Nonlinear Model Predictive Control

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(1) Introduction

What is Model Predictive Control (MPC)?

Setup

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \ \ x_{\mathbf{u}}(0) = x_0$$

or, briefly

$$x^+ = f(x, u)$$

with $x \in X$, $u \in U$

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Additionally, we impose state constraints $x_\mu(n) \in \mathbb{X}$
and control constraints $\mu(x(n)) \in \mathbb{U}$

for all $n \in \mathbb{N}$ and given sets $\mathbb{X} \subseteq X$, $\mathbb{U} \subseteq U$

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Idea of MPC: use an optimal control problem which minimizes the distance to x_* in order to synthesize a feedback law μ

The idea of MPC

For defining the MPC scheme, we choose a stage cost $\ell(x,u)$ penalizing the distance from x_* and the control effort, e.g., $\ell(x,u) = \|x - x_*\|^2 + \lambda\|u\|^2$ for $\lambda \geq 0$

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Notation in what follows:

- general feedback laws will be denoted by μ
- the MPC feedback law will be denoted by μ_N

The basic MPC scheme

Formal description of the basic MPC scheme:

At each time instant n solve for the current state $x_{\mu_N}(n)$

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_{\mu_N}(n), \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_{\mu_N}(n)$$

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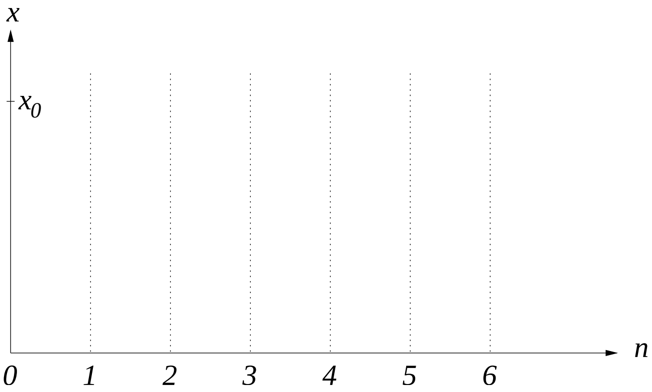

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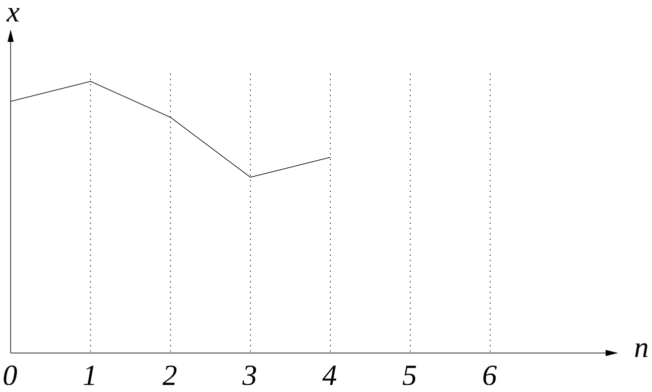

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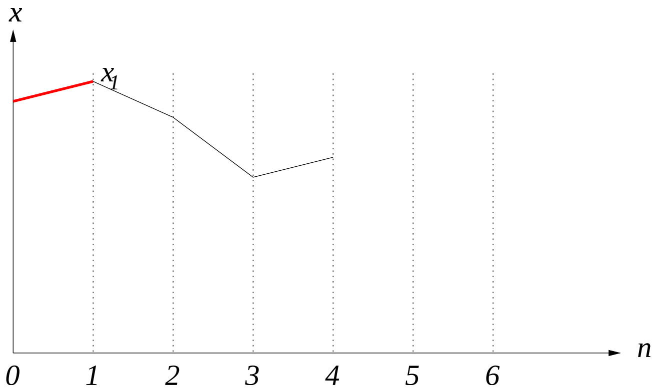

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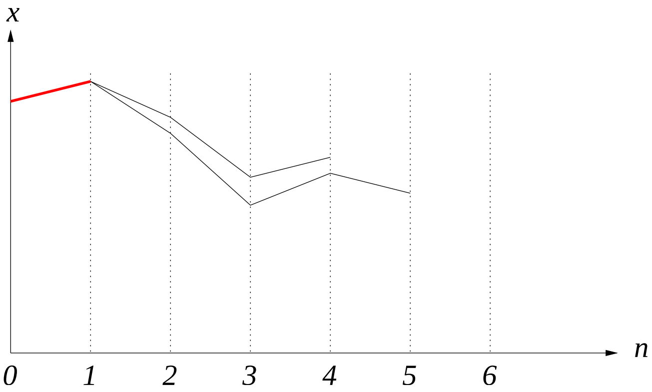

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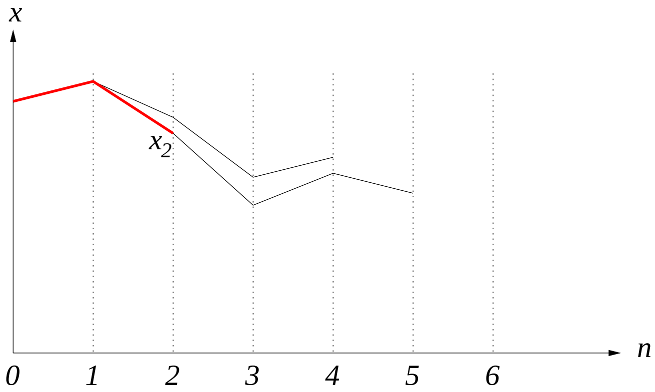

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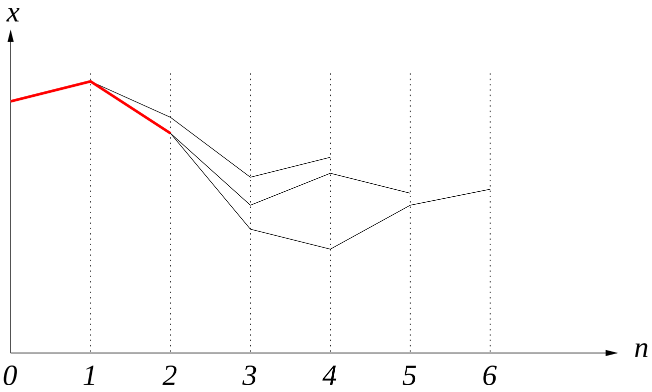

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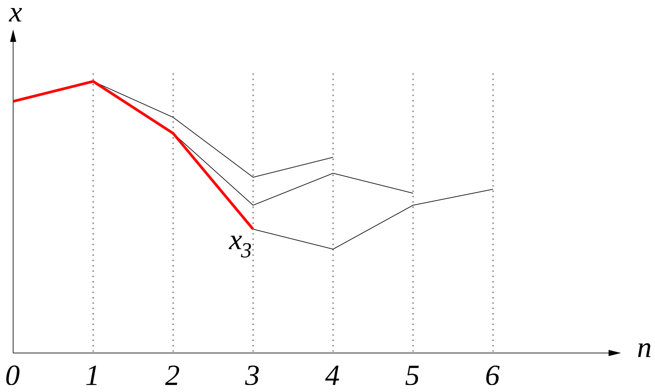

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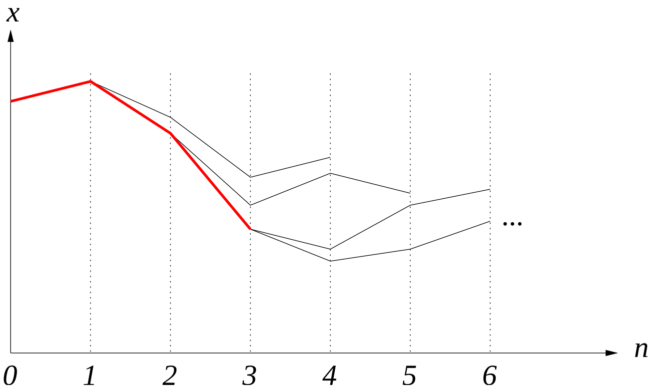

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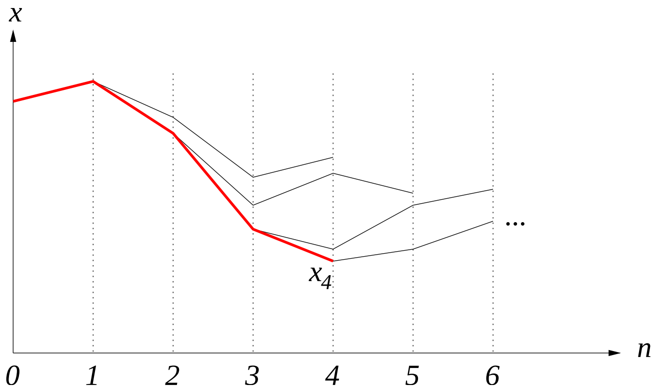

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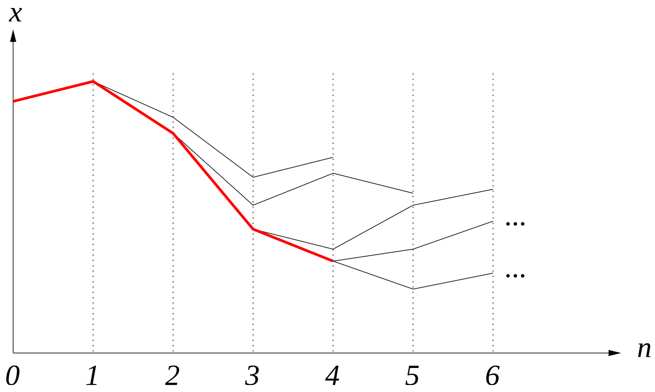

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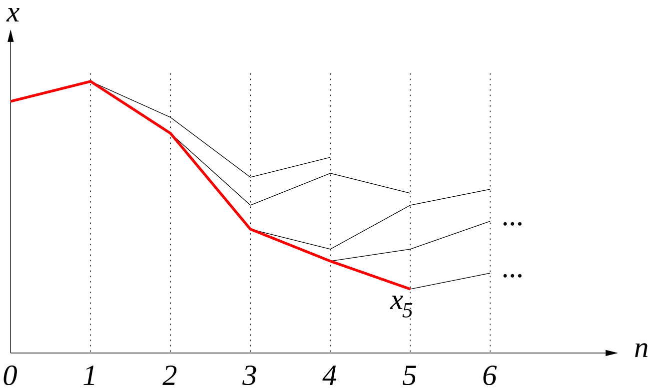

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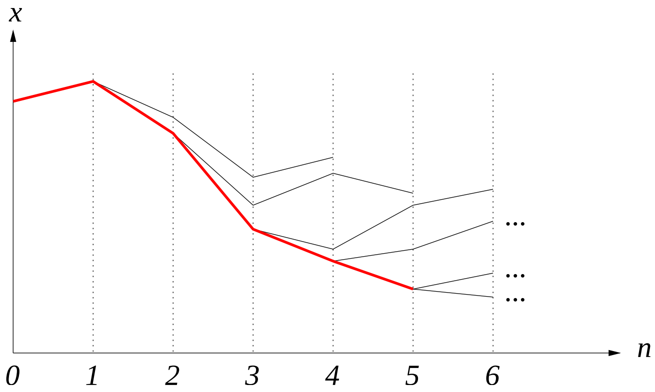

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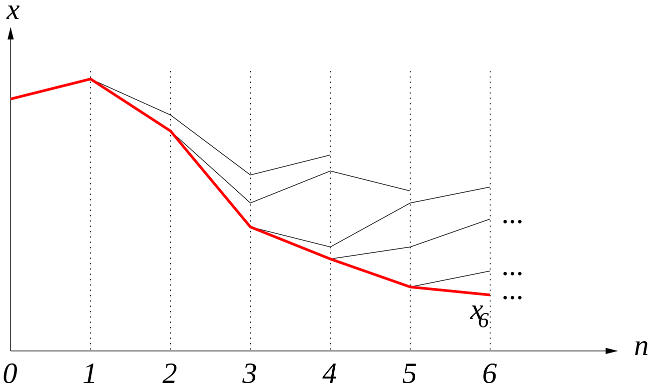

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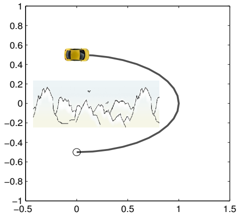
and, of course, the development of good algorithms (not topic of this course)

An example



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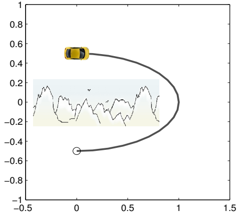

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with $\varphi = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$

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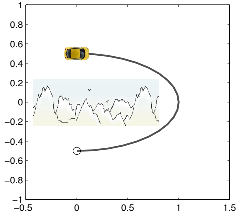

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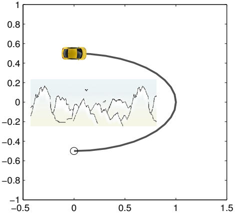

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MPC with $\ell(x,u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$ yields asymptotic stability for $N = 11$ but not for $N \leq 10$

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- the feedback value $\mu_N(x_0)$ is the first element of the resulting optimal control sequence

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- MPC computes the feedback law by iteratively solving finite horizon optimal control problems using the current state $x_0 = x_{\mu_N}(n)$ as initial value
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  $\rightsquigarrow$ model reduction in time
- the car-and-mountain example shows that MPC does not always yield an asymptotically stabilizing feedback law

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(2a) Background material:

Lyapunov functions

Purpose of this section

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with $x \in X$ or, in long form

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(later we will apply the results to $g(x) = f(x, \mu_N(x))$)

Note: we do not require g to be continuous

Comparison functions

For $\mathbb{R}_0^+ = [0, \infty)$ we use the following classes of comparison functions

$$\mathcal{K} := \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,\middle|\, \begin{array}{l} \alpha \text{ is continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array} \right\}$$

$$\mathcal{K}_\infty := \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,\middle|\, \alpha \in \mathcal{K} \text{ and } \alpha \text{ is unbounded} \right\}$$

$$\mathcal{KL} := \left\{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,\middle|\, \begin{array}{l} \beta \text{ is continuous,} \\ \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \in \mathbb{R}_0^+ \\ \text{and } \beta(r, \cdot) \text{ is strictly de-} \\ \text{creasing to } 0 \text{ for all } r \in \mathbb{R}_0^+ \end{array} \right\}$$

Asymptotic stability revisited

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We say that x_* is asymptotically stable for $x^+ = g(x)$ on a forward invariant set Y if there exists $\beta \in \mathcal{KL}$ such that

$$\|x(n) - x_*\| \leq \beta(\|x(0) - x_*\|, n)$$

holds for all $x \in Y$ and $n \in \mathbb{N}$

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How can we check whether this property holds?

Lyapunov function

Let $Y \subseteq X$ be a forward invariant set and $x_* \in X$. A function $V : Y \to \mathbb{R}_0^+$ is called a Lyapunov function for $x^+ = g(x)$ if the following two conditions hold for all $x \in Y$:

(i) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$$

(ii) There exists $\alpha_V \in \mathcal{K}$ such that

$$V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$$

Stability theorem

Theorem: If the system $x^+ = g(x)$ admits a Lyapunov function V on a forward invariant set Y, then x_* is an asymptotically stable equilibrium on Y

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The bounds $\alpha_1(\|x - x_*\|) \le V(x) \le \alpha_2(\|x - x_*\|)$ imply that asymptotic stability holds with $\beta(r,t) = \alpha_1^{-1}(\tilde{\beta}(\alpha_2(r), t))$

Lyapunov functions — discussion

While the convergence $x(n) \to x_*$ is typically non-monotone for an asymptotically stable system, the convergence $V(x(n)) \to 0$ is strictly monotone

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But it is in general difficult to find a candidate for a Lyapunov function

For MPC, we will use the optimal value functions which we introduce in the next section

(2b) Background material: Dynamic Programming

Purpose of this section

We define the optimal value functions V_N for the optimal control problem

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

used within the MPC scheme (with $x_0 = x_{\mu_N}(n)$)

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We present the dynamic programming principle, which establishes a relation for these functions and will eventually enable us to derive conditions under which V_N is a Lyapunov function

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Note: an optimal $\mathbf{u}^\star$ does not need to exist in general. In the sequel we assume that $\mathbf{u}^\star$ exists if x_0 is feasible

Dynamic Programming Principle

Theorem: (Dynamic Programming Principle) For any feasible $x_0 \in \mathbb{X}$ the optimal value function satisfies

$$V_N(x_0) = \inf_{\substack{u \in \mathbb{U} \\ f(x_0,u) \in \mathbb{X}}} \{\ell(x_0, u) + V_{N-1}(f(x_0, u))\}$$

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Moreover, if $\mathbf{u}^\star$ is an optimal control, then

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Idea of Proof: Follows by taking infima in the identity

$$\begin{aligned} J_N(x_0, \mathbf{u}) &= \ell(x_\mathbf{u}(0), \mathbf{u}(0)) + \sum_{k=1}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)) \\ &= \ell(x_0, \mathbf{u}(0)) + J_{N-1}(f(x_0, \mathbf{u}(0)), \mathbf{u}(\cdot + 1)) \end{aligned}$$

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Corollary: Let $x^\star$ be an optimal trajectory of length N with optimal control $u^\star$ and $x^\star(0) = x$.

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$$\mu_N(x) \in \operatorname*{argmin}_{u \in \mathbb{U}} \{\ell(x,u) + V_{N-1}(f(x,u))\}$$

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and

$$u^\star(k) = \mu_{N-k}(x^\star(k)), \quad k = 0, \ldots, N-1$$

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Moreover, for simple systems the principle can be used for computing V_N and μ_N — we will see an example in the excercises

(2c) Background material:

Relaxed Dynamic Programming

Infinite horizon optimal control

Just like the finite horizon problem we can define the infinite horizon optimal control problem

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \;\; J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \;\; x_{\mathbf{u}}(0) = x_0$$

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If we could compute an optimal feedback μ_∞ for this problem (which is — in contrast to computing μ_N — in general a very difficult problem), we would have solved the stabilization problem

Infinite horizon dynamic programming principle

Recall the corollary from the finite horizon dynamic programming principle

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Relaxing dynamic programming

Unfortunately, an equation of the type

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"relaxed dynamic programming inequality" [Rantzer et al. '06ff]

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What can we conclude from this inequality?

Relaxed dynamic programming

We define the infinite horizon performance of the MPC closed loop system $x^+ = f(x, \mu_N(x))$ as

$$J_\infty^{cl}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \;\; x_{\mu_N}(0) = x_0$$

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Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a forward invariant set for the MPC closed loop and assume that

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

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Relaxed dynamic programming

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$$J_\infty^{cl}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \;\; x_{\mu_N}(0) = x_0$$

Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a forward invariant set for the MPC closed loop and assume that

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

Then for all $x \in Y$ the infinite horizon performance satisfies

$$J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$$

Relaxed dynamic programming

Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the inequalities

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N.

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Proof: The assumed inequalities immediately imply that $V = V_N$ is a Lyapunov function for $x^+ = g(x) = f(x, \mu_N(x))$ with

$$\alpha_1(r) = \alpha_3(r), \quad \alpha_V(r) = \alpha\, \alpha_3(r)$$

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$\Rightarrow$ asymptotic stability

Relaxed dynamic programming

For proving the performance estimate $J_\infty^{cl}(x_0, \mu_N) \le V_N(x_0)/\alpha$, the relaxed dynamic programming inequality implies

$$\begin{aligned}
&\alpha \sum_{n=0}^{K-1} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \\
\le\ & \sum_{n=0}^{K-1} \Big(V_N(x_{\mu_N}(n)) - V_N(x_{\mu_N}(n+1)) \Big) \\
=\ & V_N(x_{\mu_N}(0)) - V_N(x_{\mu_N}(K)) \ \le\ V_N(x_{\mu_N}(0))
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- The performance of the MPC controller can be measured by looking at the infinite horizon value along the MPC closed loop trajectories
- Relaxed dynamic programming gives us conditions under which both asymptotic stability and performance results can be derived

Application of background results

The main task will be to verify the assumptions of the relaxed dynamic programming theorem, i.e.,

$$V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

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To this end, we present two different approaches:

- modify the optimal control problem in the MPC loop by adding terminal constraints and costs
- derive assumptions on f and ℓ under which MPC works without terminal constraints and costs

(3) Stabilizing Model Predictive Control with stabilizing terminal conditions

V_N as a Lyapunov Function

Problem: Prove that the MPC feedback law μ_N is stabilizing

V_N as a Lyapunov Function

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Approach: Verify the assumptions

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of the relaxed dynamic programming theorem for the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \ \ x_{\mathbf{u}}(0) = x_0$$

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(where “small” means that the error can be compensated replacing $\ell(x, \mu_N(x))$ by $\alpha\ell(x, \mu_N(x))$ with $\alpha \in (0, 1)$)

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$\rightsquigarrow$ additional stabilizing constraints were proposed

Terminal constraints and cost

Optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

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Idea: add local Lyapunov function $F : \mathbb{X}_0 \to \mathbb{R}_0^+$ as terminal cost

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N))$$

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F is defined on a region $\mathbb{X}_0$ around x_* which is imposed as terminal constraint $\quad x(N) \in \mathbb{X}_0$

[Chen & Allgöwer '98, Jadbabaie et al. '98 . . .]

