$ of (20) which is sufficiently close to $\eta(\cdot)$. Let $\tau_1^\eta \in (t_j, t_{j+1})$ its (possible) first jump time and set $x^k(t) = \eta(t)$ on $[0, \tau_1^\eta]$. The first jump of $x^k(\cdot)$ will be at t_{j+1}. Using the linear growth condition **F**, the Lipschitz continuity of S_1 and Lemma 3 we derive $|\eta(t_{j+1}) - x^k(t_{j+1} + 0)| \leq \mu\beta(1 + C)h$ with the bound C on the reachable set (recall that $h = \frac{1}{k}$), if we extend $x^k(\cdot)$ by $\dot{x}^k(t) = \dot{\eta}(t)$ on $[\tau_1^\eta, t_{j+1}]$. Now we consider the problems (20) and (17) on $[t_{j+1}, 1]$. Since $F(\cdot)$ is L–Lipschitz, one can choose $\dot{x}^k(\cdot)$ such that $\langle \eta(t) - x^k(t), \dot{\eta}(t) - \dot{x}^k(t)\rangle \leq L|\eta(t) - x^k(t)|^2 + 2\mu\beta(1 + C)h|\eta(t) - x^k(t)|$ up to the second jump time τ_2^η.

On every impulsive point τ_i^η we will add $\mu\beta(1 + C)$ to the estimation of $|\eta(t) - x^k(t)|$. Since the solutions $\eta(\cdot)$ and $x^k(\cdot)$ differ by $\mathcal{O}(h)$, this estimation also holds for the jump times τ_i^η and $\tau_i^{x^k}$ by **A1**. The rest of the proof is standard and will be omitted. $\square$

The following theorem is a corollary of Theorem 2:

Theorem 3. *If all the conditions of Theorem 1 hold, then we have for sufficiently large k:*

For every solution $x(\cdot)$ of (1)–(2) there exists a solution $x^k(\cdot)$ of (4)–(5) such that $\rho(x(\cdot\,), x^k(\cdot\,)) \leq O\left(\frac{1}{k}\right)$ and vice versa.

Proof. For the system without impulses the proof is trivial. It is known that $(t-s)F(x(t_i)) = \int_s^t F(x(t_i))d\eta$ by the convexity of Aumann's integral. Since the strategy of postponing jumps in (20) and (5) coincides too, the result is just a reformulation of Theorem 2. □

Due to Definition 1 of $\rho(\cdot,\cdot)$, the jump times of solutions in Theorem 3 also differ as $O\left(\frac{1}{k}\right)$.

Let us consider the following nonlinear example:

Example 2. *(Kenderov's example) Consider the following two dimensional system*

$$\dot{x}(t) \in F(x(t)) \stackrel{def}{=} \{(A+uB)\,x(t) : \; u \in [-1,1]\}\,, \; x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

*on $I = [0,1]$, where we add the impulse hyperplane $\tau(x) = \frac{1}{16}|x|$ and the jump function $S(x) = -\frac{x}{2}$. Let the matrixes $A = \begin{pmatrix} k^2-1 & k\sqrt{1-k^2} \\ -k\sqrt{1-k^2} & k^2-1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -2k\sqrt{1-k^2} \\ 2k\sqrt{1-k^2} & 0 \end{pmatrix}$ be given, where $k \in (0,1)$ is a fixed number (we will take $k = \frac{1}{3}$). Notice that without impulses this example is due to Kenderov and can be found in [7, Example 5.2.1]. It is easy to see that **A1, A2** are true with $N = \frac{1}{16}$. We will show that $NC < 1$ which implies that **A3** a) holds due to Remark 1.*

Hence, consider $C(u) = A + uB$. It is easy to see that $|C(u)| \leq \frac{\sqrt{34}}{3}$. Of course $F(\cdot)$ is not uniformly bounded, but it is sufficient to check the assumptions on a compactum containing all solution values as we noticed earlier.

