Abstracts

Economic MPC and the role of exponential turnpike properties

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1. Problem formulation

We consider discrete time control systems with state $x \in X$ and control values $u \in U$, where X and U are normed spaces with norms denoted by $\|\cdot\|$. The control system under consideration is given by

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

with $f : X \times U \to X$. For a given control sequence $u = (u(0), \ldots, u(K-1)) \in U^K$ or $u = (u(0), u(1), \ldots) \in U^\infty$, by $x_u(k, x)$ we denote the solution of (1) with initial value $x = x_u(0, x) \in X$.

For given admissible sets of states $\mathbb{X} \subseteq X$ and control values $\mathbb{U} \subseteq U$ and an initial value $x \in \mathbb{X}$ we call the control sequences $u \in \mathbb{U}^K$ satisfying

$$x_u(k, x) \in \mathbb{X} \quad \text{for all } k = 0, \ldots, K$$

admissible. The set of all admissible control sequences is denoted by $\mathbb{U}^K(x)$. Similarly, we define the set $\mathbb{U}^\infty(x)$ of admissible control sequences of infinite length. For simplicity of exposition we assume $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$, i.e., that for each initial value $x \in \mathbb{X}$ we can find a trajectory staying inside $\mathbb{X}$ for all future times.

Given a feedback map $\mu : X \to U$, we denote the solutions of the closed loop system

$$x(k+1) = f(x(k), \mu(x(k)))$$

by $x_\mu(k)$ or by $x_\mu(k, x)$ if we want to emphasize the dependence on the initial value $x = x_\mu(0)$. We say that a feedback law μ is admissible if it renders the admissible set $\mathbb{X}$ (forward) invariant, i.e., if $f(x, \mu(x)) \in \mathbb{X}$ holds for all $x \in \mathbb{X}$. Note that $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$ immediately implies that such a feedback law exists.

Our goal is now to find an admissible feedback controller which yields approximately optimal average performance. To this end, for a given stage cost $\ell : X \times U \to \mathbb{R}$ we define the averaged functionals and optimal value functions

$$\begin{aligned} J_N(x,u) &:= \frac{1}{N}\sum_{k=0}^{N-1} \ell(x_u(k,x), u(k)), & V_N(x) &:= \inf_{u \in \mathbb{U}^N(x)} J_N(x,u), \\ J_\infty(x,u) &:= \limsup_{N\to\infty} J_N(x,u) \quad \text{and} & V_\infty(x) &:= \inf_{u \in \mathbb{U}^\infty(x)} J_\infty(x,u). \end{aligned}$$

We assume that ℓ is bounded from below on $\mathbb{X}$, i.e., that $\ell_{\min} := \inf_{x\in\mathbb{X}, u\in\mathbb{U}} \ell(x,u)$ is finite. This assumption immediately yields $J_N(x,u) \geq \ell_{\min}$ and $J_\infty(x,u) \geq \ell_{\min}$ for all admissible control sequences. In order to simplify the exposition in what follows, we assume that (not necessarily unique) optimal control sequences for J_N exist which we denote by $u^\star_{N,x}$ or briefly by $u^\star$.

Similarly to the open loop functionals, we can define the average cost of the closed loop solution for any feedback law μ by

$$\begin{aligned} J_K(x,\mu) &= \frac{1}{K}\sum_{k=0}^{K-1} \ell(x_\mu(k,x),\mu(x_\mu(k,x))) \\ J_\infty(x,\mu) &= \limsup_{K\to\infty} J_K(x,\mu). \end{aligned}$$

In order to construct the desired feedback law, henceforth denoted by μ_N, we employ a model predictive control (MPC) approach: in each time instant k, we compute an optimal control $u^\star_{N,x_0}$ for the initial value $x_0 = x_{\mu_N}(k,x)$ and define the feedback value as $\mu_N(x_0) := u^\star_{N,x_0}$, i.e., as the first element of the finite horizon optimal control sequence.

2. Value and trajectory convergence results

The presented results hold for averaged optimal control problems exhibiting an optimal steady state, i.e., for which there exists a point $x^e \in \mathbb{X}$ and a control value $u^e \in \mathbb{U}$ with

$$f(x^e,u^e) = x^e \quad \text{and} \quad V_\infty(x) \geq \ell(x^e,u^e)$$

for all $x \in \mathbb{X}$.