Terminal constraints and cost

We thus change the optimal control problem to

$$\underset{\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)}{\text{minimize}} \quad J_N(x_0,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k),\mathbf{u}(k)) \; + \; F(x_{\mathbf{u}}(N))$$

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Which properties do we need for F and $\mathbb{X}_0$ in order to ensure

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With suitable assumptions we can even avoid the “small error”

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Assumptions on $F : \mathbb{X}_0 \to \mathbb{R}_0^+$ and $\mathbb{X}_0$

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There exists a controller $\kappa : \mathbb{X}_0 \to \mathbb{U}$ with the following properties:

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Simplest choice: $\mathbb{X}_0 = \{x_*\}$, $F \equiv 0$, $\kappa \equiv 0$ (if $f(x_*, 0) = x_*$)

Prolongation of control sequences

Let $\tilde{\mathbf{u}} \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$

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By (ii) the stage cost of the prolongation is bounded by

$$\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1)) \leq F(x_{\mathbf{u}}(N-1)) - F(x_{\mathbf{u}}(N))$$

Reversal of $V_{N-1} \leq V_N$

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Let $\tilde{\mathbf{u}}^\star \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ be the optimal control for J_{N-1}, i.e.,

$$V_{N-1}(x_0) = J_{N-1}(x_0, \tilde{\mathbf{u}}^\star)$$

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\end{aligned}$$

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$\rightsquigarrow$ as desired, we obtain $V_{N-1} \geq V_N$

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Feasible sets

Define the feasible set

$$\mathbb{X}_N := \{x \in \mathbb{X} \,|\, \mathbb{U}^N_{\mathbb{X}_0}(x) \neq \emptyset\}$$

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$\rightsquigarrow$ the MPC control is only defined on $\mathbb{X}_N$!

Stability theorem

Theorem: Consider the MPC scheme with regional terminal constraint $x_{\mathbf{u}}(N) \in \mathbb{X}_0$ and Lyapunov function terminal cost F compatible with ℓ.

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Proof: Combine dynamic programming with $V_{N-1} \geq V_N$

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In Section (4) we will see how stability can be proved without stabilizing terminal constraints

Suboptimality

Recall: For both stabilizing terminal constraints the relaxed dynamic programming theorem yields the estimate

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Without terminal constraints, the inequality $V_N \leq V_\infty$ is immediate

However, the terminal constraints also reverse this inequality, i.e., we have $V_N \geq V_\infty$ and the gap is very difficult to estimate

Suboptimality — example

We consider two examples with $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \mathbb{R}$ for $N = 2$

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General estimates for fixed N appear difficult to obtain. But we can give an asymptotic result for $N \to \infty$

Asymptotic Suboptimality

Theorem: For both types of terminal constraints the assumptions of the stability theorems ensure

$$V_N(x) \to V_\infty(x)$$

and thus

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Idea of proof: uses that any approximately optimal trajectory for J_∞ converges to x_* and can thus be modified to meet the constraints with only moderately changing its value

Summary of Section (3)

- terminal conditions reverse the usual inequality $V_{N-1} \leq V_N$ to $V_{N-1} \geq V_N$

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- this enables us to derive the relaxed dynamic programming inequality (with $\alpha = 1$) from the dynamic programming principle
- the operating region is restricted to the feasible set $\mathbb{X}_N$
- $J^{cl}_\infty(x, \mu_N) \leq V_N(x)$ holds and $V_N \to V_\infty$ for $N \to \infty$, but $V_N \gg V_\infty$ is possible

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(4) Stabilizing Model Predictive Control without stabilizing terminal conditions

MPC without stabilizing terminal constraints

We return to the basic MPC formulation

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_0,u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0 = x_{\mu_N}(n)$$

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In order to motivate why we want to avoid terminal constraints and costs, we consider an example of P double integrators in the plane

Motivation for avoiding terminal conditions

Example: [Annunziato/Borzì '10ff., Fleig '14ff.] The Fokker-Planck Equation

$$
\begin{aligned}
\partial_t y(x,t) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\Big(a_{ij}(x,t)y(x,t)\Big) + \sum_{i=1}^{d} \partial_{x_i}\Big(b_i(x,t;u))y(x,t)\Big) &= 0 \\
y(\cdot,0) &= y_0
\end{aligned}
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is a parabolic PDE describing the evolution of a probability density function for stochastic control systems

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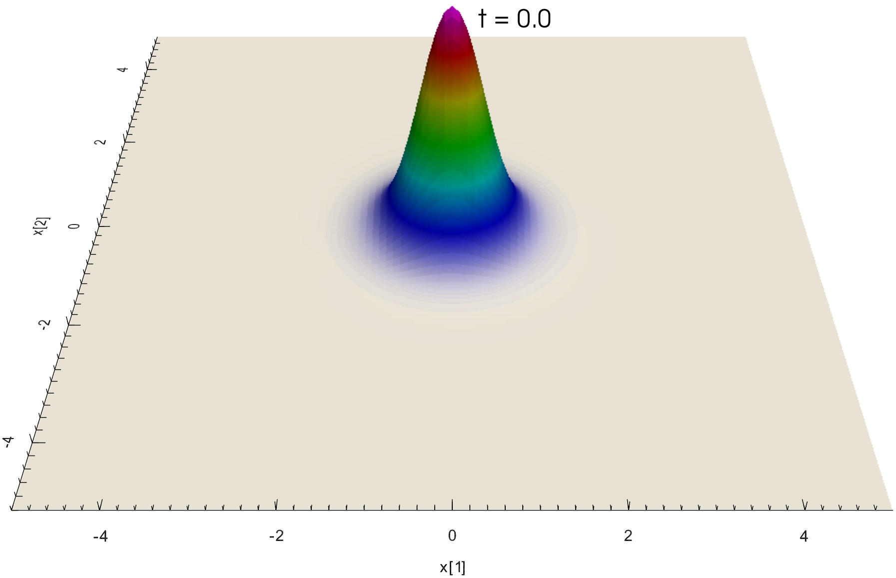
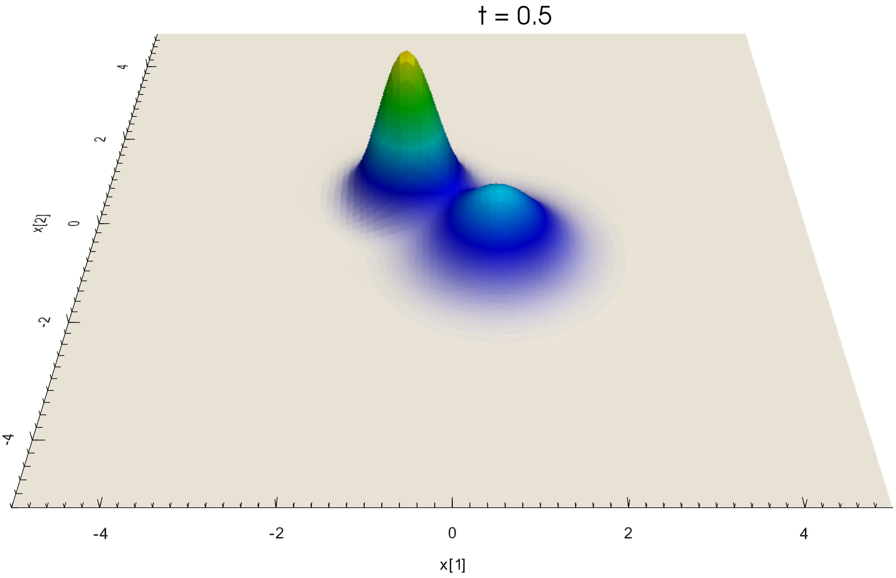
Reference density function



Reference density function



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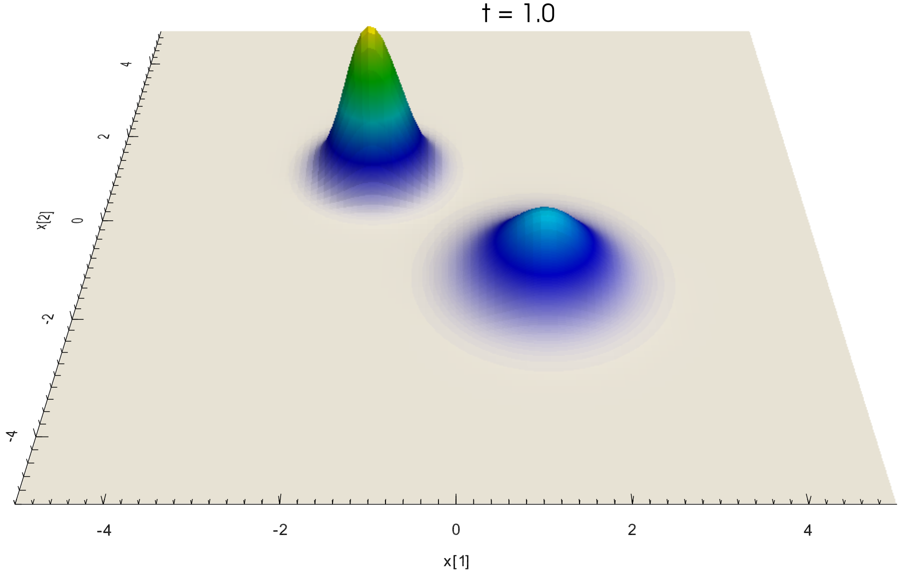

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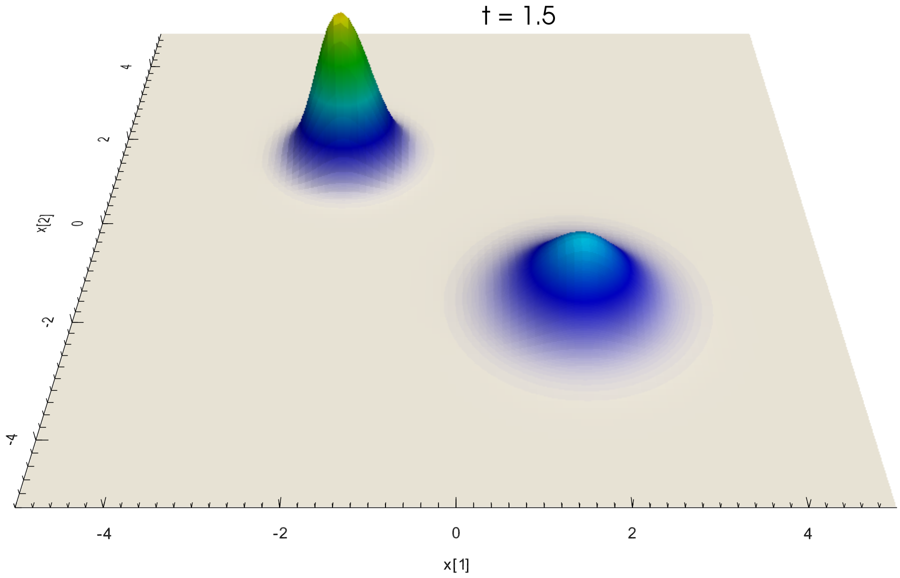

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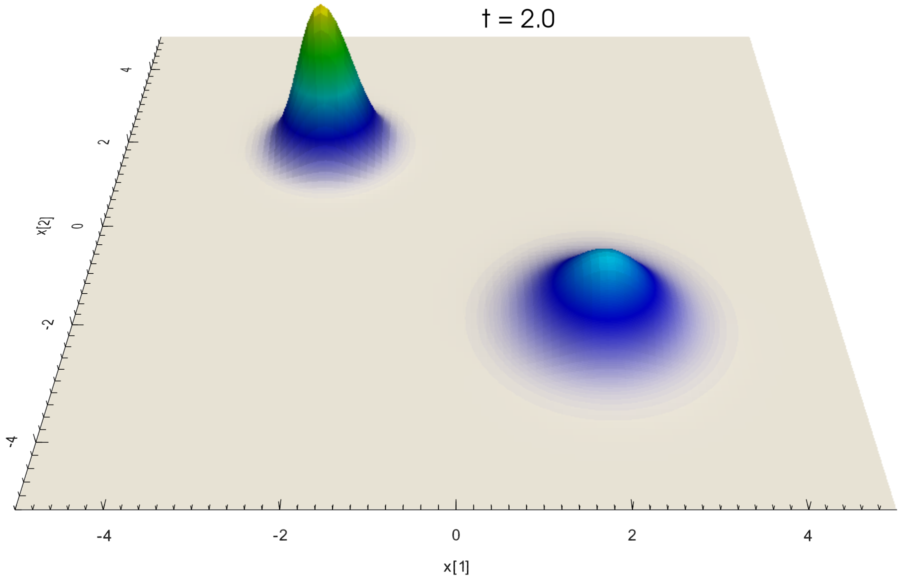

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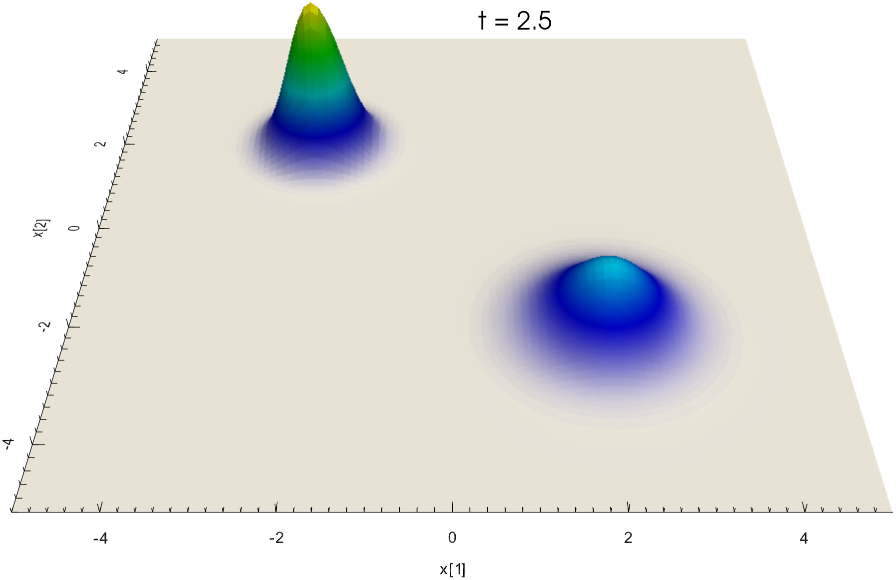

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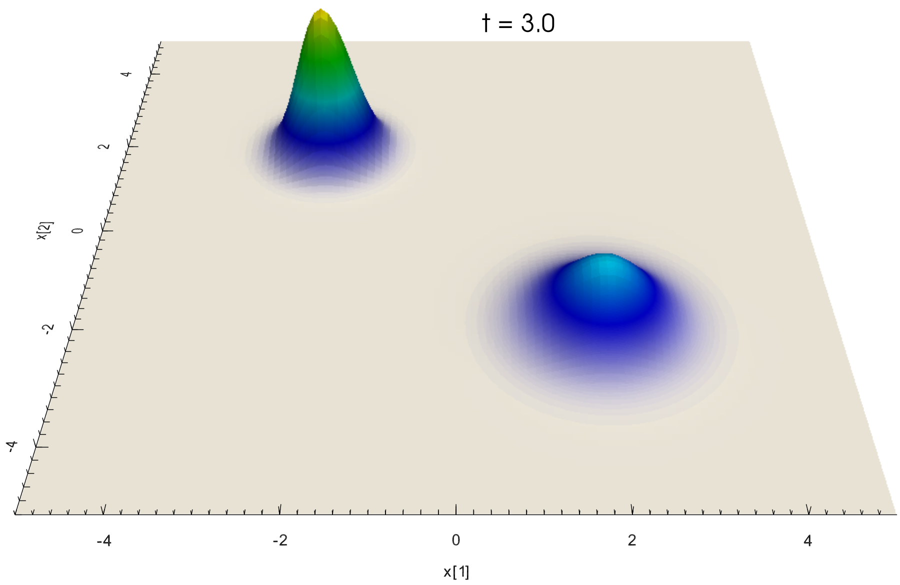

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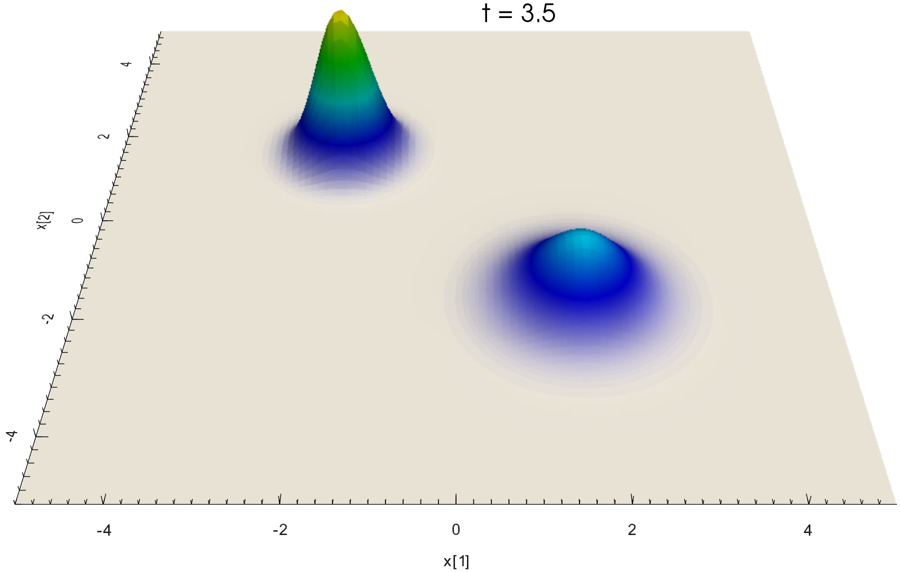