The reachable set $\mathcal{D}_t$ $(t \in I)$ for the system without impulses has the form (in polar coordinates (r,ϕ)):

$$\mathcal{D}_t = \left\{ r(t) \begin{pmatrix} \cos\phi_u(t) \\ \sin\phi_u(t) \end{pmatrix} \mid r(t) = r_0 e^{(k^2-1)t}, \right.$$
$$\left. \phi_u(t) = \phi_0 + k\sqrt{1-k^2}\cdot(2u-1)t, \;\; 0 \leq u \leq 1 \;\; \right\}.$$

In our case $(r_0, \phi_0) = \left(\sqrt{2}, \frac{\pi}{4}\right)$.

Obviously, if $x(t) \in \mathcal{D}_t$, then $x(t) \in r(t)S_{n-1}$, where S_{n-1} is the unit sphere. Although the switching function $\tau(x) = \dfrac{|x|}{16}$ depends on the state x, it appears that all the solutions have the same jump time

$$\tau^* = \tau(x(\tau^*)) = \frac{|x(\tau^*)|}{16} = \frac{r_0}{16} e^{(k^2-1)\tau^*}$$

and $\tau^* \approx 0.0821630987895$, cf. Figure 2. The dashed lines show how the solutions jump. Hence, $x(\tau^* + 0) = \dfrac{x(\tau^*)}{2}$ and $\mathcal{D}_{\tau^*+0} = \dfrac{1}{2}\mathcal{D}_{\tau^*}$.

As indicated in Section 3 in more details, we will approximate numerically the solution funnel by calculating finitely many solutions of (4)–(5).

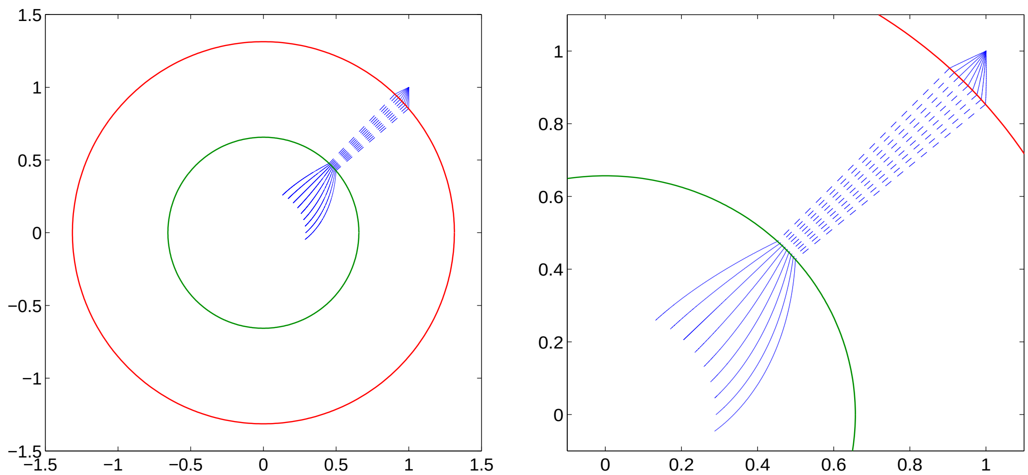

Figure 2: reachable set for Example 2 (full picture resp. details)

All trajectories have the same jump time τ^* which is approximated by τ^*_N, the common jump time of all Euler iterates, cf. Table 1. The logarithmic least squares approximation by $\log(Ch^p_N)$ gives the values $C = 0.5523505$, $p = 0.9996063$.

N	τ^*_N	$\tau^*_N - \tau^*$
10	$---$	$---$
20	0.1	0.0178369
40	0.1	0.0178369
80	0.0875	0.0053369
160	0.0875	0.0053369
320	0.084375	0.0022119
640	0.0828125	0.0006494
1280	0.0828125	0.0006494
2560	0.082421875	0.0002588
5120	0.0822265625	0.0000635