For such problems, it was shown in [1, 2, 4] that the receding horizon controller μ_N shows optimal infinite horizon averaged performance if the terminal constraint $x_u(N,x) = x^e$ is added as an additional condition to the finite horizon problem employed for computing μ_N.

Here, we consider the MPC formulation without such terminal constraints. Motivation for doing so is on the one hand that removing the terminal constraint also removes the need to compute x^e beforehand and on the other hand that not imposing terminal constraints increases the region of feasibility for the MPC problem.

The central result from [5] shows that under appropriate conditions the feedback μ_N indeed shows approximately optimal performance and that the gap to optimality, i.e., the difference $|J_\infty(x,\mu_N) - V_\infty(x)|$ decreases to 0 for $N \to \infty$. The assumptions for this result are

(i) Uniform continuity of V_N in a neighborhood of x^e for all sufficiently large N
(ii) A turnpike property, which describes the fact that the finite time optimal trajectory enters a neighborhood of the optimal equilibrium x^e which shrinks to 0 as $N \to \infty$

Both properties can, e.g., be ensured by suitable controllability and dissipativity properties involving both the dynamics and the stage cost, for details and a formal version of (i) see [5]. The turnpike property (ii) is formally expressed as follows:

There is $\sigma(N)$ such that any optimal trajectory $x_{u^\star}(k)$ with horizon N satisfies

$$\min_{k=0,\ldots,N} \|x_{u^\star}(k) - x^e\| \leq \sigma(N), \quad \text{with } \sigma(N) \to 0 \text{ as } N \to \infty.$$

In addition to the value convergence result, important additional results are proved in [5] under the condition that $\sigma(N)$ tends to 0 faster than $1/N$. More precisely, under this additional condition, convergence of the MPC closed loop trajectory to a neighborhood of x^e (shrinking to x^e as $N \to \infty$) and an approximate optimality condition during the transient phase can be shown.

Several numerical examples show that it is a reasonable condition to expect that $\sigma(N)$ tends to 0 faster than $1/N$. More precisely, in many examples $\sigma(N) \approx C\theta^N$ for constants $C > 0$ and $\theta \in (0, 1)$ can be observed, i.e., an exponential turnpike property.

3. Exponential turnpike properties

Since exponential turnpike properties play an important role in Economic MPC, it is of considerable importance to find conditions which ensure this property for a given example. The following condition for an exponential turnpike property will be presented and discussed in [3] (to which we also refer for the precise technical assumptions and the proof). For its formulation, for a modified stage cost $\tilde{\ell}$ defined in [5] we define

$$\widetilde{J}_N(x, u) := \frac{1}{N+1} \sum_{k=0}^{N} \tilde{\ell}(x_u(n), u(n)),$$

and the optimal value function of the terminal constrained problem

$$\widetilde{V}_N(x_0, x_N) := \inf_u \widetilde{J}_N(x, u), \qquad \text{s.t. } x_u(0) = x_0,\ x_u(N) = x_N$$

Then, an exponential turnpike property holds if there exists $\gamma \geq 1$ and $\delta \geq 1$ such that for all $x_0, x_N \in \mathbb{X}$ and $N \in \mathbb{N}$ the inequality

$$V_N(x_0, x_N) \leq \frac{\gamma \min_u \tilde{\ell}(x_0) + \delta \min_u \tilde{\ell}(x_N)}{N+1}$$

holds.

References

[1] D. Angeli, R. Amrit, and J. B. Rawlings, *Receding horizon cost optimization for overly constrained nonlinear plants*, in Proceedings of the 48th IEEE Conference on Decision and Control – CDC 2009, Shanghai, China, 2009, pp. 7972–7977.

[2] D. Angeli and J. B. Rawlings, *Receding horizon cost optimization and control for nonlinear plants*, in Proceedings of the 8th IFAC Symposium on Nonlinear Control Systems – NOLCOS 2010, Bologna, Italy, 2010, pp. 1217–1223.

[3] T. Damm, L. Grüne, M. Stieler, and K. Worthmann, *An exponential turnpike theorem and examples (working title)*. In preparation, 2012.

[4] M. Diehl, R. Amrit, and J. B. Rawlings, *A Lyapunov function for economic optimizing model predictive control*, IEEE Trans. Autom. Control, 56 (2011), pp. 703–707.

[5] L. Grüne, *Economic receding horizon control without terminal constraints*. Preprint, University of Bayreuth, 2011. Provisionally accepted for Automatica. Preprint available from `num.math.uni-bayreuth.de/de/publications/2011/gruene_ecompc_2011` .

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