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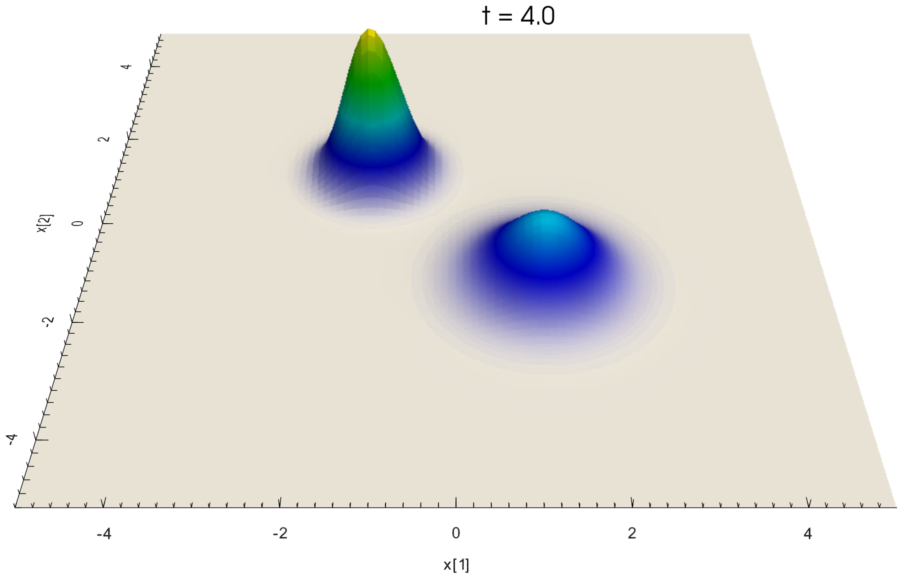
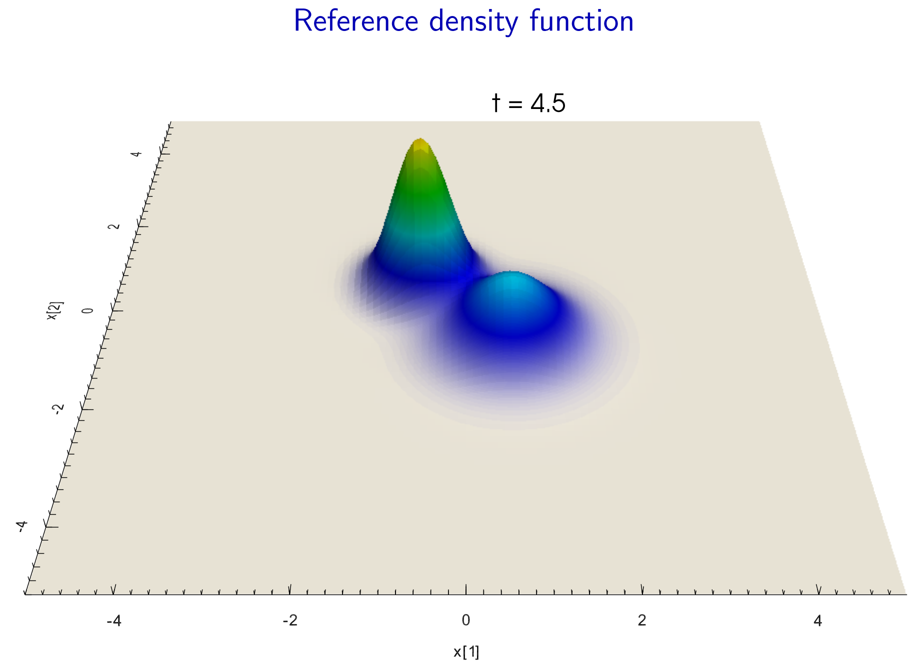
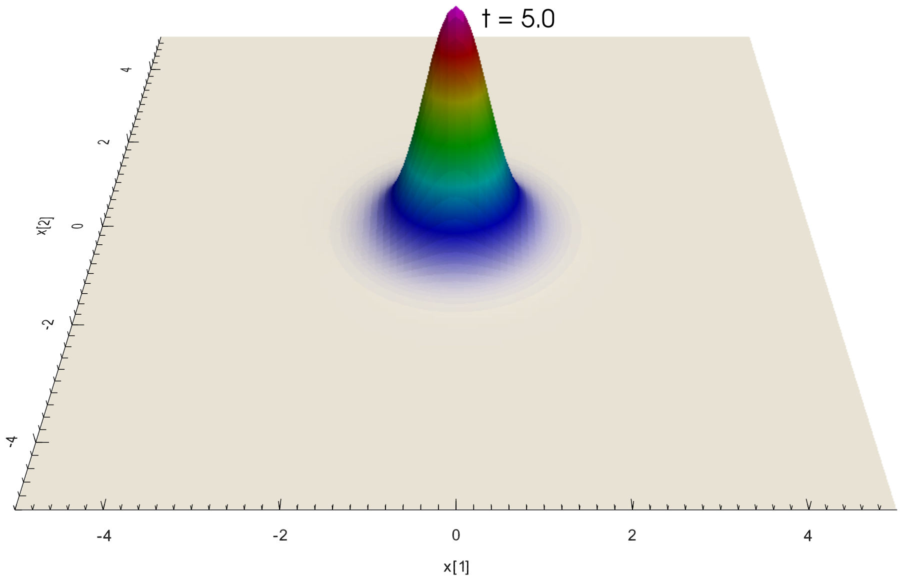
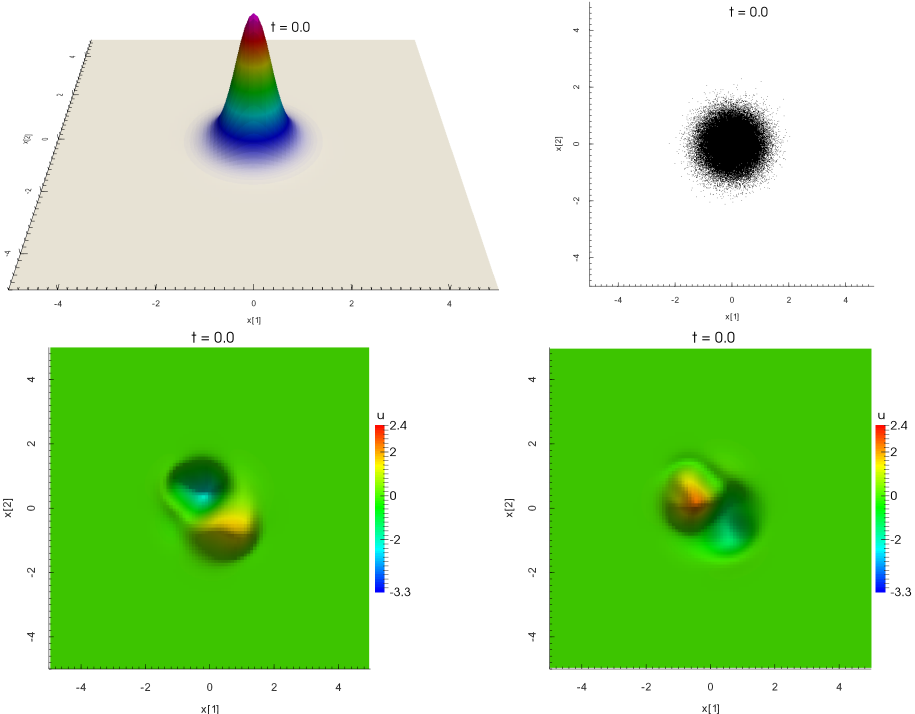


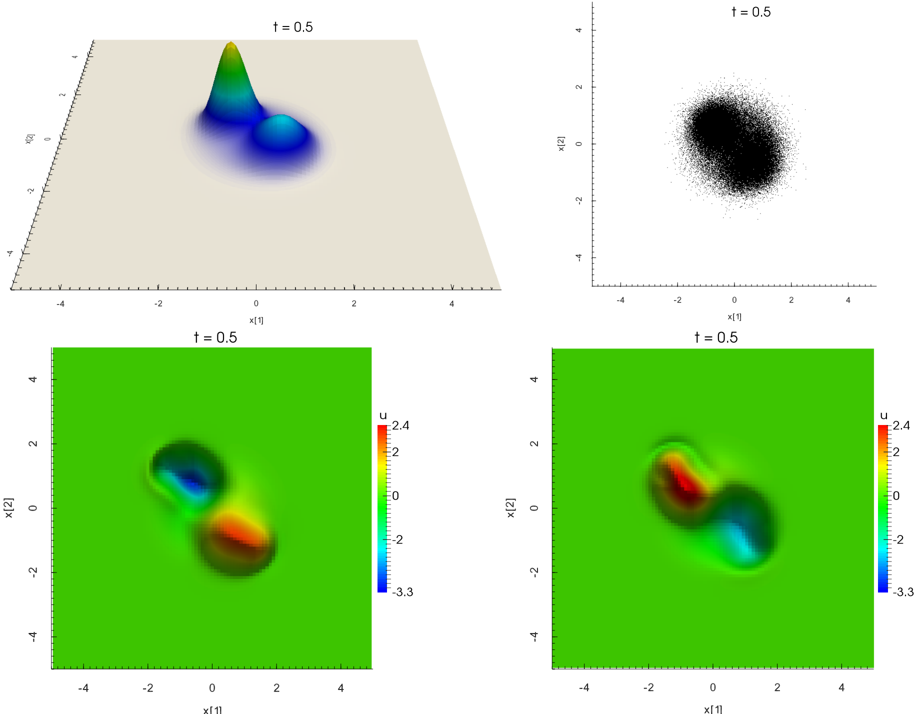

Reference density function
t = 4.5
x[2]
4
2
0
-2
-4
x[1]
-4
-2
0
2
4

Reference density function





t = 0.0
x[1]
x[2]
u
2.4
-3.3


t = 0.5
x[1]
x[2]
t = 0.5
x[1]
x[2]
t = 0.5
u
2.4
2
0
-2
-3.3
x[1]
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2.4
2
0
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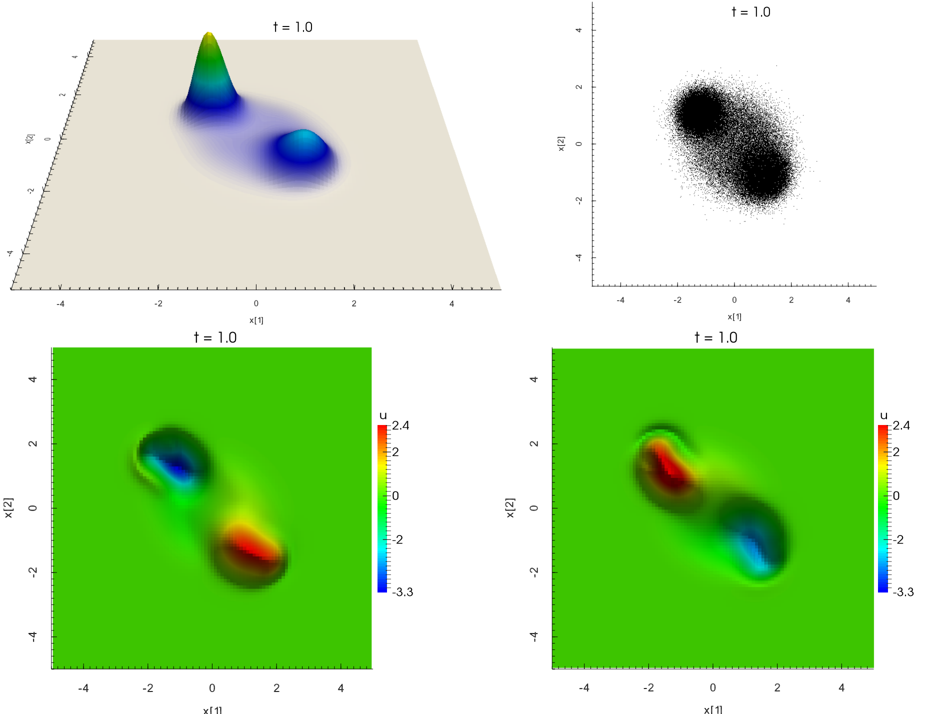
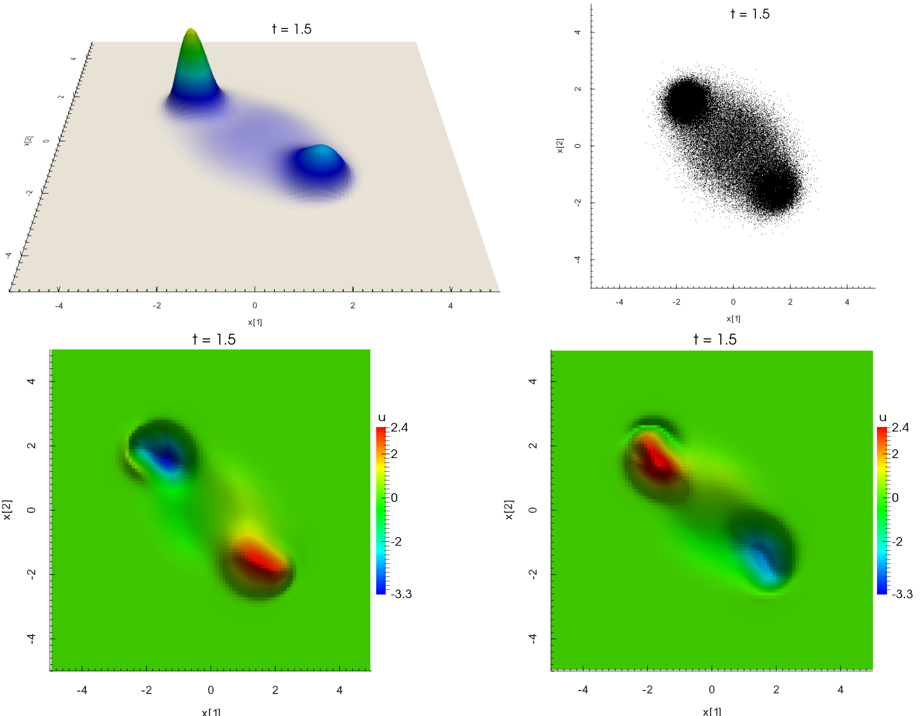
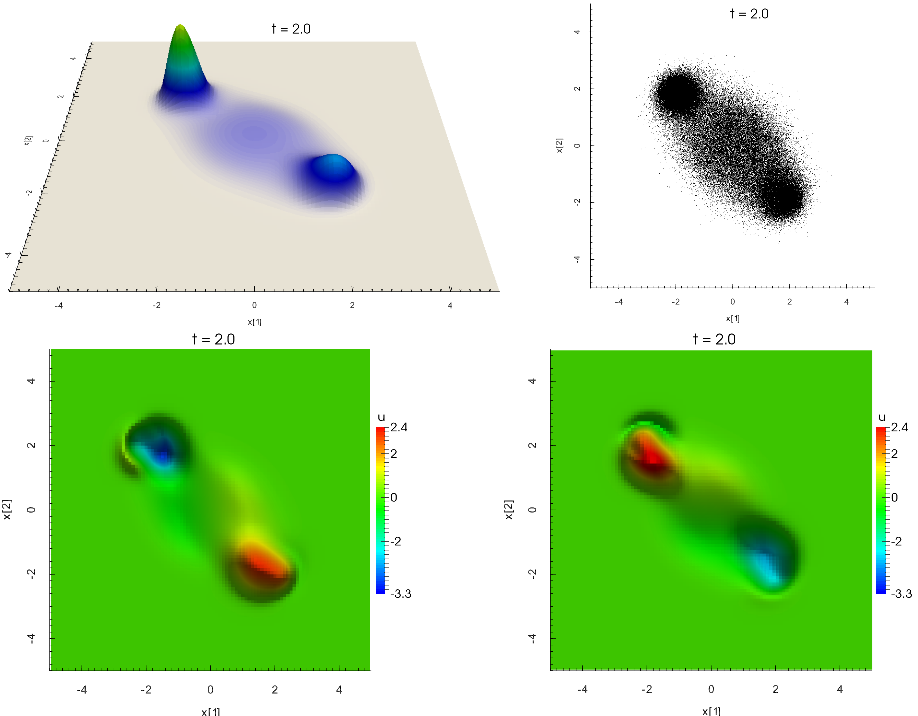
x[1]
x[2]



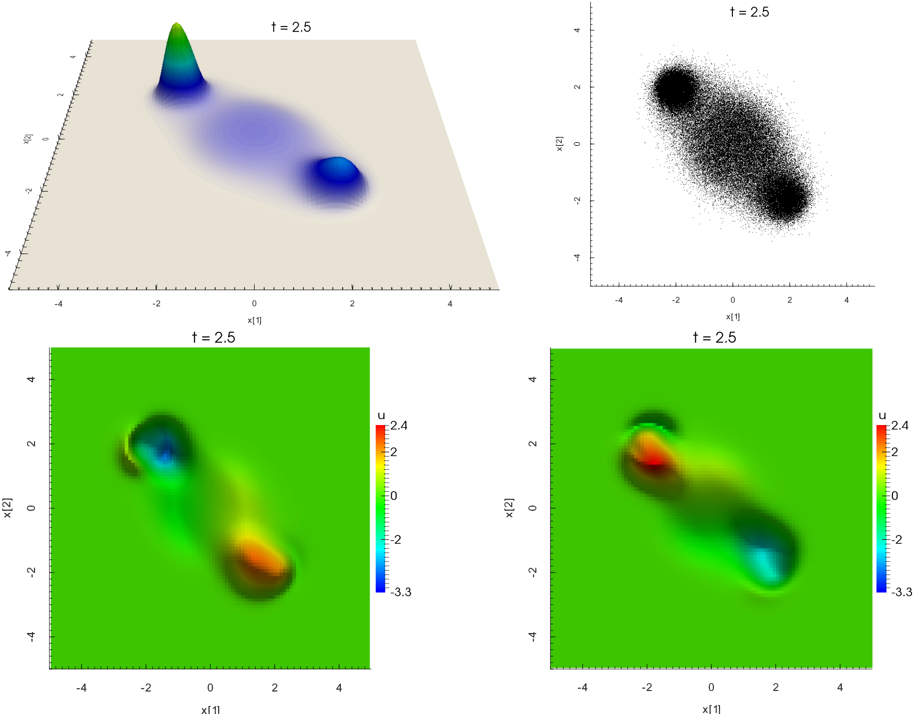
t = 1.0
x[1]
x[2]
u
2.4
2
0
-2
-3.3



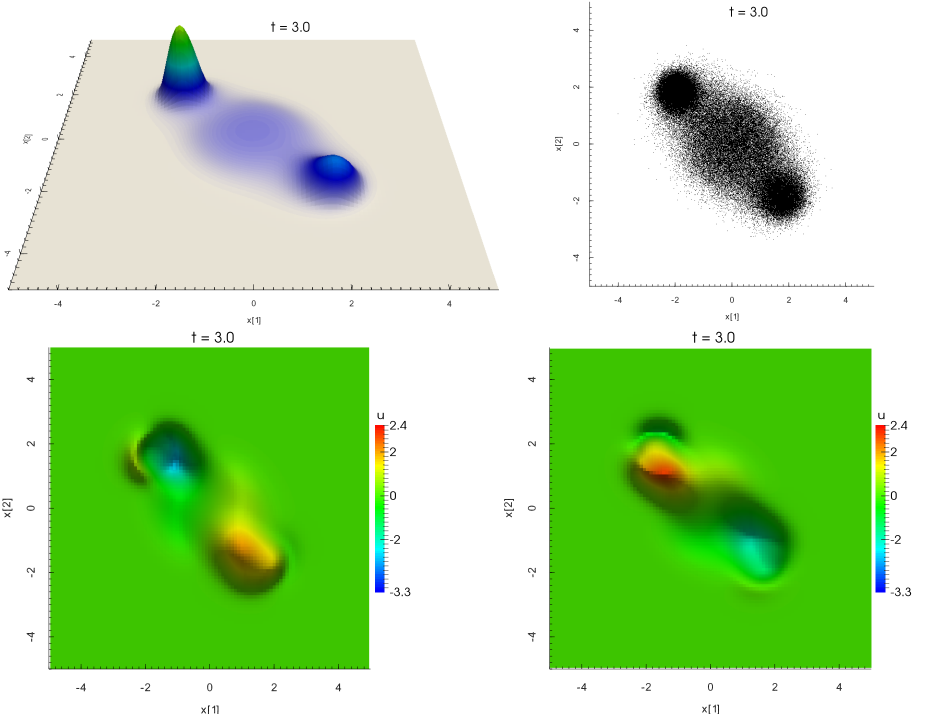
t = 1.5
x[1]
x[2]
u
2.4
2
0
-2
-3.3

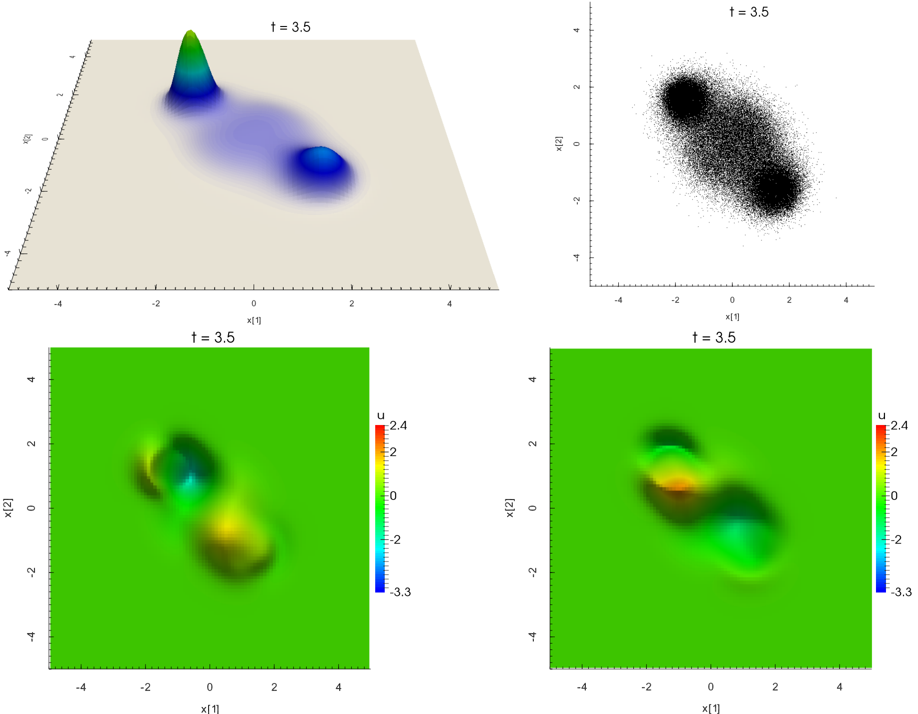


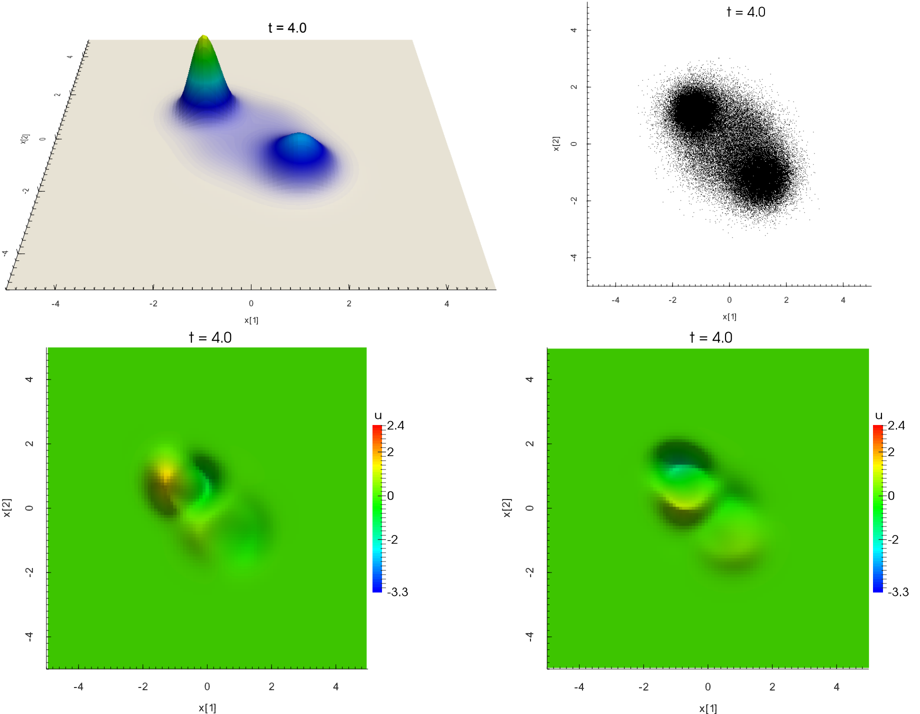
t = 2.0
x[1]
x[2]
t = 2.0
x[1]
x[2]
t = 2.0
u
2.4
2
0
-2
-3.3
x[1]
x[2]
t = 2.0
u
2.4
2
0
-2
-3.3
x[1]
x[2]



