STABILITY, PERFORMANCE AND ROBUSTNESS OF SENSITIVITY-BASED MULTISTEP FEEDBACK NMPC*

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EXTENDED ABSTRACT. In recent decades, Nonlinear Model Predictive Control (NMPC) has proven to be an important tool in control of nonlinear systems in modern technological applications. NMPC is an approach to feedback design that is based on the solution, at each controller update step, of an optimal control problem (OCP). Increased attention in the study of NMPC over the years has been continuously bringing results that address challenges in the performance of this method, the stability of the closed-loop system and robustness of NMPC schemes.

Let us consider a plant with dynamics given by the discrete-time model $x(k+1) = f\left(x(k), u(k)\right)$ where $x(k)$ represents the plant state and $u(k)$ denotes the control at time step t_k with $0 < k \in \mathbb{Z}$. Let X be the state space, U be the control value space, and $x_u(\cdot, x_0)$ be the trajectory for control sequence u and initial state x_0. For the general NMPC algorithm with finite time horizon length $N \geq 2$, at each sampling time t_n, $n = 0, 1, 2, \ldots, N-1$, we measure the state $x(n) \in X$ of the system. We set $x_0 = x(n)$ and solve the following OCP:

$$\min J_N\left(z\right) := \sum_{k=0}^{N-1} \omega_{N-k} \ell\left(n+k, x_u(k, x_0), u(k)\right) + F\left(n+N, x_u(N, x_0)\right)$$

$$\text{with respect to the optimization variable}$$
$$z := (x_u(0, x_0)^\intercal, \ldots, x_u(N, x_0)^\intercal, u(0)^\intercal, \ldots, u(N-1)^\intercal)^\intercal$$

$$\text{subject to the initial value} \qquad x_u(0, x_0) = x_0,$$

$$\text{dynamics} \quad x_u(k+1, x_0) = f\left(x_u(k, x_0), u(k)\right), \quad k = 0, \ldots, N-1,$$

$$\text{other equality constraints} \qquad G(z) = 0,$$

$$\text{and inequality constraints} \qquad H(z) \geq 0.$$

In this formulation, ℓ represents the running cost function, ω_{N-k} are the weights of the running cost function and F is the terminal cost function. In the optimal solution, we refer to the obtained optimal control sequence as $u^\star(\cdot) \in \mathbb{U}^N_{\mathbb{X}_0}(n, x_0)$ where $\mathbb{U}^N_{\mathbb{X}_0}(n, x_0)$ denotes the set of admissible finite horizon control sequences for terminal constraint set $\mathbb{X}_0$, where we assume that the terminal constraint $x_u(N, x_0) \in \mathbb{X}_0$ is contained

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in the constraints found in $G(z) = 0$ and $H(z) \geq 0$. Our approach applies to MPC formulations both with and without terminal constraint, in the latter case we set $\mathbb{X}_0 = \mathbb{X}$. We define the closed-loop NMPC-feedback law as $\mu_N(n, x(n)) := u^\star(0) \in U$, i.e., as the first element of the obtained optimal control sequence and use this control value in the next sampling period to obtain $x(n+1)$.

Rigorous statements on the system theoretical aspects of existing NMPC schemes such as various ideas exemplifying approaches to achieve stability and robustness are considerably well-established in the literature. For example, see Grüne, Pannek [3].