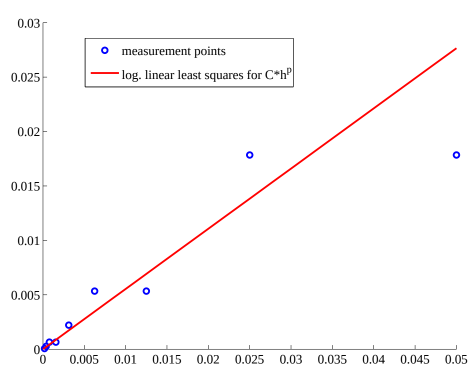

Table 1: convergence of the switching time for Euler iterates of Example 2

3 Approximation of the reachable set

The approximation of the reachable set can be realized by the approximation of the solution set in the case without impulses. In our case, however, the situation is more complicated. The problem is that the different solutions have (in general) different times of impulses. Therefore it is practically hard to approximate the reachable set or its graph. Furthermore, we have to check for every (approximate) solution $x(\cdot)$ if there is a time $\tau_i(x(\tau^*)) = \tau^* \in (t_j, t_{j+1})$. Moreover, the possible jump at the end point $\tau^* = 1$ poses some problems. One way to overcome such kind of problems is either to study (1)–(2) in a smaller interval $[0, 1-\delta]$ or to study it in the semi-open interval $[0, 1)$.

The graph of the reachable set is not closed in general. However, if we modify it by adding the values of $(t, x(t+0))$, then the graph will be closed. Denote the (modified) reachable set and its graph by

$$\overline{\mathcal{D}}_t = \{\{x(t), x(t+0)\} : \ x(\cdot) \text{ is a trajectory of (1)–(2) on } [0,t]\}, \ t \in I,$$
$$\overline{\mathcal{D}} = \{\{(t, x(t)), (t, x(t+0))\} : t \in I, \ x(\cdot) \text{ is a trajectory of (1)–(2) on } \ I\}\,.$$

Similar definitions are for the discrete system (4)–(5). The graph of the (modified) reachable set for the latter is denoted by:

$$\overline{\mathcal{D}}^k = \{(t_j, x^k(t_j)) : j = 0, \ldots, k, \ x^k(\cdot) \text{ is a trajectory of (4)–(5) on } [0,t]\}.$$

First we will state a very useful proposition stating that the closure of the reachable set $\mathcal{D}$ is compact:

Proposition 1. *Under the conditions **A1**–**A3** the graph of the reachable set is precompact. The graph of the modified reachable set is compact.*

Proof. Since the graph of the reachable set is bounded by Lemma 3, it is precompact.

Let $(t^\nu, x^\nu) \in \overline{\mathcal{D}}$ and let $(t^\nu, x^\nu) \to (\bar t, \bar x)$. Let $x_\nu(\cdot)$ be the corresponding trajectories. One has to consider two cases:

I) $\bar t \neq \tau_i(\bar x)$ for every i.

II) $\bar t = \tau_i(\bar x)$ for some $i = 1, \ldots, p$.

The same reasoning as in [10, Proposition 3.1] shows the existence of a subsequence such that $x_{\nu_j}(.)$ tends to $z(.)$ and their jump times τ^*_{i,ν_j} tend to τ^*_i. $z(.)$ is again piecewise Lipschitz and has no more than p jumps at τ^*_i, $i = 1, \ldots, m$ $(m \le p)$.

It is also very easy to see that $z(\cdot)$ is indeed a solution. Namely on $[0, \tau^*_1]$ it is a solution since there are no jumps. Also $\tau_1(z(\tau^*_1)) = \tau^*_1$. On $(\tau^*_1, \tau^*_2]$ one can proceed in the same way starting from $z(\tau^*_i + 0)$.

In the case **I)** $\bar x = z(\bar t)$.

In case **II)** it is easy to see that $z(\bar t) = \bar x$ or $z(\bar t + 0) = \bar x$. □

For the estimation of the reachable set at time $t = 1$, we will replace the interval I by $I_k := [0, 1 - \delta_k]$, where $\delta_k > \dfrac{C}{k}$, $\delta_k \ge \delta_{k+1}$ and $\lim\limits_{k\to\infty} \delta_k = 0$. For example such is $\delta_k = \dfrac{C}{\sqrt{k}}$ (at least for $k \ge 2$).

The following result is a consequence of Theorem 2.