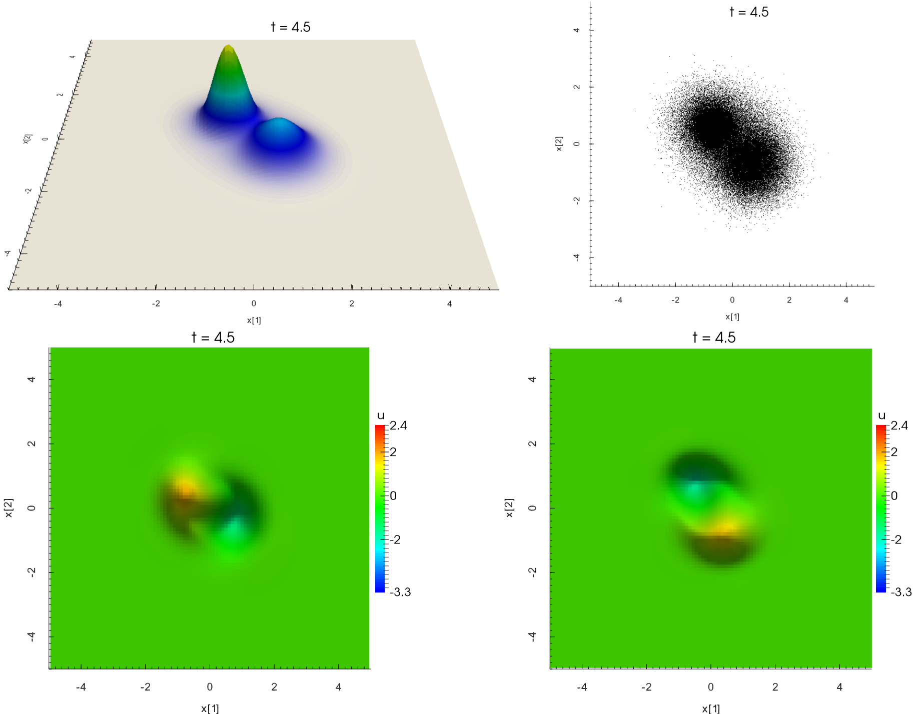
t = 2.5
x[1]
x[2]
t = 2.5
x[1]
x[2]
t = 2.5
x[1]
x[2]
u
2.4
2
0
-2
-3.3
t = 2.5
x[1]
x[2]
u
2.4
2
0
-2
-3.3

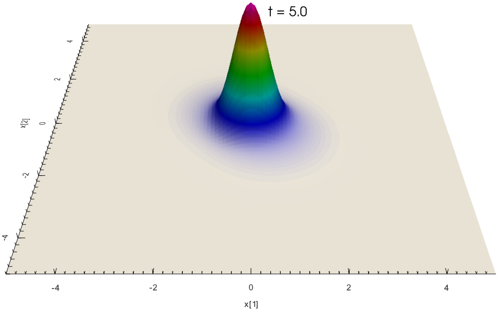

t = 3.0
x[1]
x[2]
u
2.4
2
0
-2
-3.3
-4
-2
0
2
4

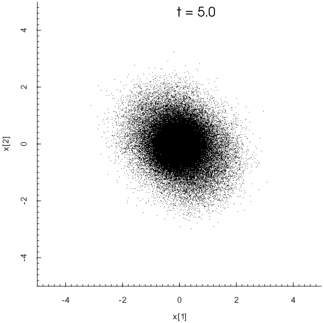

t = 3.5
x[1]
x[2]
t = 3.5
x[1]
x[2]
t = 3.5
u
2.4
2
0
-2
-3.3
x[1]
x[2]
t = 3.5
u
2.4
2
0
-2
-3.3
x[1]
x[2]



t = 4.0
x[1]
x[2]
t = 4.0
x[1]
x[2]
t = 4.0
u
2.4
2
0
-2
-3.3
x[1]
x[2]
t = 4.0
u
2.4
2
0
-2
-3.3
x[1]
x[2]


t = 4.5
x[1]
x[2]
t = 4.5
x[1]
x[2]
t = 4.5
x[1]
x[2]
u
2.4
2
0
-2
-3.3
t = 4.5
x[1]
x[2]
u
2.4
2
0
-2
-3.3


t = 5.0
x[1]
x[2]
-4
-2
0
2
4


t = 5.0
x[1]
x[2]
-4
-2
0
2
4

Stabilizing NMPC without terminal constraint

(Some) stability and performance results known in the literature:

[Alamir/Bornard '95]

[Shamma/Xiong '97, Primbs/Nevistić '00]

[Jadbabaie/Hauser '05]

[Grimm/Messina/Tuna/Teel '05, Tuna/Messina/Teel '06, Gr./Rantzer '08, Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

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use bounds on optimal value functions

Here we explain the last approach

Bounds on the optimal value function

Recall the definition of the optimal value function

$$V_N(x) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k,x), \mathbf{u}(k))$$

Boundedness assumption: there exists $\gamma > 0$ with

$$V_N(x) \le \gamma \ell^\star(x) \quad \text{for all } x \in \mathbb{X}, N \in \mathbb{N}$$

where $\ell^\star(x) := \min_{u \in \mathbb{U}} \ell(x,u)$

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(sufficient conditions for and relaxations of this bound will be discussed later)

Stability and performance index

We choose ℓ, such that

$$\alpha_3(\|x - x_*\|) \leq \ell^\star(x) \leq \alpha_4(\|x - x_*\|)$$

holds for $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ (again, $\ell(x,u) = \|x - x_*\|^2 + \lambda \|u\|^2$ works)

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Then, the only inequality left to prove in order to apply the relaxed dynamic programming theorem is

$$V_N(f(x, \mu_N(x))) \le V_N(x) - \alpha_N \ell(x, \mu_N(x))$$

for some $\alpha_N \in (0,1)$ and all $x \in \mathbb{X}$

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We can compute α_N from the bound $V_N(x) \leq \gamma \ell^\star(x)$

Computing α_N

We assume $\quad V_N(x) \le \gamma \ell^\star(x) \quad$ for all $\quad x \in \mathbb{X},\ N \in \mathbb{N} \quad (*)$

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We want $\quad V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$

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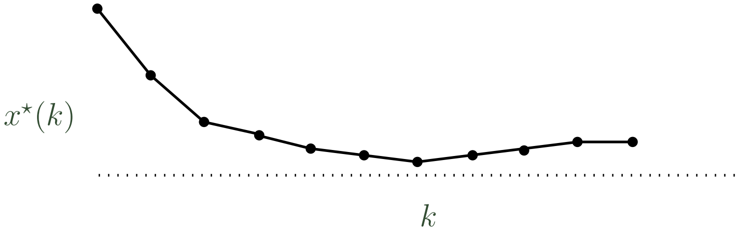

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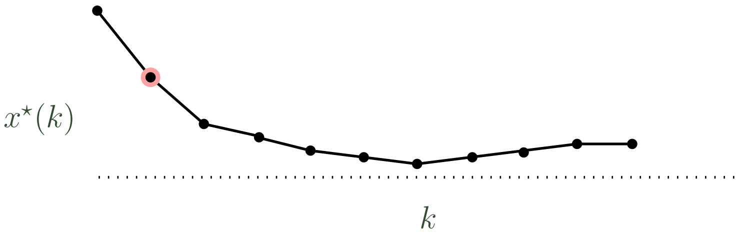

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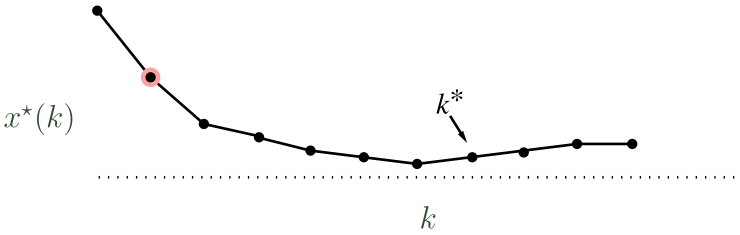
- use $(*)$ to find $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$



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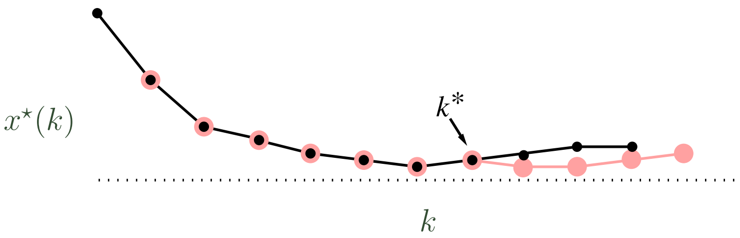

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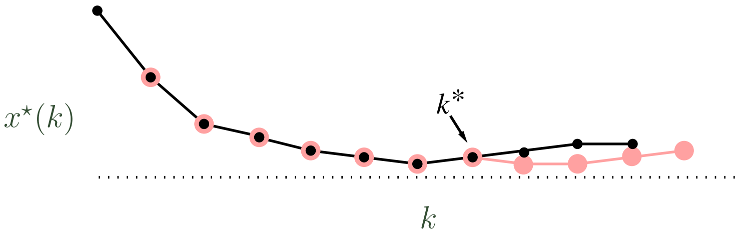

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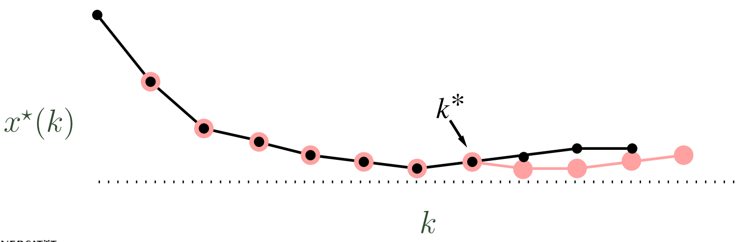
$$\Rightarrow\ V_N(x^\star(1)) \le J_N(x^\star(1), \tilde{\mathbf{u}}) \le V_N(x^\star(0)) - (1 - \gamma\eta_N)\, \ell(x^\star(0), \mathbf{u}^\star(0))$$



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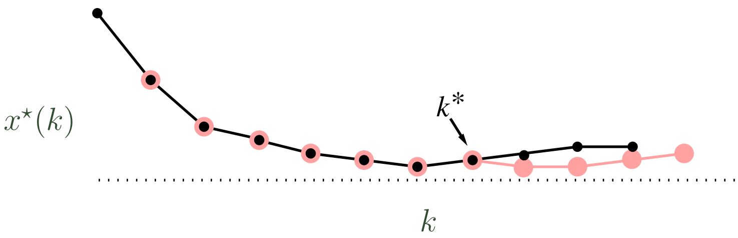

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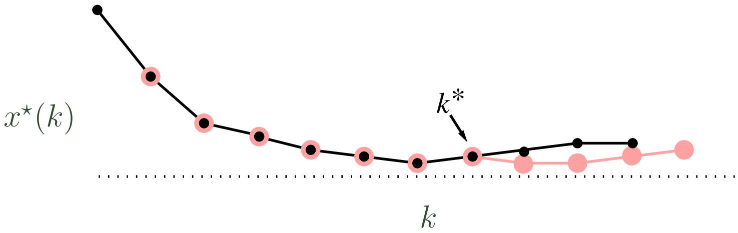

UNIVERSITÄT BAYREUTH

Decay of the optimal trajectory

We assume $V_N(x) \leq \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

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Variant 1 [Grimm/Messina/Tuna/Teel '05]

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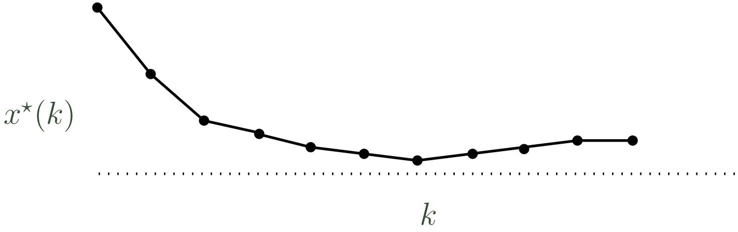

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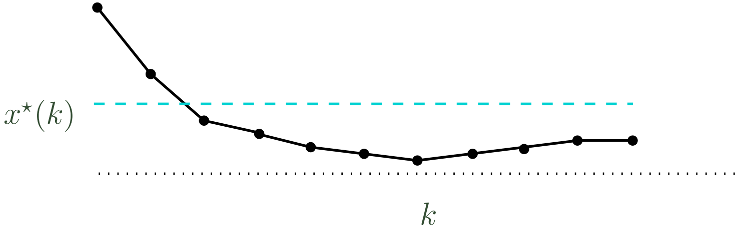
$V_N(x) \le \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \le \gamma \ell^\star(x)/N$ for at least one $k^\star$



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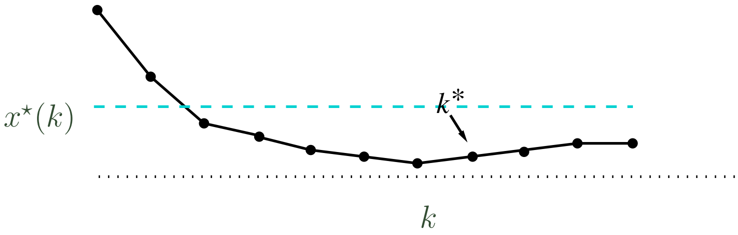

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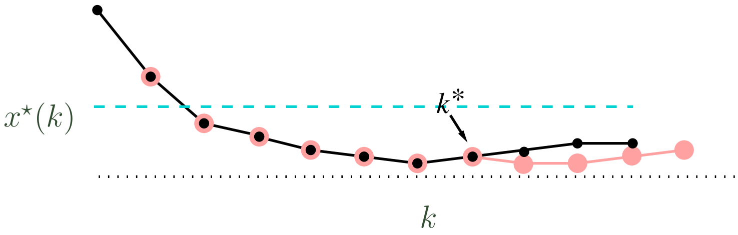

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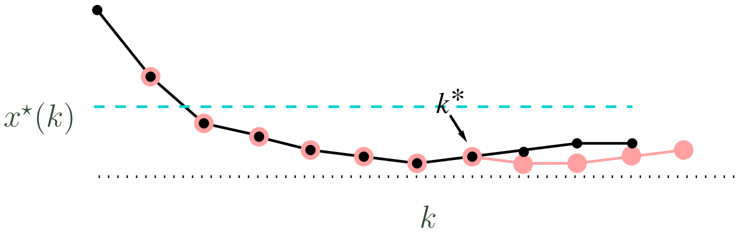
Variant 1 [Grimm/Messina/Tuna/Teel '05]

$V_N(x) \le \gamma \ell^\star(x) \;\Rightarrow\; \ell(x^\star(k), u^\star(k)) \le \gamma \ell^\star(x)/N$ for at least one $k^\star$ $\;\Rightarrow\; \alpha_N = 1 - \gamma(\gamma - 1)/N$



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Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

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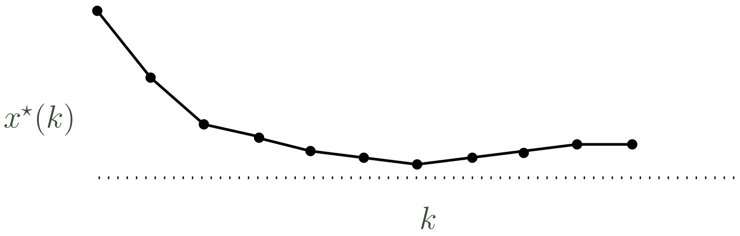
Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

$V_N(x) \leq \gamma \ell^\star(x)$



Decay of the optimal trajectory

We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

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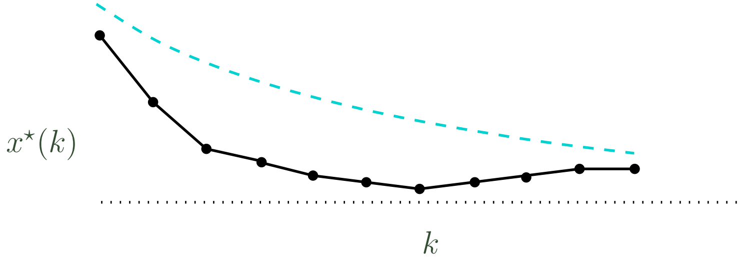
$$V_N(x) \le \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \le \gamma \left(\frac{\gamma-1}{\gamma}\right)^k \ell^\star(x)$$



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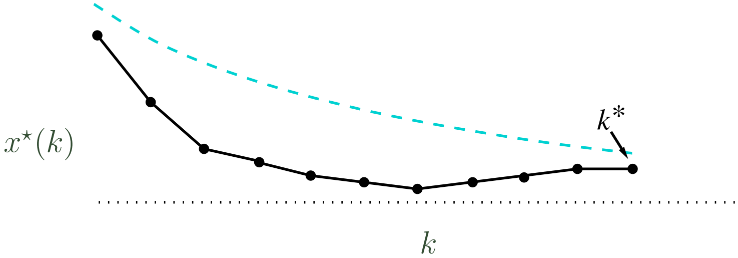

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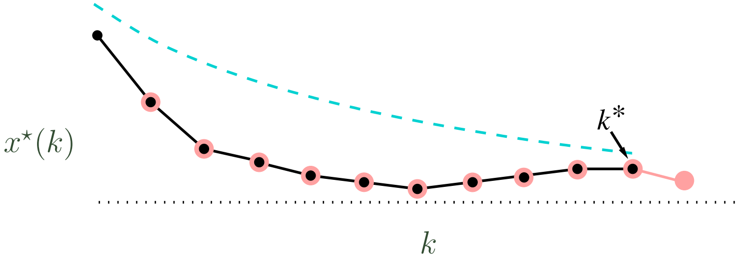

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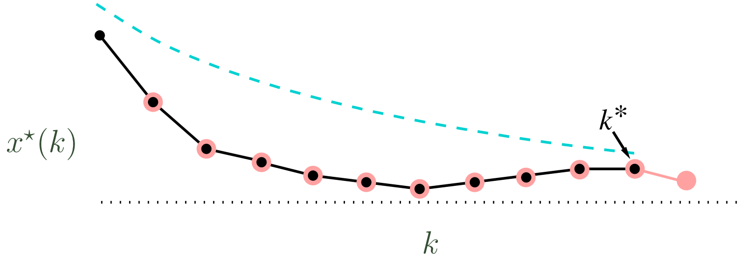

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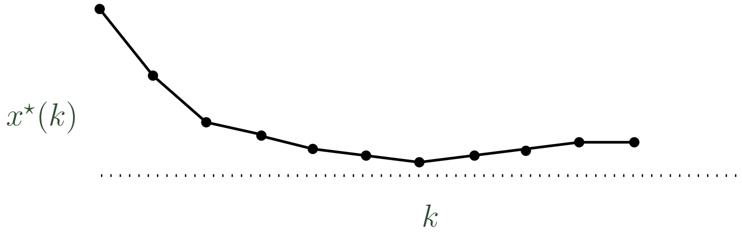

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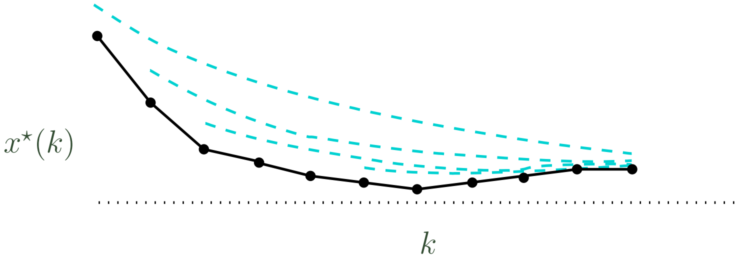
$V_N(x) \le \gamma \ell^\star(x) \quad \Rightarrow \quad$ formulate all constraints



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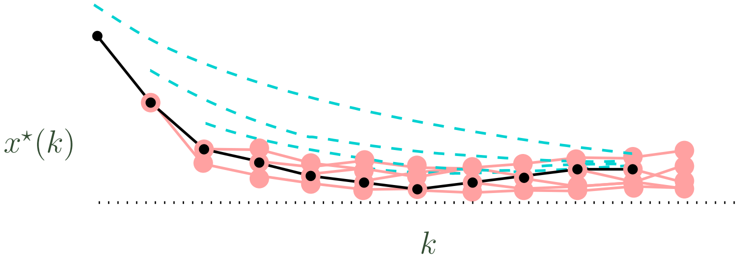

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$V_N(x) \le \gamma \ell^\star(x)$ $\Rightarrow$ formulate all constraints and trajectories

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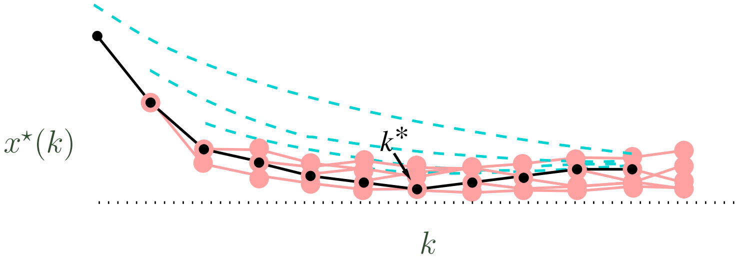

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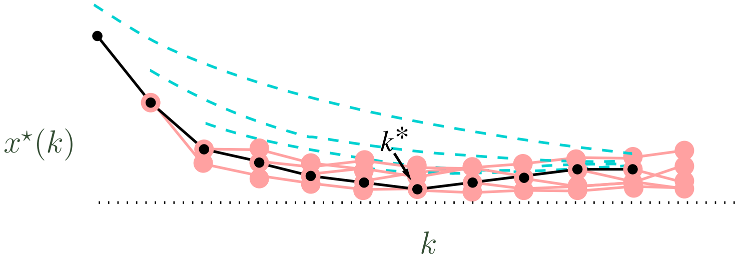

Optimization approach to compute α_N

We explain the optimization approach (Variant 3) in more detail. We want α_N such that

$$V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$$

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Verifying the relaxed Lyapunov inequality

Find α_N, such that for all optimal trajectories $x^\star$, $\mathbf{u}^\star$:

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The inequalities from the last slides translate to

$$\sum_{n=k}^{N-1} \lambda_n \leq \gamma \lambda_k, \quad k = 0, \ldots, N-2 \qquad (1)$$

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$\Rightarrow$ if α_N is such that the inequality

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This is a linear optimization problem whose solution can be computed explicitly (which is nontrivial) and reads

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}}$$

Stability and performance theorem

Theorem: [Gr./Pannek/Seehafer/Worthmann '10]: Assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$. If

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Conversely, if $N < 2 + \frac{\ln(\gamma-1)}{\ln\gamma - \ln(\gamma-1)}$, then there exists a system for which $V_N(x) \le \gamma \ell^\star(x)$ holds but the NMPC closed loop is not asymptotically stable.