The described OCP can be viewed as a nonlinear programming (NLP) problem that depends on the parameter x_0. First, let

$$\tilde{G}(z) = \begin{bmatrix} x_u(0, x_0) - x_0 \\ x_u(k+1, x_0) - f\left(x_u(k, x_0), u(k)\right), \quad k = 0, \ldots, N-1 \\ G(z) \end{bmatrix}.$$

Suppose $\tilde{G} \in \mathbb{R}^{n_e}, H \in \mathbb{R}^{n_i}$. Let $C := \left[\tilde{G}, H\right]^\intercal \in \mathbb{R}^{n_c}$ where $n_c = n_e + n_i$. Then we can write the NLP problem as $\mathcal{P}(p)$ which is parametric in the initial state $p := x_0 = x(n)$ via

$$\min J_N\left(z, p\right) \quad \text{subject to} \quad z \in \Sigma(p)$$

where

$$\Sigma(p) = \{z \mid C_i(z,p) = 0, i = 1, \ldots, n_e,\ C_i(z,p) \geq 0, i = n_e + 1, \ldots, n_c\}$$

denotes the admissible set,

$$\mathcal{L}(z, \mu, p) = J_N(z,p) + \mu^\intercal C(z,p)$$

denotes the Lagrangian function,

$$\mathcal{A}(z,p) = \{1, \ldots, n_e\} \cup \{i \mid C_i(z,p) = 0,\ i = n_e + 1, \ldots, n_c\}$$

denotes the index set of active constraints and $C_{\mathcal{A}(z,p)}$ denotes the active constraints and $\mu_{\mathcal{A}(z,p)}$ are the corresponding multipliers.

Now the NLP Sensitivity Theorem in Fiacco [1] states sufficient conditions for the differentiability of an optimal solution $z(p)$ with respect to p. The theorem states that if J_N and $C_i, i = 1, \ldots, n_c$ are twice differentiable in a neighborhood of the nominal solution $z^\star(p_0)$ and second order sufficient conditions (SOSC), linear independent constraint qualification (LICQ) and strict complementarity hold at $z^\star(p_0)$, then for p in a neighborhood of p_0, there exists a unique, continuous and differentiable $z^\star(p)$ which is a local miminizer satisfying SOSC and LICQ for the problem $\mathcal{P}(p)$. Moreover, $z(p)$ and $\mu(p)$ are continuously differentiable functions of p in the said neighborhood of p_0 and based on the implicit function theorem

$$\begin{bmatrix} \nabla^2_{zz}\mathcal{L}(z^\star, \mu^\star, p_0) & \nabla_z C_{\mathcal{A}(z^\star,p_0)}(z^\star, p_0)^\intercal \\ \nabla_z C_{\mathcal{A}(z^\star,p_0)}(z^\star, p_0) & 0 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial z}{\partial p}(p_0) \\ \dfrac{\partial \mu_{\mathcal{A}(z^\star,p_0)}}{\partial p}(p_0) \end{bmatrix} = -\begin{bmatrix} \nabla^2_{zp}\mathcal{L}(z^\star, \mu^\star, p_0) \\ \nabla_p C_{\mathcal{A}(z^\star,p_0)}(z^\star, p_0) \end{bmatrix}$$

holds. The matrix $\left[\dfrac{\partial z}{\partial p}(p_0),\ \dfrac{\partial \mu_{\mathcal{A}(z^\star,p_0)}}{\partial p}(p_0)\right]^\mathsf{T}$ is called the sensitivity matrix which consists of the sensitivity of the solution z and the sensitivity of the multipliers corresponding to active contraints both with respect to the parameter p evaluated at p_0. The sensitivity $\dfrac{\partial z}{\partial p}(p_0)$ gives rise to a first-order approximation of the optimal solution for a perturbed parameter via

$$\tilde{z}(p) = z^\star(p_0) + \frac{\partial z}{\partial p}(p_0)\,(p - p_0)\,.$$

One of the main challenges in NMPC applications is reducing the computational effort brought about by the online solution of an OCP at every time step t_n. Reducing the computational load due to these NLP problems should be accomplished without sacrificing statements on the stability, performance and robustness of the resulting control algorithm. To solve this problem, a straightforward approach is using a multistep feedback, i.e., using more than just the first element of the resulting finite horizon optimal control sequence and thus performing the optimization less often. Instead of only the first element of the obtained optimal control sequence, we implement the first m elements, i.e. $u^\star(0), u^\star(1), \dots, u^\star(m-1)$, and then proceed with the next optimization. We call the number m the control horizon. With this, we write the feedback law $\mu_N(x,k) := u^\star(k),\ k = 0, \dots, m-1$ which we refer to as a multistep NMPC-feedback law.