Theorem 4. *Under the assumptions of Theorem 3, $D_H(\mathcal{D}^k, \mathcal{D}) = \mathcal{O}(h)$ in the sense that for every $\delta > 0$ the estimates $\operatorname{dist}(\mathcal{D}^k|_{I_\delta}, \mathcal{D}) = \mathcal{O}(h)$ and $\operatorname{dist}(\mathcal{D}|_{I_\delta}, \mathcal{D}^k) = \mathcal{O}(h)$ hold, where $\mathcal{D}^k|_{I_\delta}$ is $\mathcal{D}^k$ restricted on $I_\delta = [0, 1-\delta]$.*

Since there can be solutions of (1)–(2) or (4)–(5) with a jump at $t = 1$ we have to restrict the interval.

First we will describe the graph of the reachable set of Example 1.

Example 3. *Let us consider the impulsive differential inclusion of Example 1.*

Case 1) *Using standard calculations one can describe the reachable set:*

$$\overline{\mathcal{D}}_t = \begin{cases} [t+\frac{1}{3}, \frac{1}{3}e^{3t}] & (0 \le t < \tau^*), \\ [\frac{\tau^*+1}{2} + t - \tau^*, \frac{\tau^*+1}{2}e^{3(t-\tau^*)}] \bigcup [t+\frac{1}{3}, t+1] & (\tau^* \le t < \tilde{\tau}), \\ [t+\frac{1}{3}, t+1] & (\tilde{\tau} \le t) \end{cases}$$

Here, $\tau^* \approx 0.501747$ and $\tilde{\tau} \approx 0.559449$. In Figure 3, the reachable sets are shown for both cases. The red dashed line indicates the jump surface, the green dashed line the values of solutions after the jump.

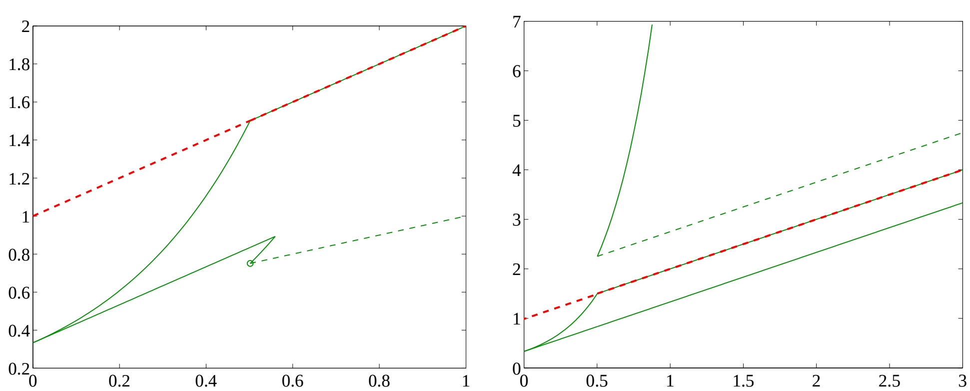

Figure 3: reachable set for Example 1, case 1) resp. 2)

Case 2) *Let $x(\cdot)$ be a solution. It is easy to see that in this case the function $f(t) = \tau(x(t)) - t$ defined in the proof of Lemma 3 is strongly increasing and hence every equation $t = \tau(x(t))$ has no more than one solution.*

*It is clear that τ^* is the same as in **1)**, the reachable set can be represented as*

$$\overline{\mathcal{D}}_t = \begin{cases} [t+\frac{1}{3}, \frac{1}{3}e^{3t}] & (0 \le t < \tau^*), \\ [\frac{3}{2}(\tau^*+1) + t - \tau^*, \frac{3}{2}(\tau^*+1)e^{3(t-\tau^*)}] \bigcup [t+\frac{1}{3}, t+1] & (\tau^* \le t). \end{cases}$$

All the numerical tests use special selections of the right-hand side. Since in all examples the right-hand side of the differential inclusion is parameterized by functions $f(t, x, u)$, several convex combinations of certain exposed points of the set-valued right-hand side are used for the selection process. This simulation process gives a first impression of the trajectory funnel. For each trajectory, the possible hitting of the jump surfaces is tested. The so-called generalized Steiner selections were also studied for Castaing representations of set-valued integrands and in the approximation of reachable sets of linear differential inclusions, cf. [4]. In the linear case, the reachable set (without impulses) can be rewritten as Aumann's integral and these selections fill out the whole funnel of all integrable selections.