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Horizon dependent γ-values

The theorem remains valid if we replace the bound condition

$$V_N(x) \le \gamma \ell^\star(x)$$

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$$\rightsquigarrow \quad \alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)}$$

Horizon dependent γ-values

The theorem remains valid if we replace the bound condition

$$V_N(x) \le \gamma \ell^\star(x)$$

by

$$V_N(x) \le \gamma_N \ell^\star(x)$$

for horizon-dependent bounded values $\gamma_N \in \mathbb{R}$, $N \in \mathbb{N}$

$$\rightsquigarrow \quad \alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)}$$

This allows for tighter bounds and a refined analysis

Controllability condition

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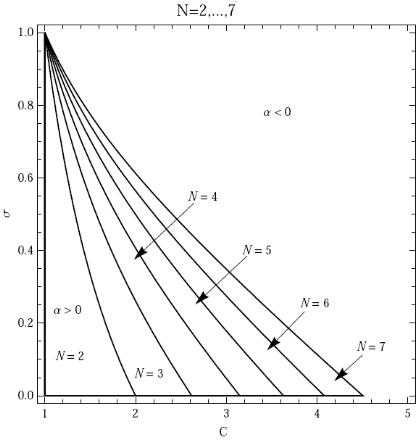
This allows to compute the minimal stabilizing horizon

$$\min\{N \in \mathbb{N} \,|\, \alpha_N > 0\}$$

depending on C and σ

Stability chart for C and σ



(Figure: Harald Voit)

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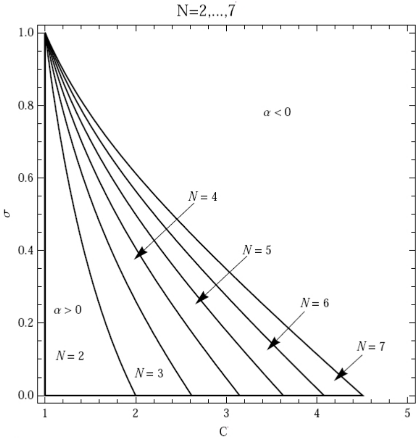

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Conclusion: for short optimization horizon N it is

more important: small C ("small overshoot")

less important: small σ ("fast decay")

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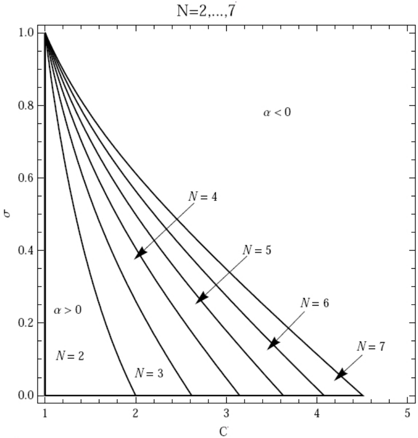

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(we will see at the end of the section how to use this information)

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- under appropriate uniformity assumptions, the results are easily carried over to tracking time variant references $x_{\text{ref}}(n)$ instead of an equilibrium x_* [Gr./Pannek '11]

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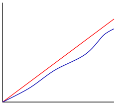

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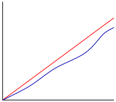

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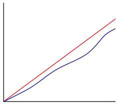

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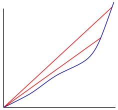

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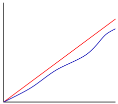

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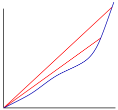

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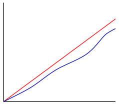

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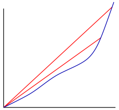
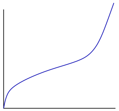


- $\rho \in \mathcal{K}_\infty$ arbitrary



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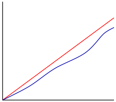

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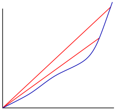

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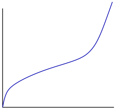
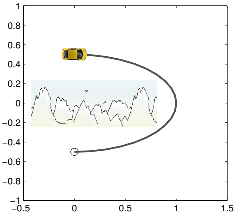


[Grimm/Messina/Tuna/Teel '05, Gr./Pannek '11]

The car-and-mountains example reloaded



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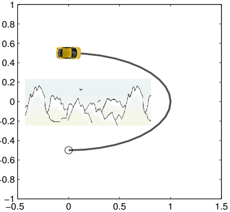

MPC with $\ell(x,u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$
$\rightsquigarrow$ asymptotic stability for $N = 11$ but not for $N \le 10$

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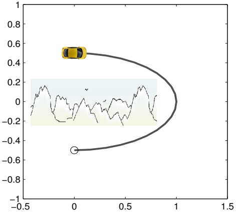

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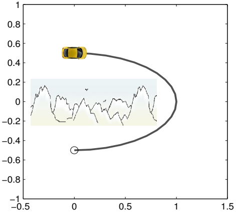

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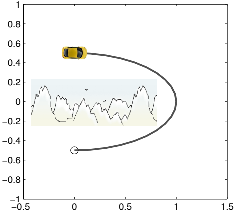

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$\ell(x,u) = (x_1 - x_{*,1})^2 + 5(x_2 - x_{*,2})^2 + |u|^2$ $\rightsquigarrow$ as. stab. for $N = 2$

A PDE example

We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0, 1]$

solution $y = y(t, x)$

boundary conditions $y(t, 0) = y(t, 1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

and distributed control $u : \mathbb{R} \times \Omega \to \mathbb{R}$

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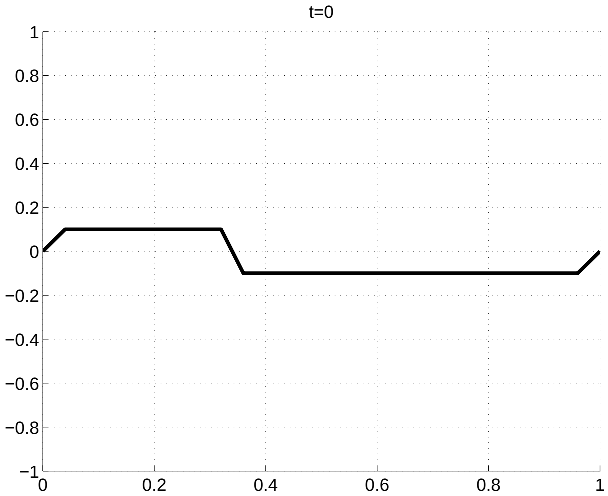
Discrete time system: $y(n) = y(nT, \cdot)$, sampling time $T = 0.025$

The uncontrolled PDE



uncontrolled ($u \equiv 0$)

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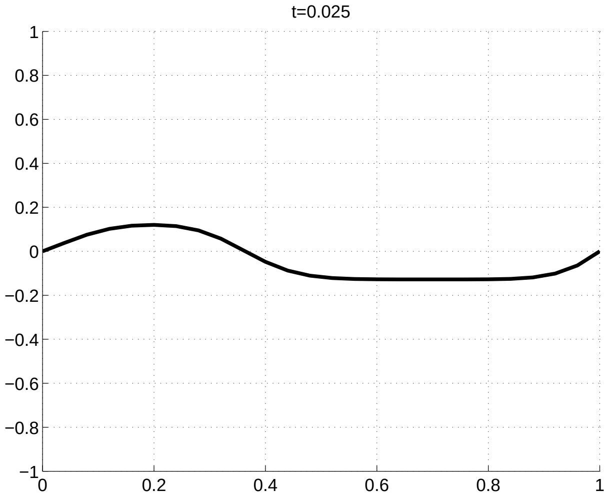

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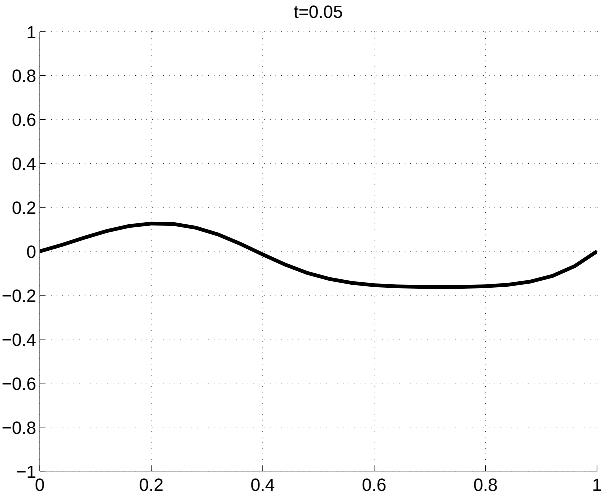

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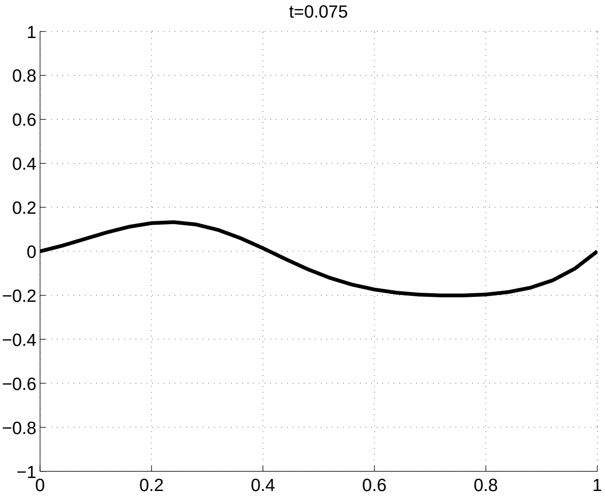

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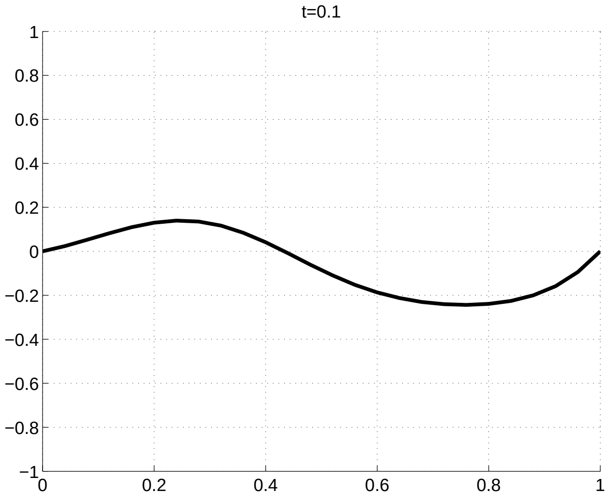

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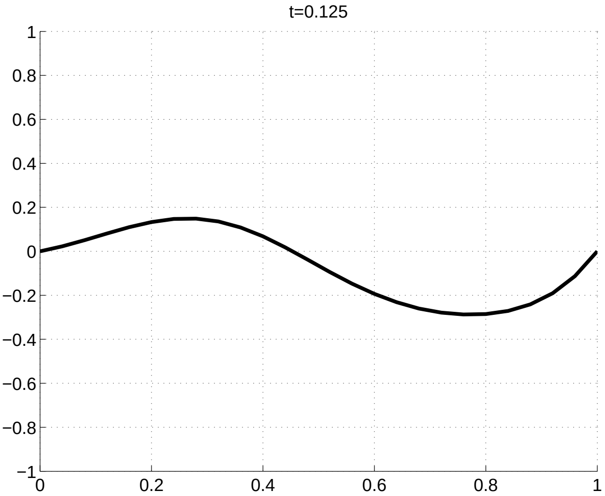

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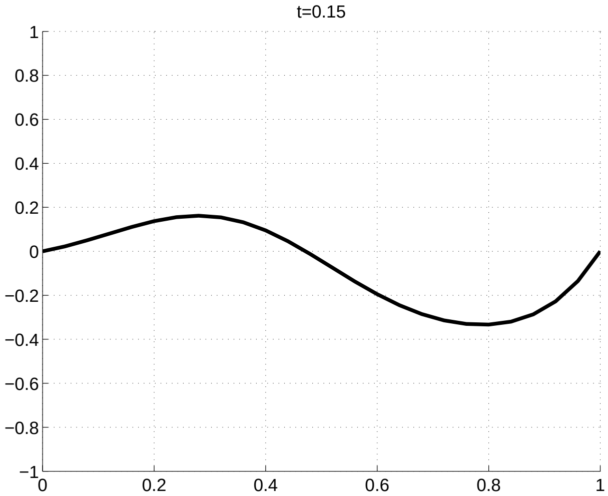

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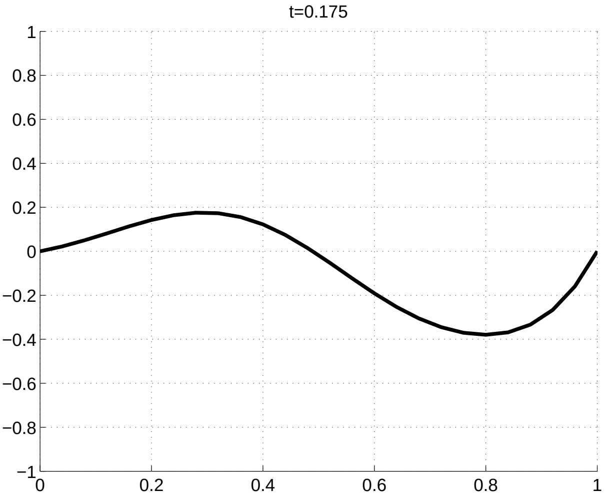

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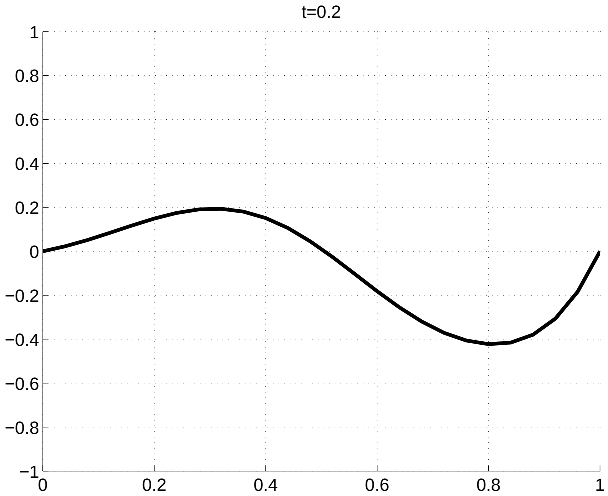

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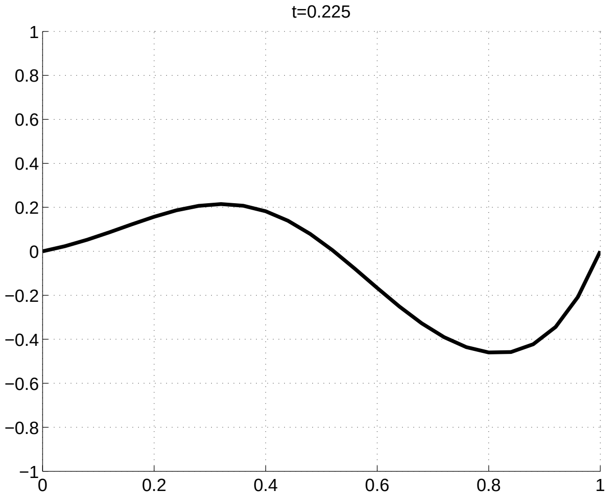

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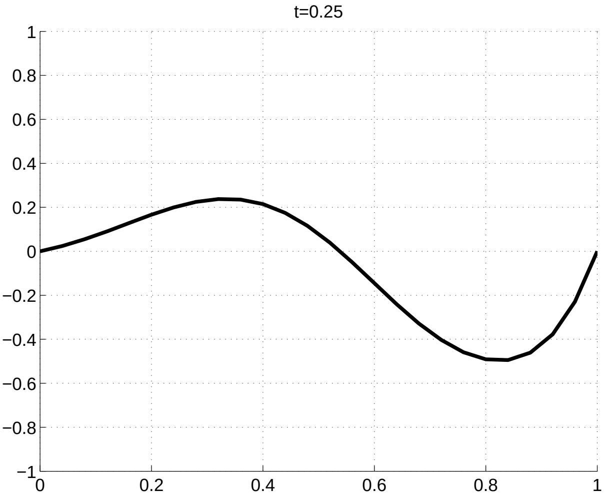

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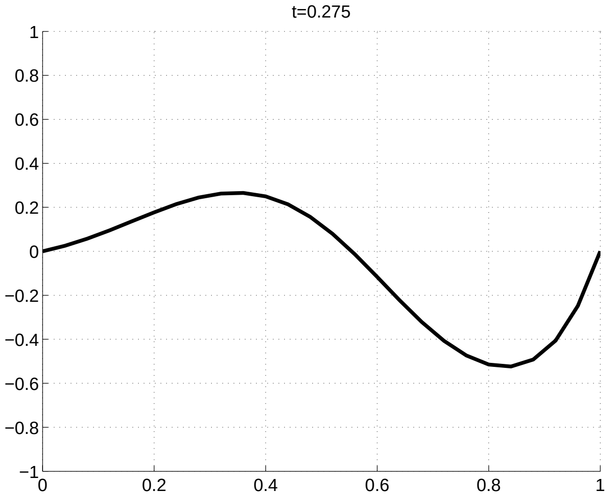

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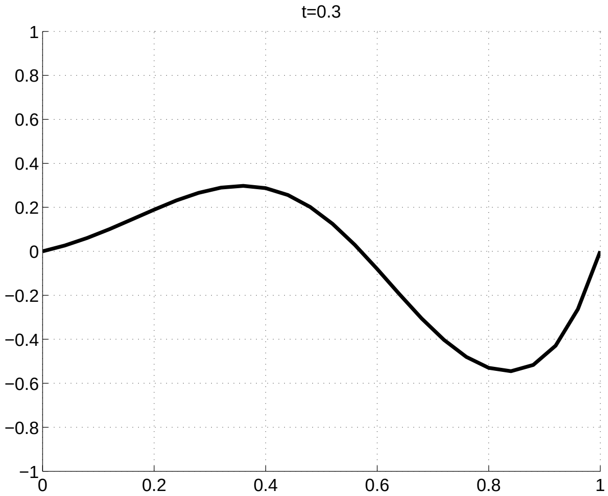