Consider the OCP with initial state $x_0 \in X$ and optimization horizon $N \in \mathbb{N}_0$. Let $V_N(x_0)$ denote the finite horizon optimal value function, $V_\infty^{\mu_{N,m}}(x_0)$ denote the multistep feedback optimal value function and $V_\infty(x_0)$ denote the infinite horizon optimal value function. In the simplest case where neither terminal costs nor terminal constraints are imposed, via relaxed dynamic programming, [2] establishes the estimate

$$\alpha V_\infty(x_0) \le \alpha V_\infty^{\mu_{N,m}}(x_0) \le V_N(x_0)$$

for some $\alpha \in (0,1]$ describing the suboptimality of the multistep feedback $\mu_{N,m}$ for the infinite horizon problem. Through this, [2] further establishes a suitable bound for $V_N(x_{\mu_{N,m}}(n))$ for all n and that $\mu_{N,m}$ yields asymptotic stability of the MPC closed-loop system. Hence, upon using longer control horizons, stability and performance results still remain valid.

However, longer control horizon may reduce robustness since the use of more elements of the control sequences suggests that the system runs in open loop for a longer time. More precisely, due to external perturbations, modeling errors etc. the measured states $x_k^{(m)} := x(n+k)$, $k = 1, \dots, m-1$, will in general deviate from the predicted states $x_{u^\star}(k, x(n))$ and since in a multistep feedback law we do not use this information the controller cannot react to this deviation.

A remedy for this issue is incorporating sensitivity, which can be used to update the next entry of the multistep feedback which is actually the first element of the tail of the optimal control sequence, injecting the updated control value to the system to generate the next state, and repeating this process to the remaining succeeding entries of the multistep feedback before finally performing the next optimization solving the next NLP problem at time t_{n+m}. In a different context, it has already been demonstrated by Zavala and Biegler [5] that sensitivity techniques are well suited in order to perform such an update based on recent measurements.

We thus propose the following strategy. With initial value $x_0 = x(n)$ we solve the problem $\mathcal{P}(x_0)$ and obtain the optimal control sequence $u_0^\star, u_1^\star, u_2^\star, \dots, u_{N-1}^\star$. We then implement $u_0^\star$ to obtain $x_1^\star := x_{u^\star}(1, x_0)$. Now by Cor. 3.16 in [3], the tail $u_1^\star, u_2^\star, \dots, u_{N-1}^\star$ is the optimal control sequence for the problem $\mathcal{P}(x_1^\star)$ with initial value $x_1^\star$, time instant t_1 and optimization horizon $N-1$. The idea is, since as an initial value, $x_1^\star$ corresponds to $u_1^\star, u_2^\star, \dots, u_{N-1}^\star$, we can improve $u_1^\star$ using the sensitivity $\dfrac{\partial u_1^\star}{\partial x_1^\star}$, which tells us how $u_1^\star$ changes with respect to $x_1^\star$, and the perturbation $x_1^{(m)} - x_1^\star$, where $x_1^{(m)}$ denotes the measured state at time t_{n+1}. We thus obtain the correction

$$\tilde{u}_1 = u_1^\star + \frac{\partial u_1^\star}{\partial x_1^\star}\left(x_1^{(m)} - x_1^\star\right).$$

We perform the same procedure to update $u_2^\star, \dots, u_m^\star$ before once again performing an optimization to solve the succeeding NLP.

For this proposed algorithm, we present numerical results from various examples and investigate the effects of this approach on the robustness of the NMPC method. Moreover, we indicate how this procedure can be incorporated into the stability and performance analysis of the MPC closed loop.

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