The focus is laid on selections, since the jump conditions depend on the trajectory and must be tested separately for each trajectory. Due to Theorem 3, the evaluation can be simply performed with the piecewise linear interpolation.

The following three examples are linear.

Example 4. *Consider the following linear example which stems from an impulsive system in [25, Example 2.2.3], [3, Section II.F] with the gravity constant g:*

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -g \quad (t \in I = [0,2]) \end{aligned}$$

This example does not seem to fit in our problem class, since jumps $x_2(\tau+0) = -\alpha \cdot x_2(\tau)$ appear at a time t, if $x_2(t) = 0$ and $x_1(t) \le 0$. Introducing an artificial variable $x_3(\cdot)$ with $x_3'(t) = 1$, $x_3(0) = 0$ which measures the time, we set

$$\tau(x) = \begin{cases} x_1 + x_2^2 + x_3, & \text{if } x_1 \ge 0, \\ x_2^2 + x_3, & \text{else.} \end{cases}$$

Now, $\tau(x(t)) = t$ if and only if $x_2(t) = 0$, $x_1(t) \le 0$, since $x_3(t) = t$.

The following model also incorporates uncertainty (modeled by $\varepsilon \cdot U = 0.00335 \cdot [-1,1]$) on the value $g = 9.80665$:

$$\dot{x}(t) \in Ax(t) + BU \quad (t \in I = [0,2]), \quad x(0) = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \tag{21}$$

Here, $A := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \varepsilon \\ 0 \end{pmatrix}$ and $U = [-1,1]$.

The jump is given by

$$S(x) = \begin{pmatrix} 0 \\ -(\alpha+1)x_2 \\ 0 \end{pmatrix}, \quad \alpha = \frac{1}{2}.$$

The number s_N of detected jump times of the Euler iteration depends on the number of subintervals used, cf. Table 2.

N	s_N	τ_i^*, $i = 1, \ldots, s_N$
10	3	2.0, 3.5, 5.0
100	22	1.1, 1.7, 2.05, 2.3, 2.45, 2.6, 2.75, 2.9, 3.05, 3.2, 3.35, 3.5, 3.65, 3.8, 3.95, 4.1, 4.25, 4.4, 4.55, 4.7, 4.85
1000	201	1.025, 1.545, 1.815, 1.955, 2.03, 2.07, 2.09, ... 4.985, 5.0

Table 2: number of jump times for Euler iteration for Example 4

The dashed lines in Figures 4–5 show how the solutions jump w.r.t. the second coordinate.

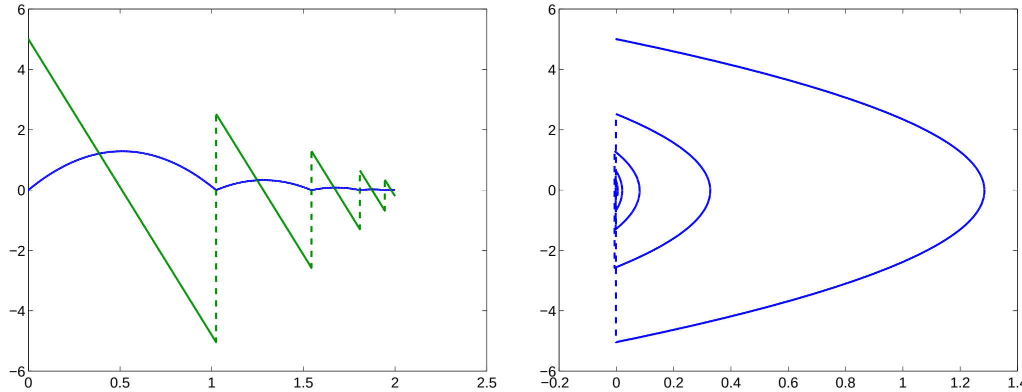

Figure 4: single trajectory for Example 4 (coordinates resp. phase portrait) and $N = 500$