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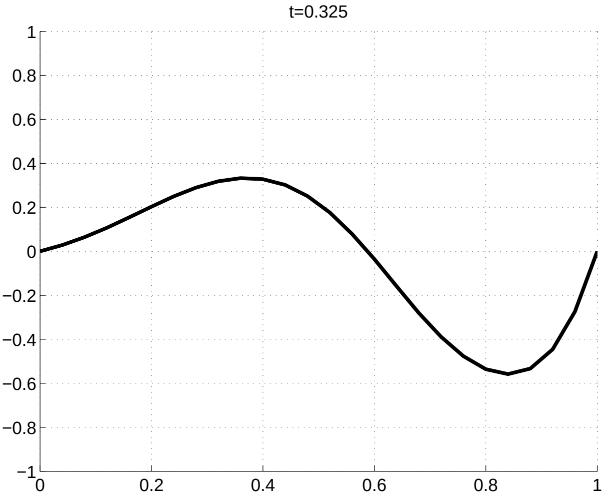

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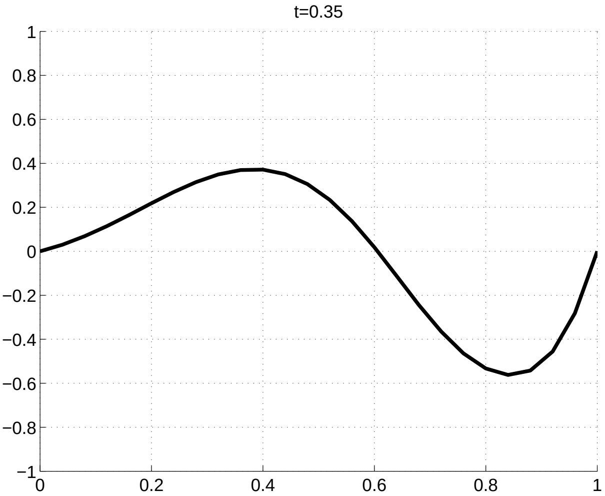

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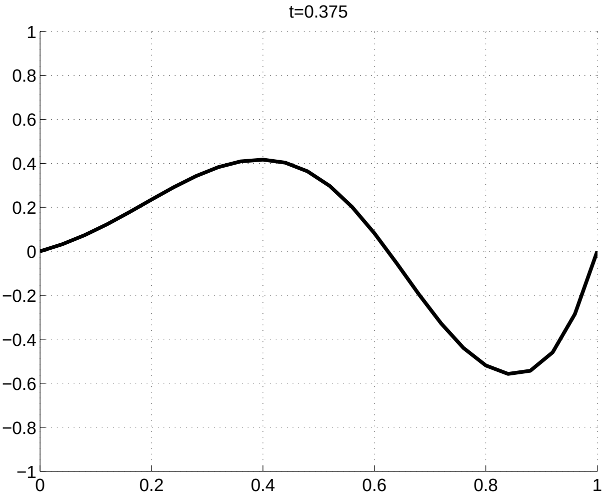

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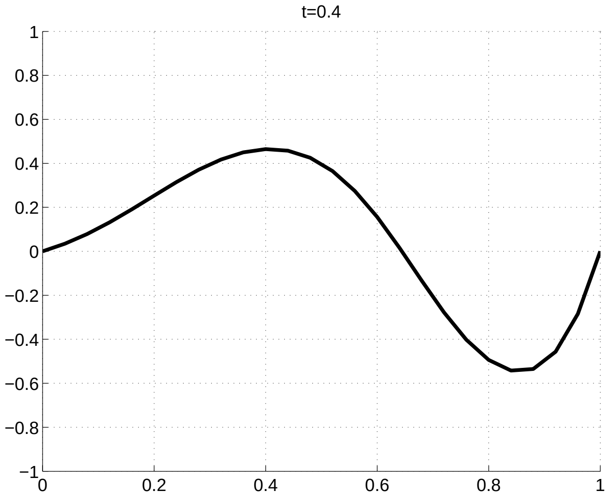

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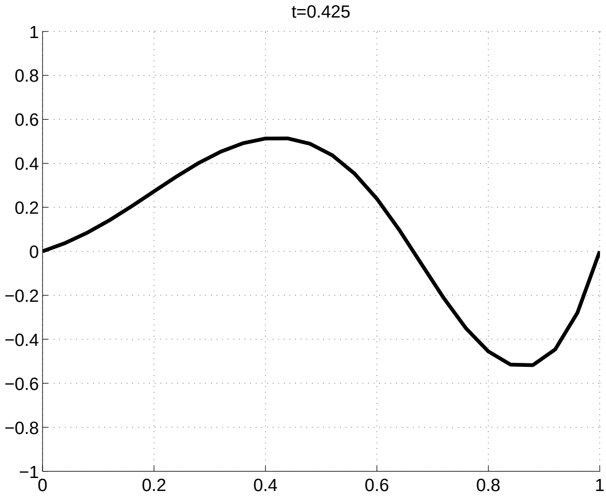

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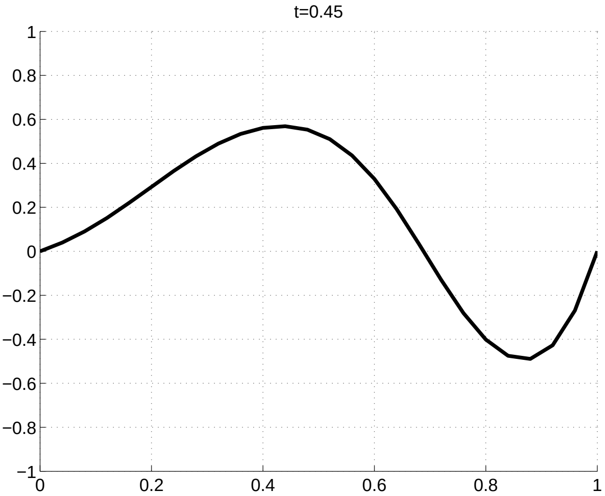

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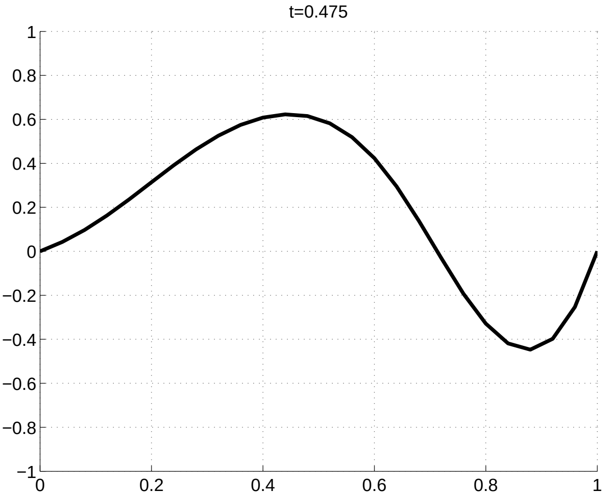

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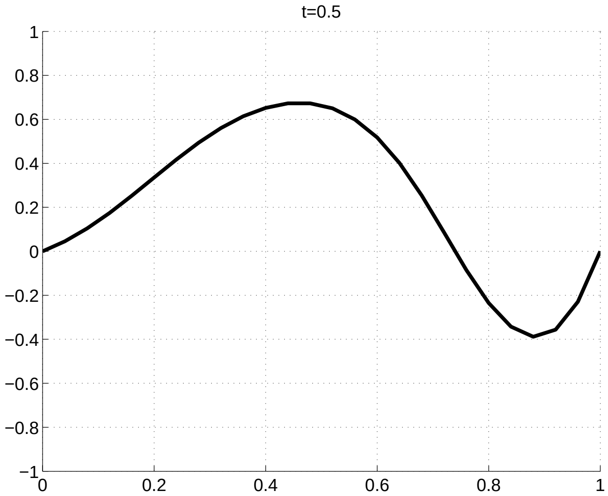

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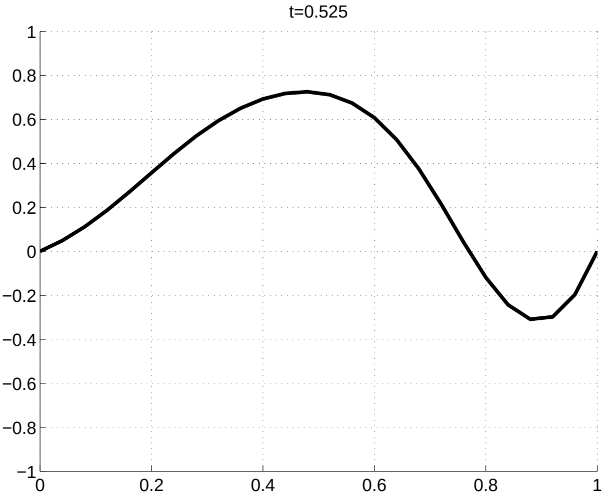

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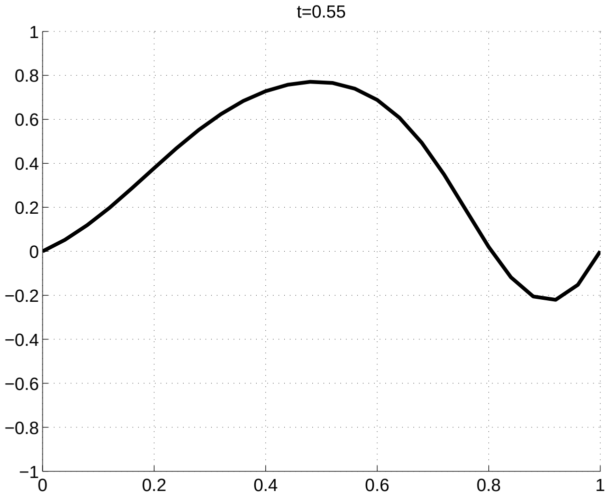

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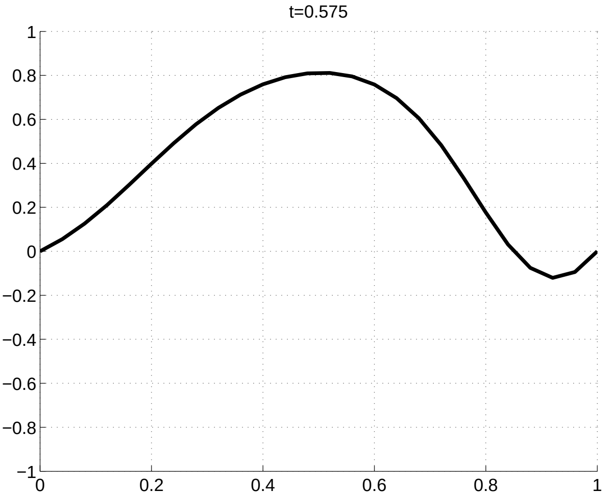
The uncontrolled PDE



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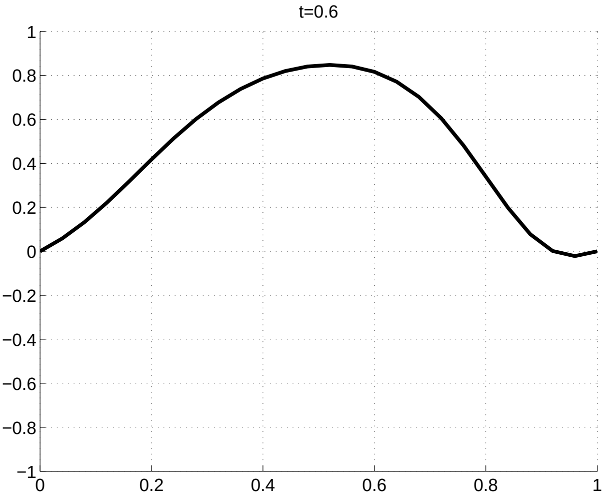

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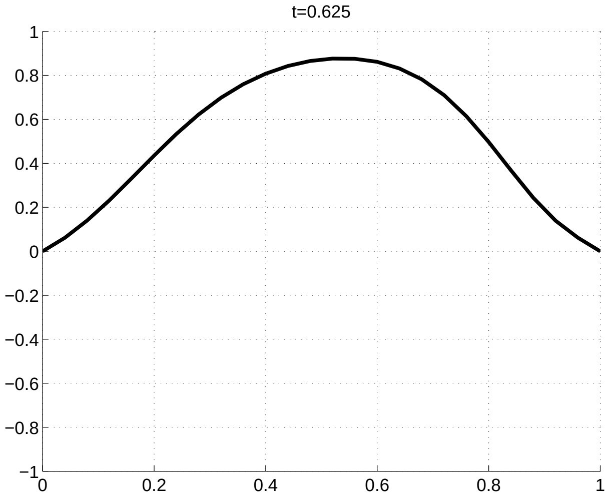

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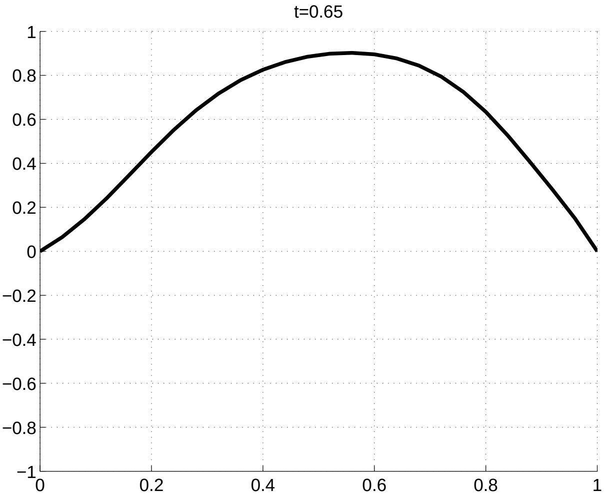

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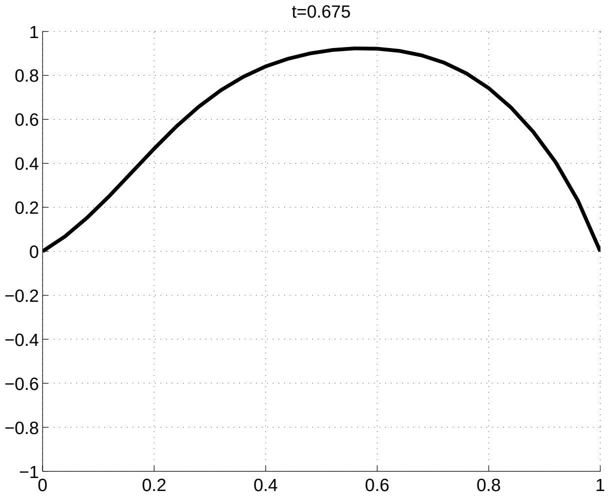

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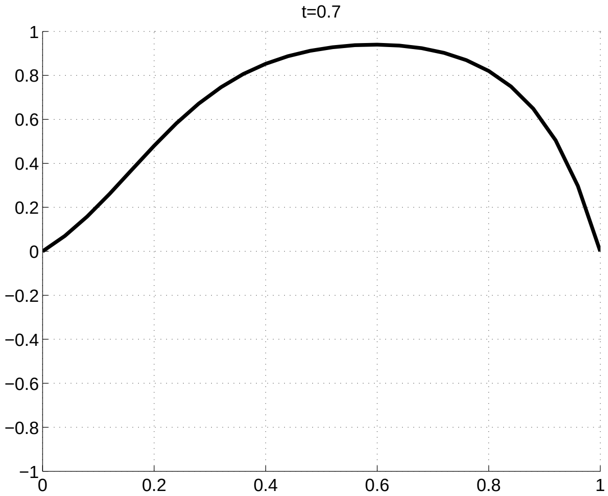

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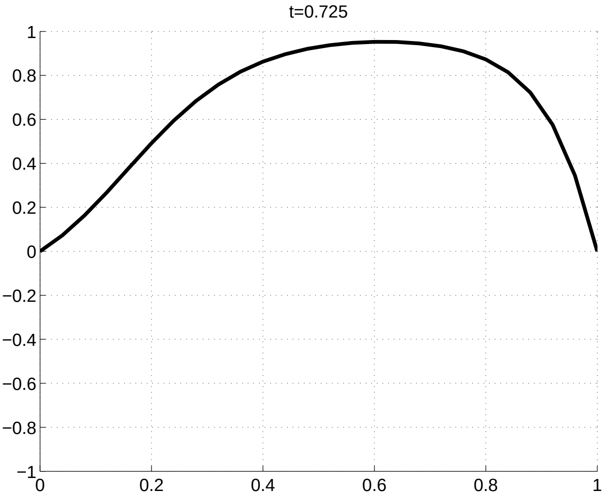

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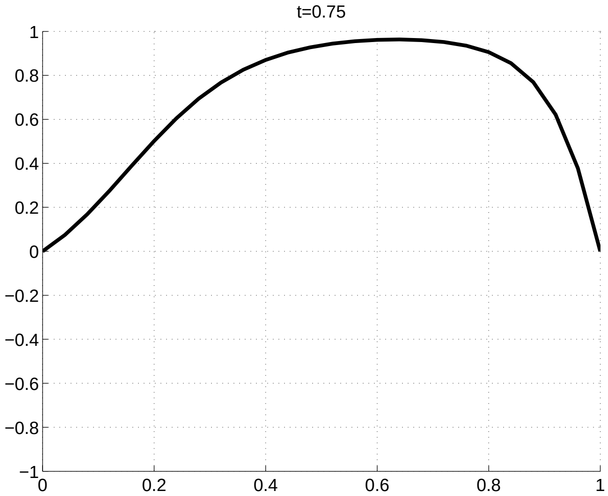

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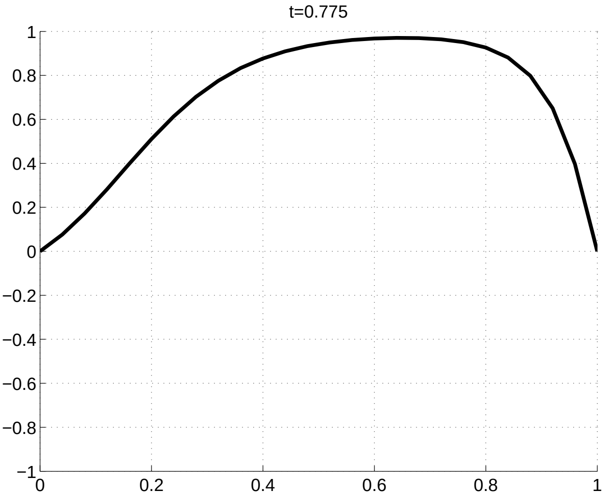

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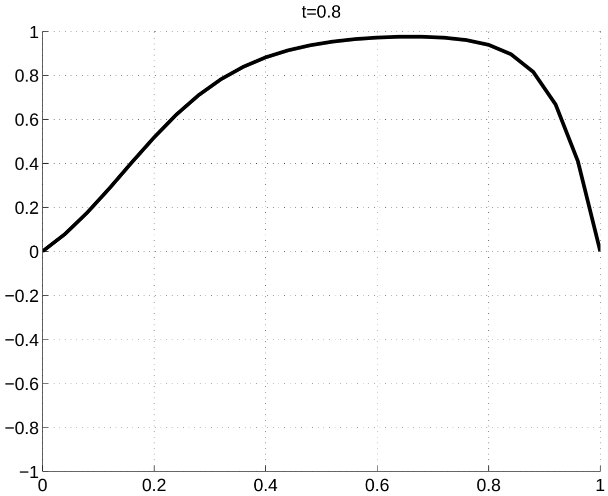

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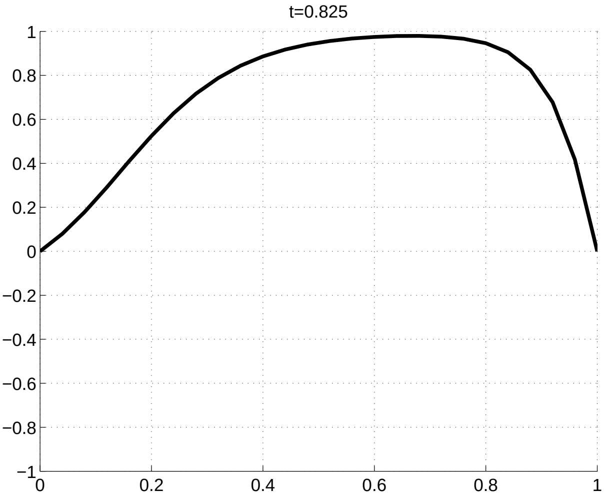

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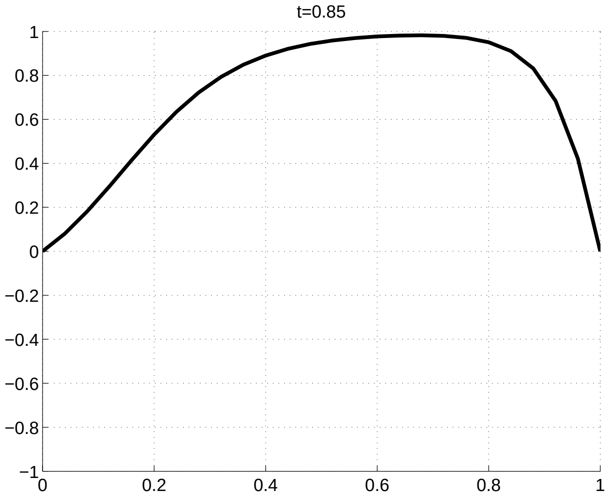