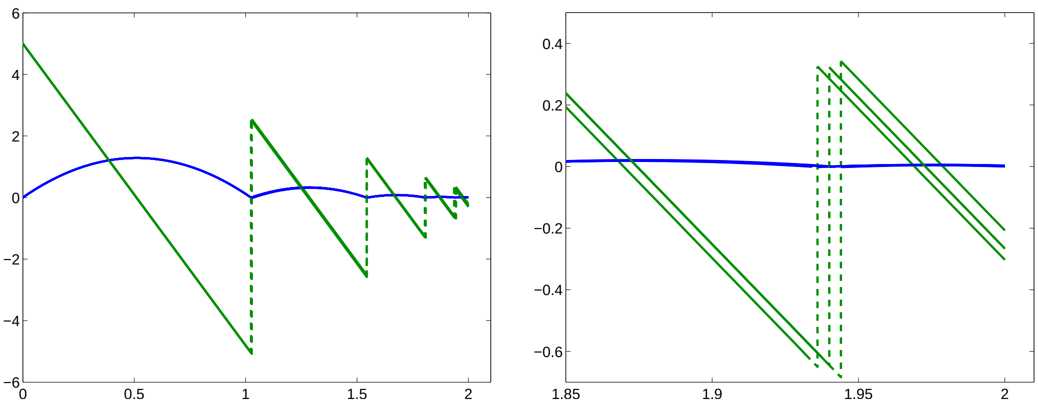

Figure 5: reachable set for Example 4 with $N = 500$ (full picture resp. details)

Example 5. *Consider the following system:*

$$\dot{x}(t) \in Ax + BU \quad (t \in [0,1]), \quad x(0) = 0, \tag{22}$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}$ *and* $U = [-1,1]$.

The impulsive surface is just one: $\tau(x) = \dfrac{|x|}{4} + 0.1$ *and* $S(x) = -\dfrac{x}{4}$. *The trajectories start from the origin and have at most one jump, cf. Figure 6.*

It is easy to see that all our conditions hold.

For the special trajectory $x(\cdot)$ *with the control function* $u(t) = 1$ *we get the following approximation of the switching time* τ^* *by the Euler iterates, cf. Table 3. The logarithmic least squares approximation by* $\log(Ch_N^p)$ *gives the values* $C = 1.294387$, $p = 1.142452$.

Different jump times occur in this example for different solutions, cf. Table 4.

Example 6. *Consider the following non-autonomous system with* $U = [-1,1]$*:*

$$\dot{x}(t) \in \frac{1}{2}x + \varepsilon U + u^*(t) \quad (t \in [0,1]), \; x(0) = 1, \; u^*(t) = 2 \cdot e^{\frac{2}{3} \cdot t} \cdot \frac{e^{3t} - e^3}{2 + e^3}\,, \tag{23}$$

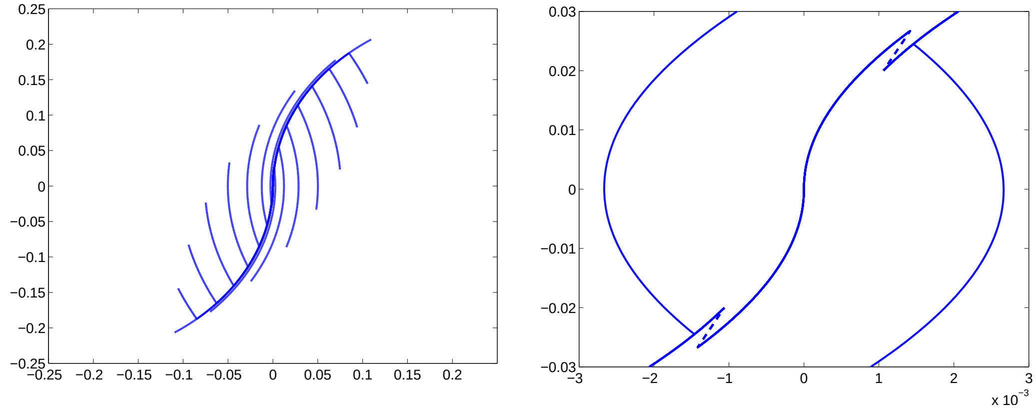

Figure 6: reachable set for Example 5 (full picture resp. details)