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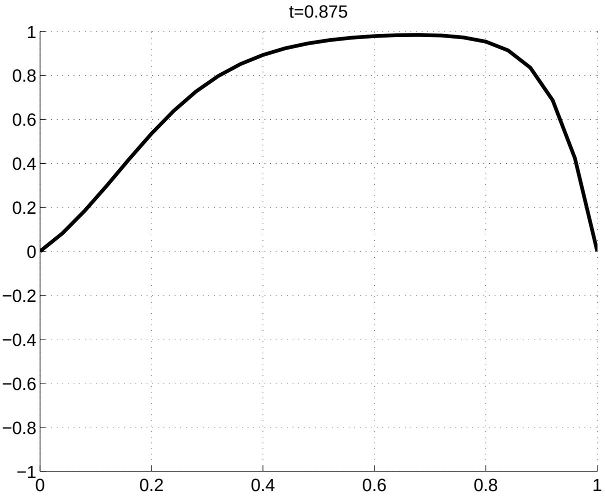

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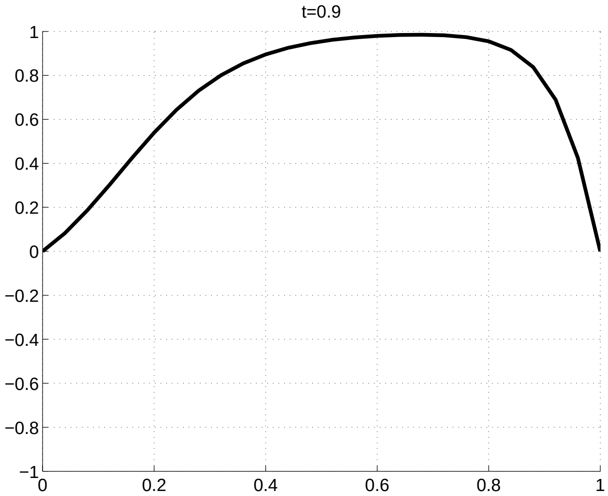

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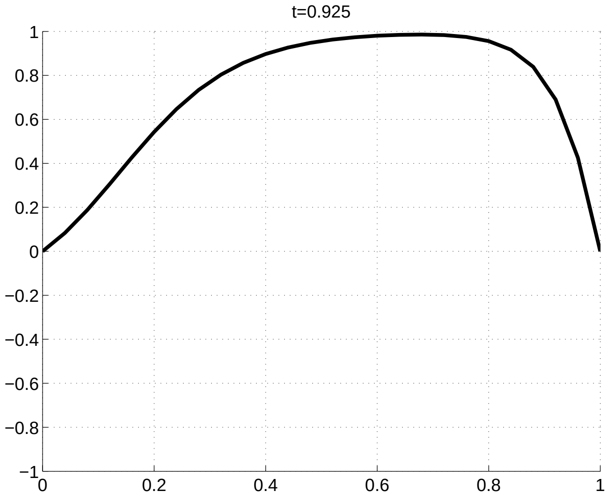

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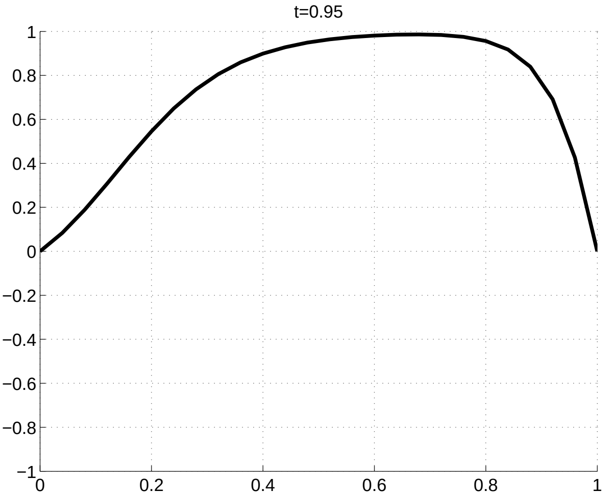

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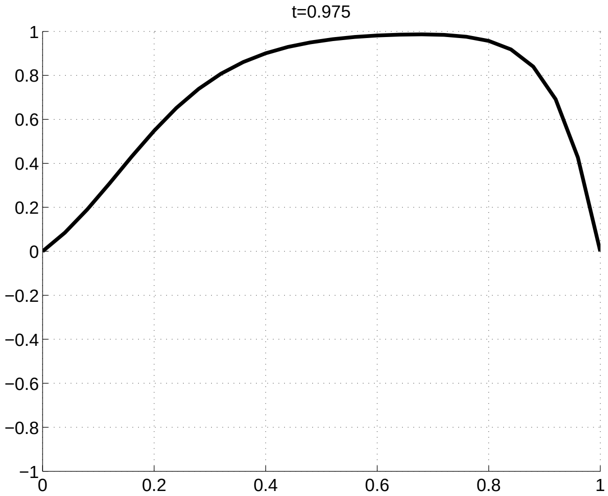

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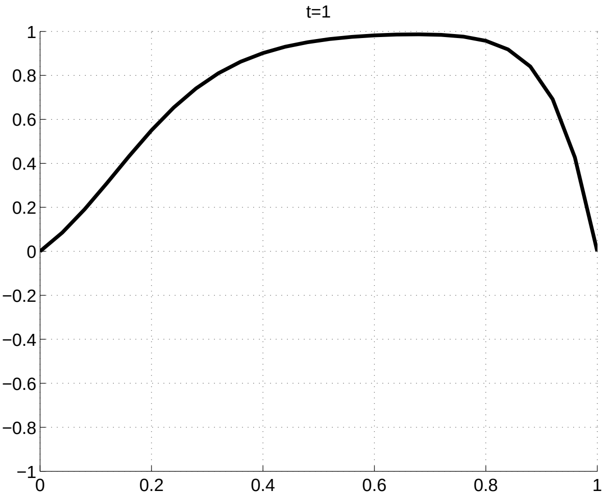

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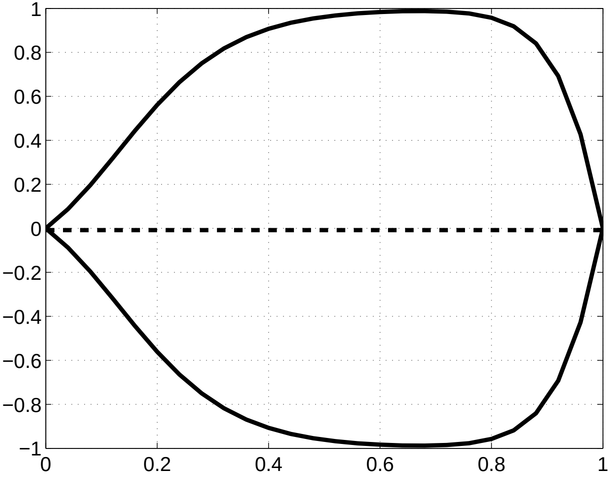

all equilibrium solutions

MPC for the PDE example

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

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$\rightsquigarrow$ controllability condition

$$\ell(y(n), u(n)) \quad \leq \quad C\sigma^n \ell^*(y(0))$$

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for $\|y_x\|_{L^2} \gg \|y\|_{L^2}$ this can only hold if $C \gg 0$

MPC for the PDE example

Conclusion: because of

$$\|y(n)\|_{L^2}^2 + \lambda\|y_x(n)\|_{L^2}^2 \leq C\sigma^n\|y(0)\|_{L^2}^2$$

the controllability condition may only hold for very large C

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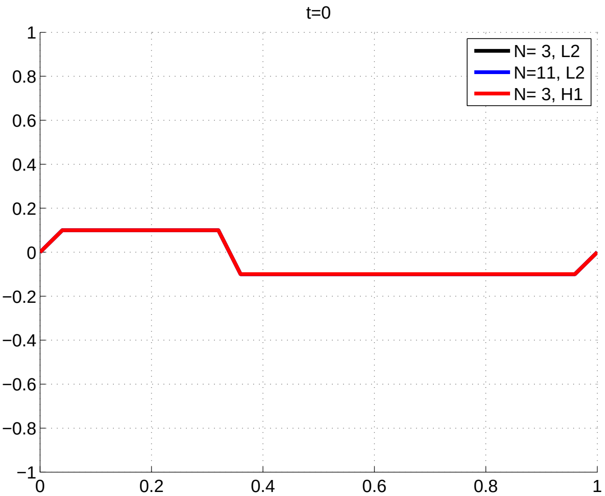
Then an analogous computation yields

$$\|y(n)\|_{L^2}^2 + (1+\lambda)\|y_x(n)\|_{L^2}^2 \le C\sigma^n\Big(\|y(0)\|_{L^2}^2 + \|y_x(0)\|_{L^2}^2\Big)$$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

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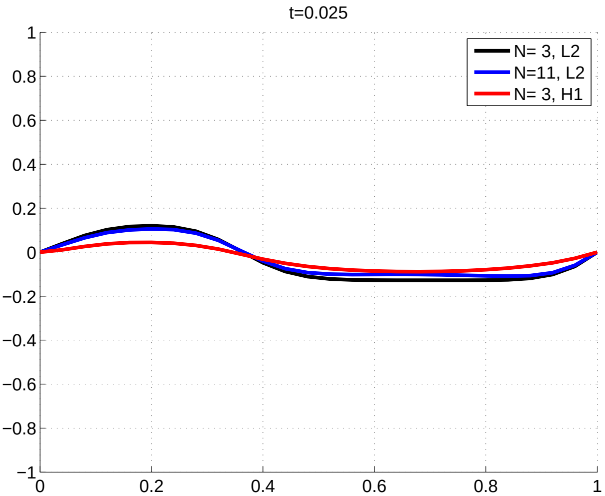

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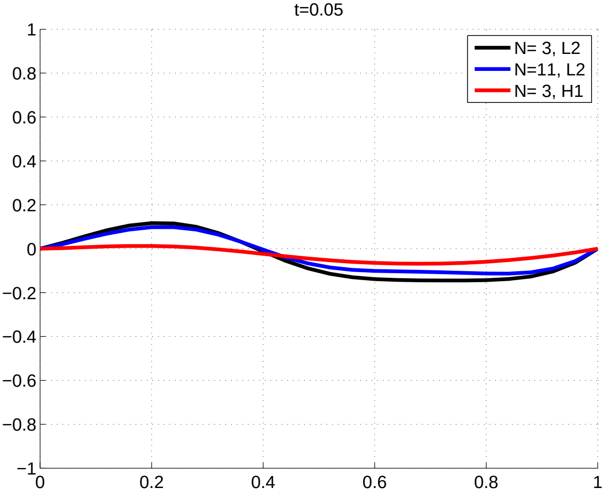

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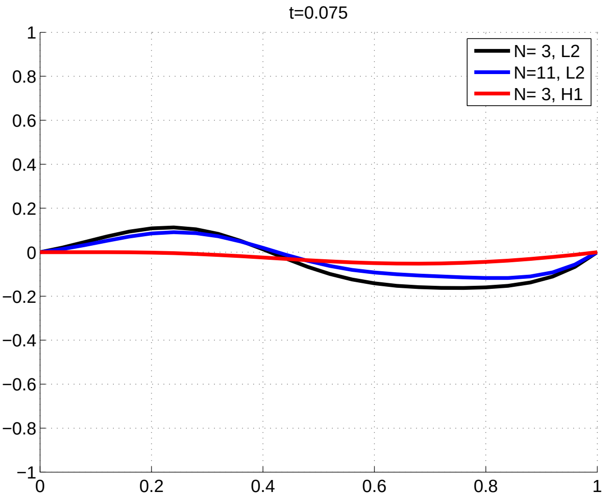

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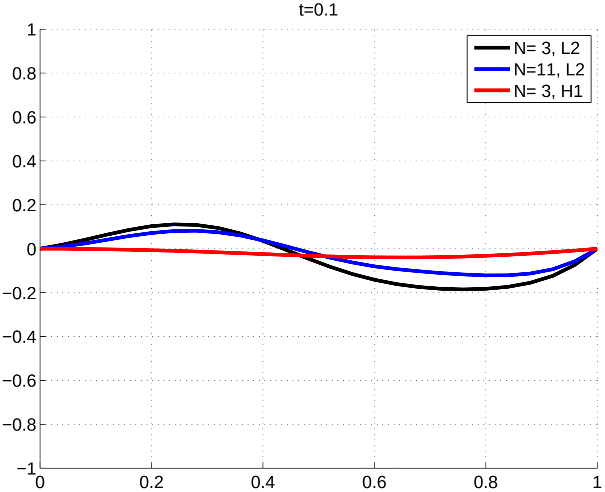

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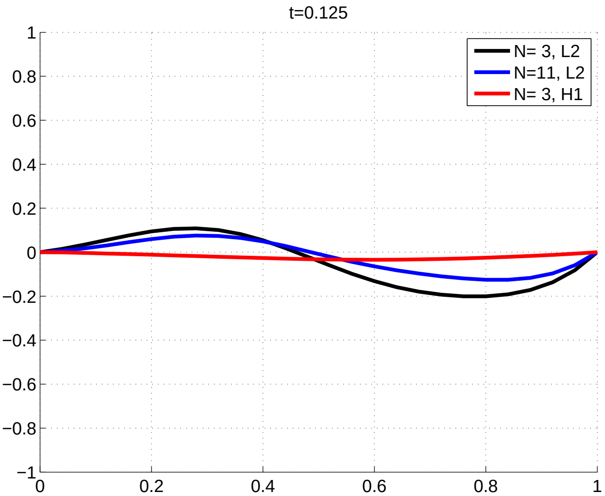

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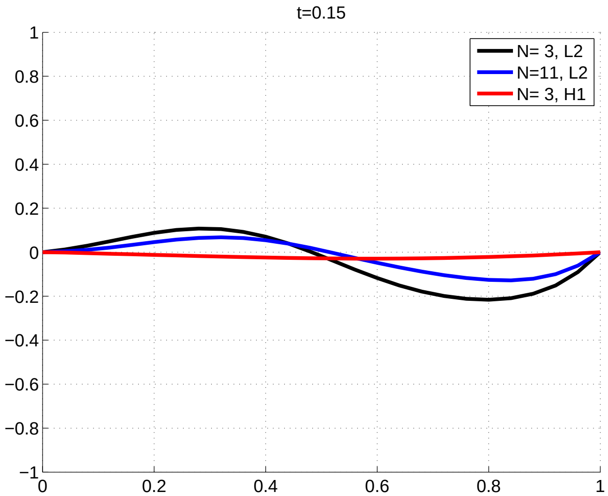

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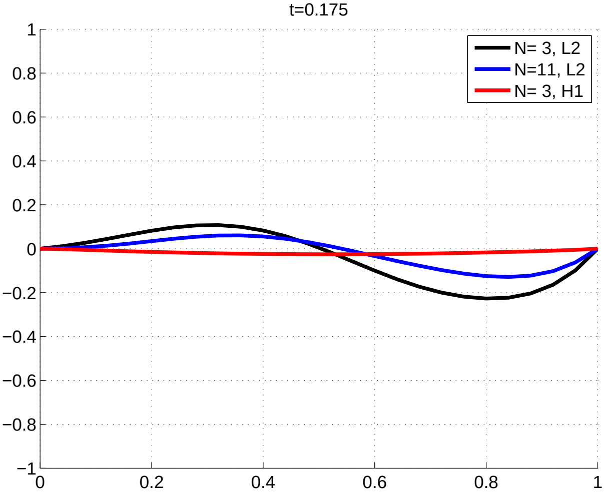

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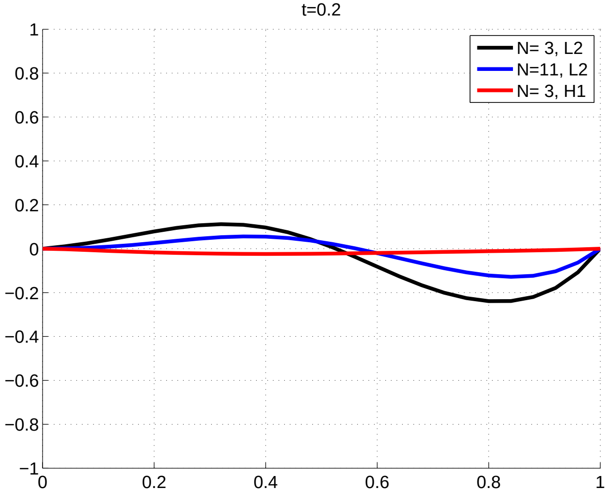

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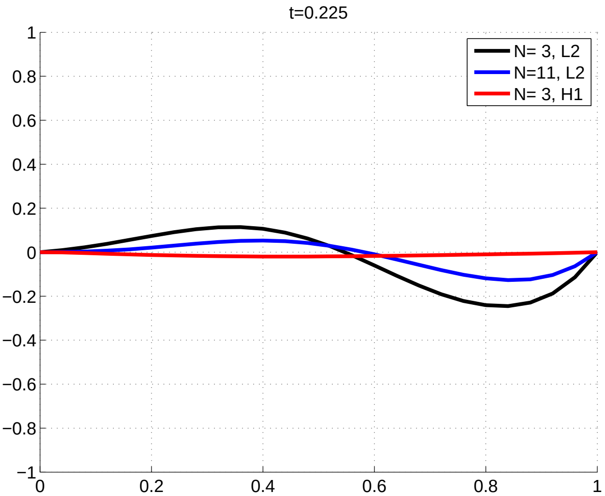

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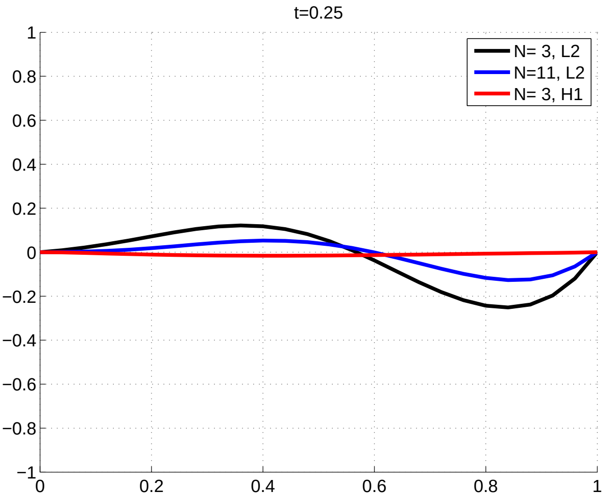

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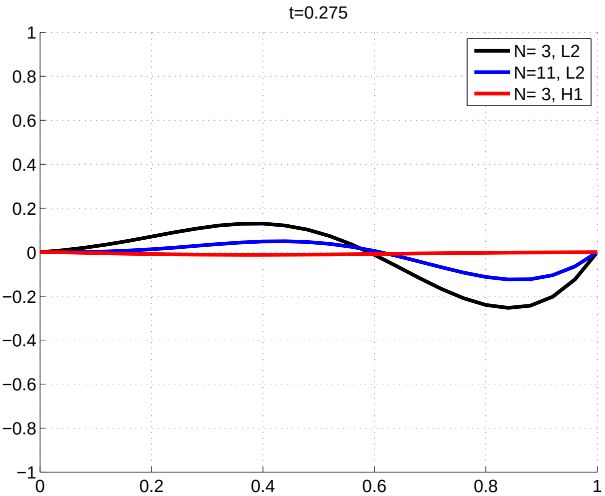

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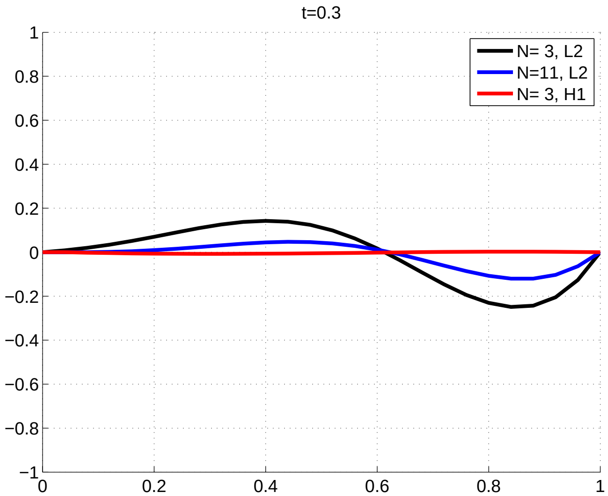