N	τ_N^*	$\tau_N^* - \tau_{5120}^*$
10	0.2	0.093164
20	0.15	0.043164
40	0.125	0.018160
80	0.1125	0.005664
160	0.1125	0.005664
320	0.109375	0.002539
640	0.1078125	0.000977
1280	0.10703125	0.000195
2560	0.10703125	0.000195
5120	0.1068359375	0.000000

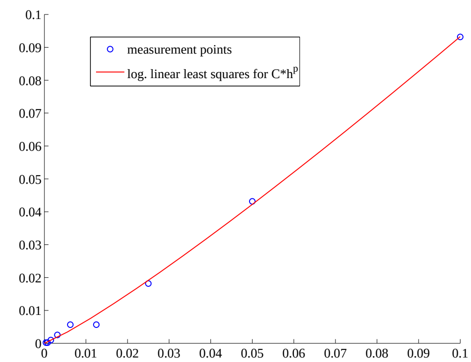

Table 3: convergence of the jump time for Euler iteration for Example 5

N	τ_N^* for $u(\cdot) = 1$	τ_N^* for $u(\cdot) = -1$
10	0.2	0.2
100	0.11	0.11
1000	0.107	0.104

Table 4: (slightly) different jump times of Euler iterates in Example 5

We consider just one impulsive surface: $\tau(x) = \frac{1}{3} \cdot (x + \frac{23}{10})$ *and* $S(x) = -\frac{1}{2} \cdot x$*. In Figures 7–8, the impulsive surface are plotted in red, the solution points for non-jump times in blue and the end points* $y(t)$ *of the solutions before the jump time* τ *and after the jump, i.e.* $y(t+0)$*, are depicted in green colour.*

In the details of Figures 9–10, the linear convergence of the Euler iteration could be clearly observed as well as the delay of the jump times to the right endpoint of the subinterval as in (5). In Figure 9, only 3 of 20 trajectories jump at $\tau_1^* = 0.9$*, the others jump at* $\tau_1^* = 1.0$*. In Figure 10 all trajectories jump before* $t = 1.0$*. The violation of the jump surface by Euler iterates tends linearly to zero in Figure 10.*

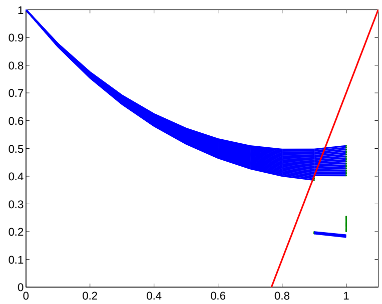

Figure 7: reachable set for Example 6 with $N = 10$ and 20 selections

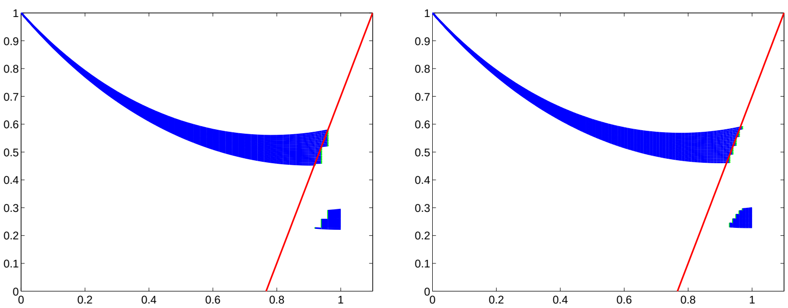

Figure 8: reachable set for Example 6 with $N = 50$ resp. $N = 100$ and 20 selections

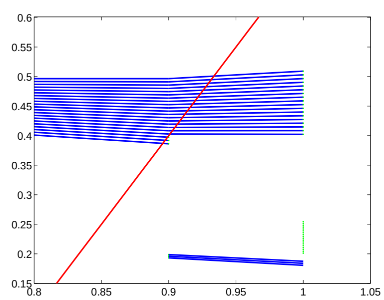

Figure 9: zoomed reachable set for Example 6 with $N = 10$ and 20 selections

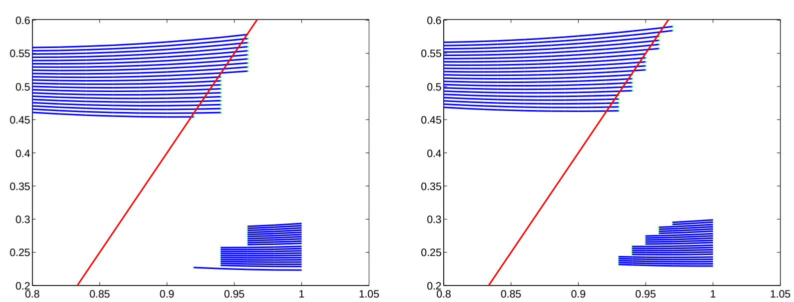

Figure 10: zoomed reachable set for Example 6 with $N = 50$ resp. $N = 100$ and 20 selections

Conclusions

The results in Section 2 show that one could apply at least three strategies to deal with jumps in trajectories when implementing a discretization. For numerical approximations, it is crucial to justify how the algorithm should react, if jump times of the original solution of the impulsive differential inclusion would not lie on grid points used for the discretization.

The first strategy is a kind of nonlinear interpolation strategy on grid points in (17), i.e. it uses an integrable selection $f_j(\cdot)$ with $f_j(t) \in F(x_j^k)$ and

$$\begin{aligned} \dot{x}^k(t) &= f_j(t) \text{ on } [t_j, t_{j+1}) \\ \text{yielding } x^k(t) &= x_j^k + \int_{t_j}^t f_j(s)ds. \end{aligned}$$

This (nonlinear) selection is used to test the jump condition

$$\tau_i(x^k(\tau_i)) = \tau_i \tag{24}$$

and allow jump times even within the subinterval $[t_j, t_{j+1}]$. Lemma 5 shows that one could reach order of convergence 1 w.r.t. step-size.

The second strategy is similar and uses the (nonlinear) selection $f_j(\cdot)$, but the jump within a subinterval $[t_j, t_{j+1}]$ is postponed to the subsequent grid point t_{j+1} as in (20). Theorem 2 tells that this simple strategy still gives us the same order of convergence than the first strategy.

In contrary to the first two approaches, the third strategy uses a piecewise linear strategy on (4)–(5) with

$$x^k(t) = x_j^k + (t - t_j)f_j \quad (t \in [t_j, t_{j+1})), \quad f_j \in F(x_j^k).$$

This interpolation is used to evaluate the jump condition (24) on the subinterval. Nevertheless, the jump is again postponed to the following grid point. Theorem 3 justifies the resulting order of convergence 1 for the simplest strategy among the three ones.

The proof of Theorem 3 was done step by step by increasing the simplicity for testing the jump condition. Lemma 5 justifies the first strategy with nonlinear selections and jumps within subintervals. It relies heavily on the Filippov-Plis lemma in Theorem 1. Within the convergence proof for the second strategy, solutions of the first strategy are approximated by the second and vice versa, also applying the result on inner perturbations of the original differential inclusion. Finally, the third strategy is justified by the convexity of Aumann's integral.

Evidently, not all the conditions of **A3** hold for Example 4 which demonstrates that this condition is important to prevent the multiple hitting of jump surfaces. However, we have only one impulsive surface which causes this beating phenomena for solutions. In this example, $f^x(\cdot)$ is still strongly monotone on the intervals of the continuity of a given solution $x(\cdot)$ and our method gives a good approximation. It is interesting to investigate the approximation of solutions with beating phenomena and we leave this interesting task for a further paper.

In a forthcoming article we want to apply our results to the Lagrange optimal control problem:

$$\min J[x] := \int_0^1 g(t, x(t))\, dt \tag{25}$$

subject to (1)–(2).

Assuming that $f(t, \cdot)$ is (locally) Lipschitz, Mordukhovich studies more general kind of problem, if the system is without impulses.

Our future aim is to extend Mordukhovich's results in [20] to the case of an impulsive system.

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