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Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

domain $\Omega = [0, 1]$

solution $y = y(t, x)$

boundary conditions $y(t, 0) = u_0(t)$, $y(t, 1) = u_1(t)$

parameters $\nu = 0.1$ and $\mu = 10$

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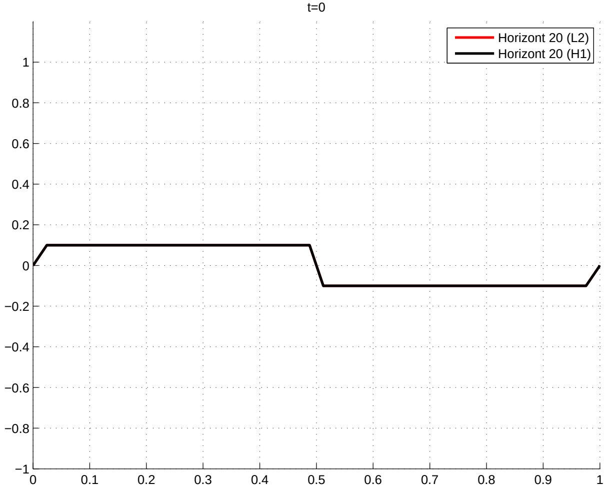
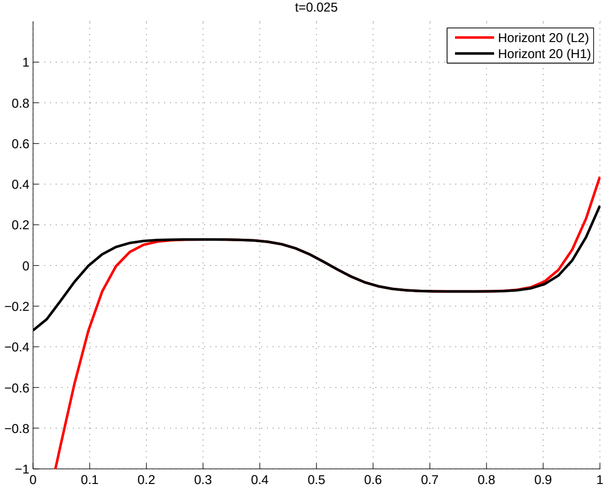
$\rightsquigarrow$ L^2 should perform better that H^1

Boundary control, L_2 vs. H_1, $N = 20$



Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$

Boundary control, L_2 vs. H_1, $N = 20$



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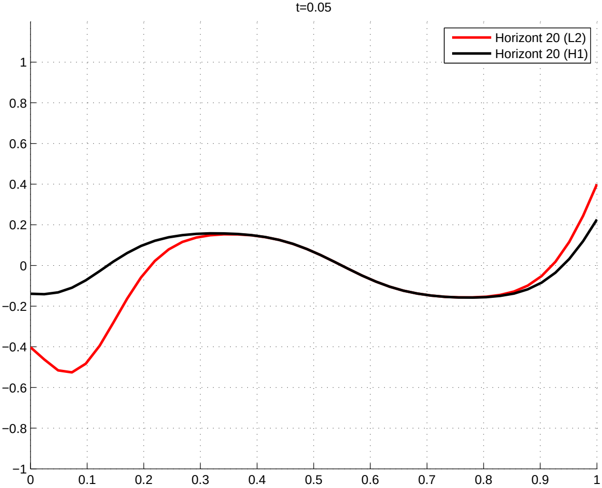

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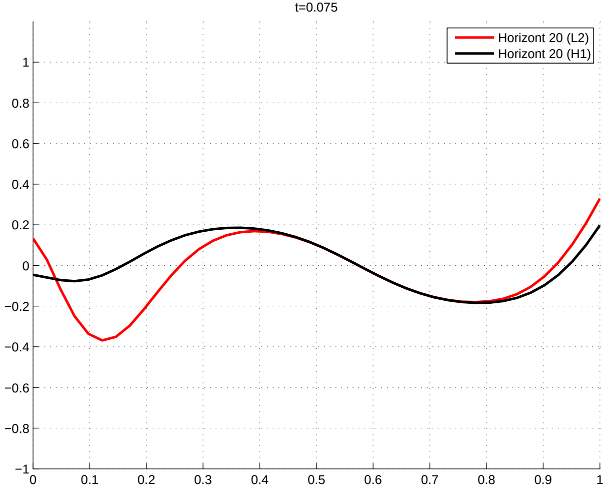

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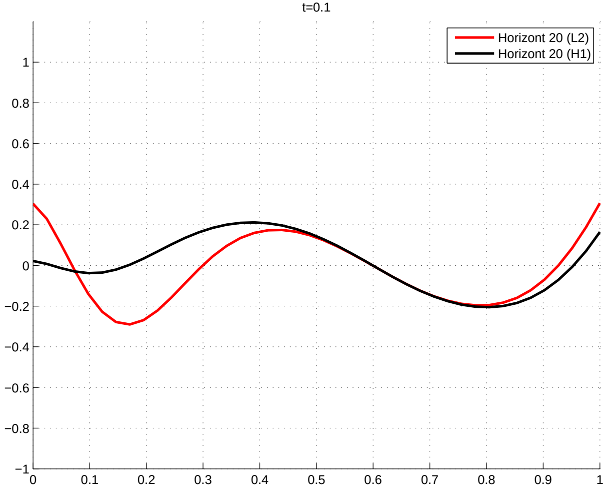

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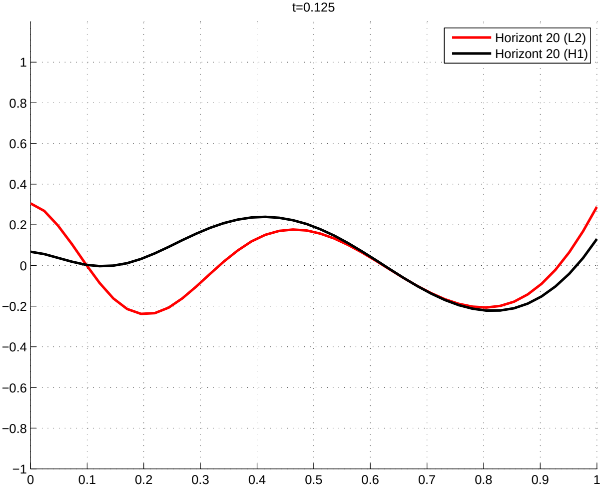

Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$

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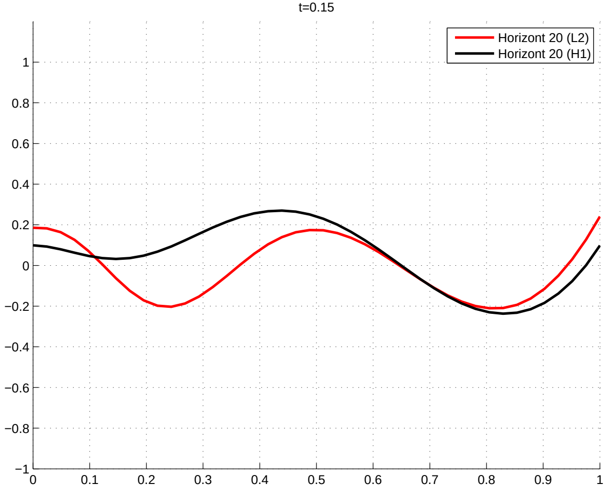

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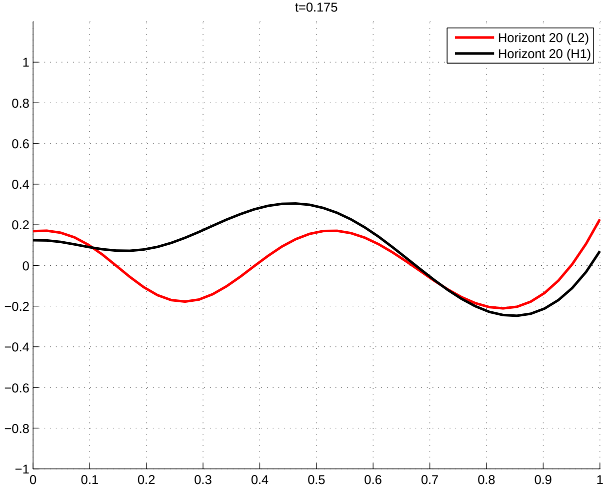

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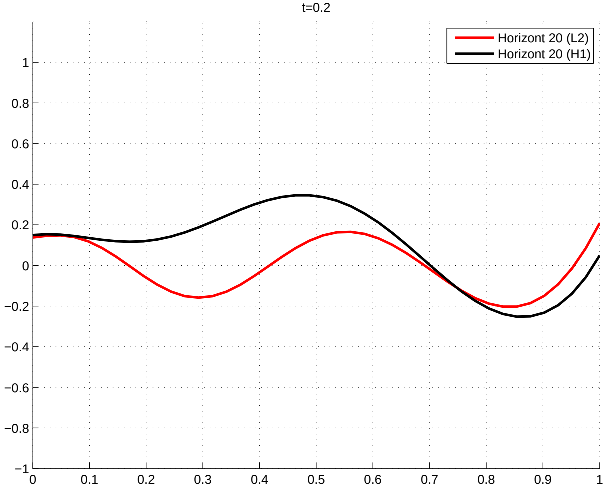

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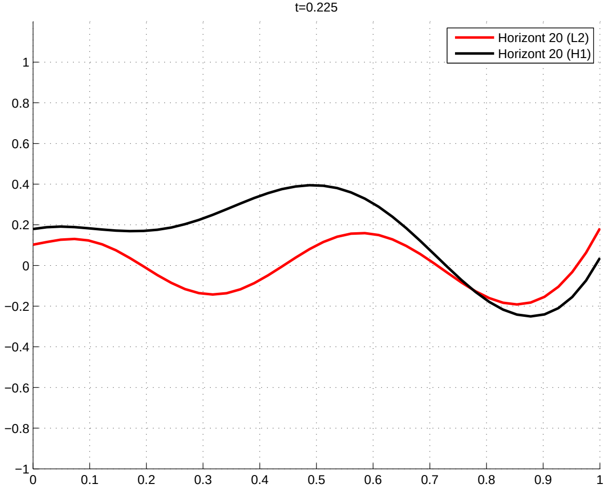

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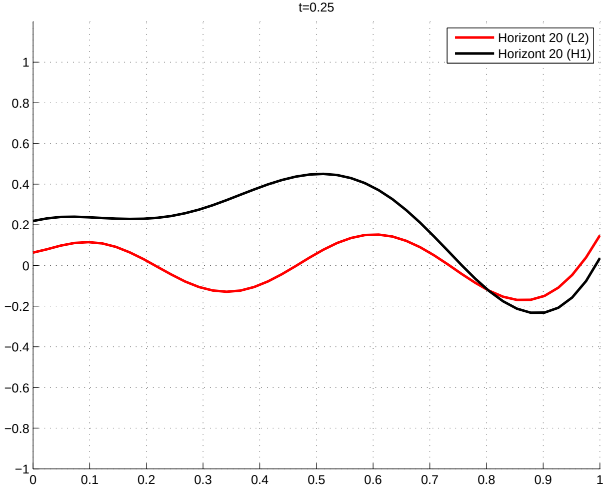

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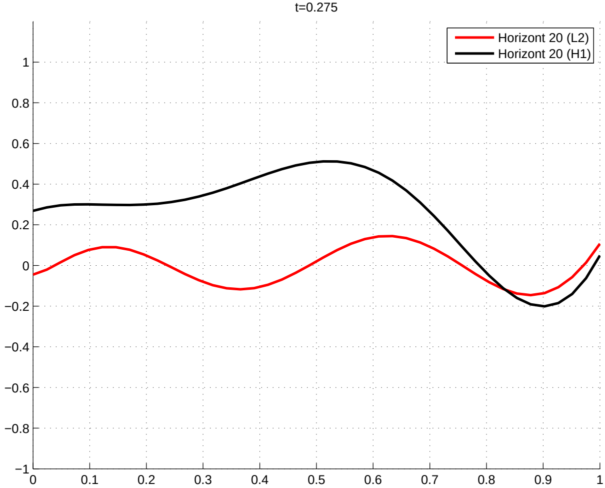

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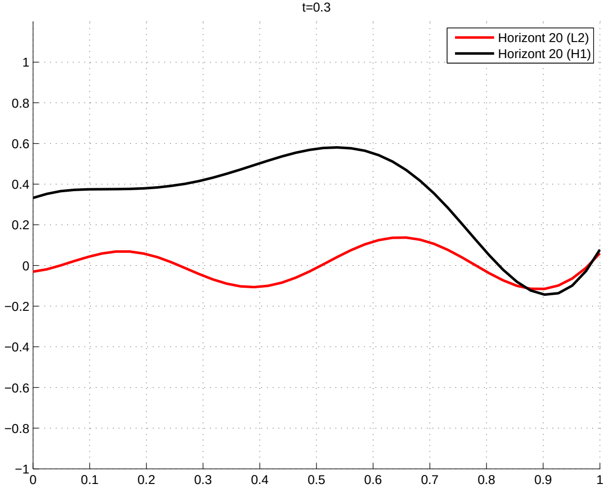

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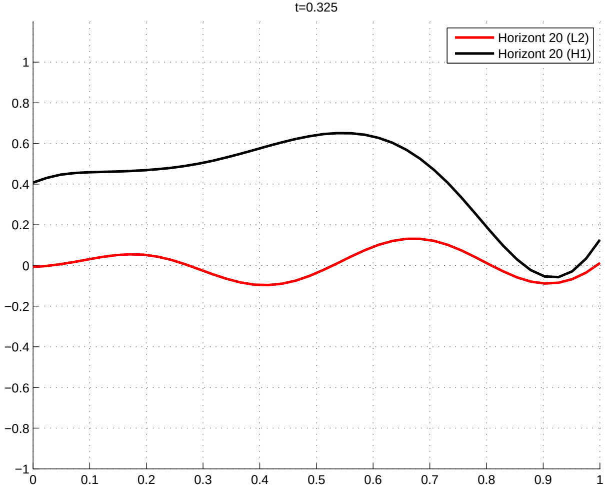

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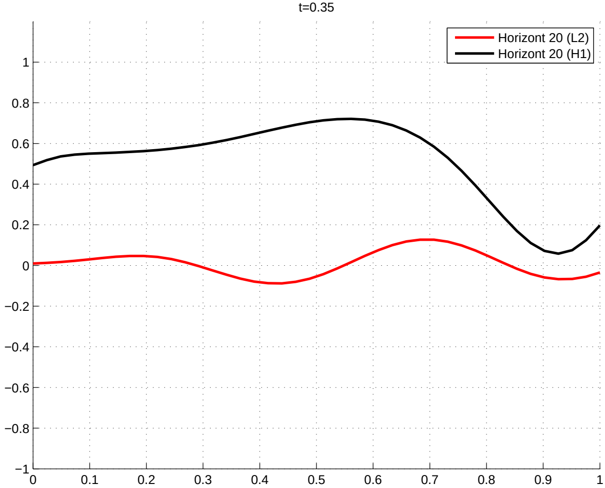

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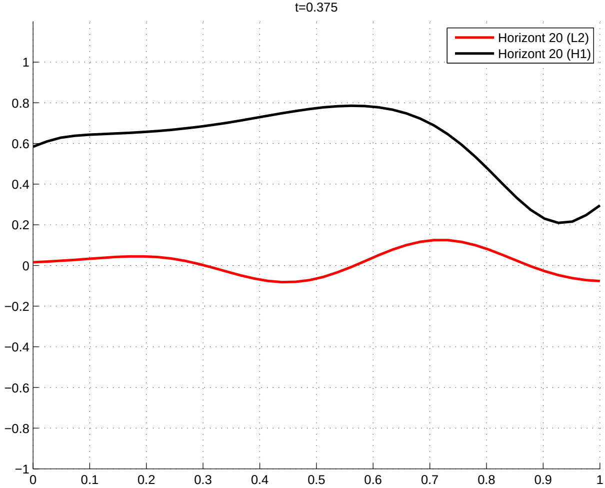

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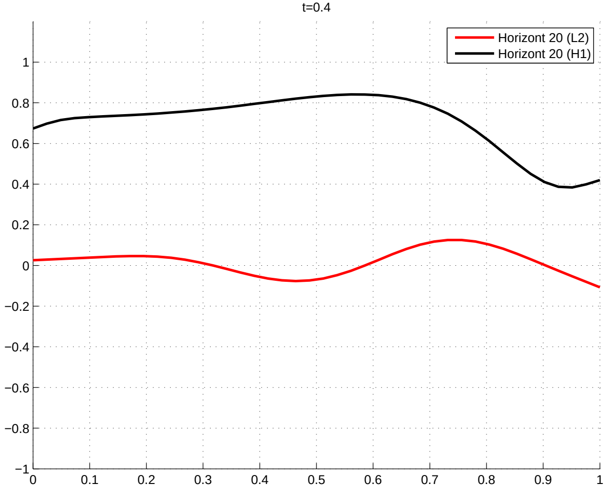

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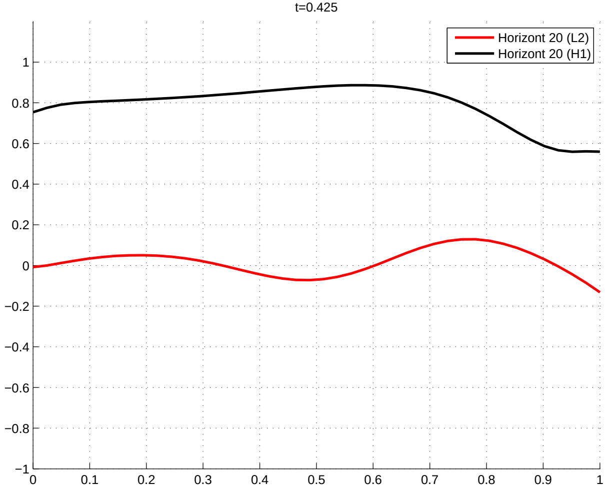

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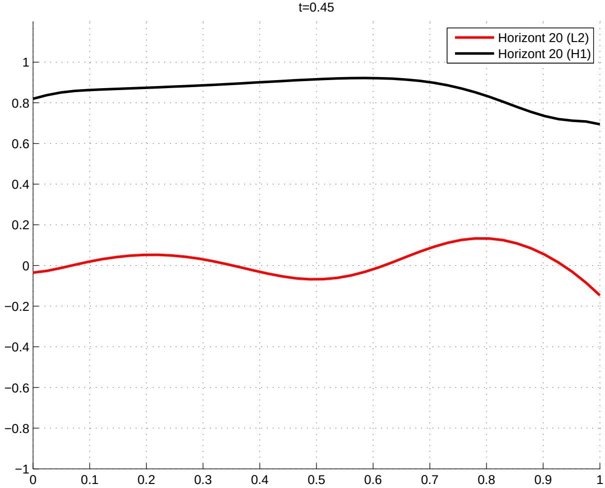

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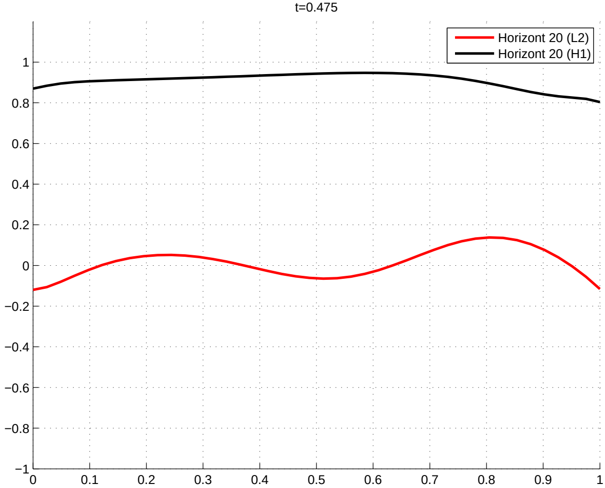

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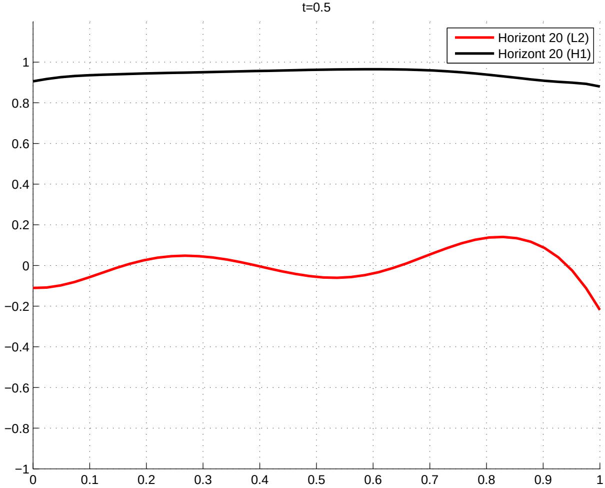

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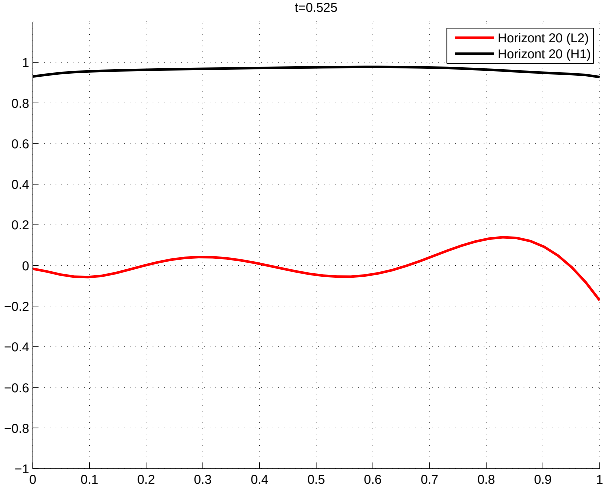

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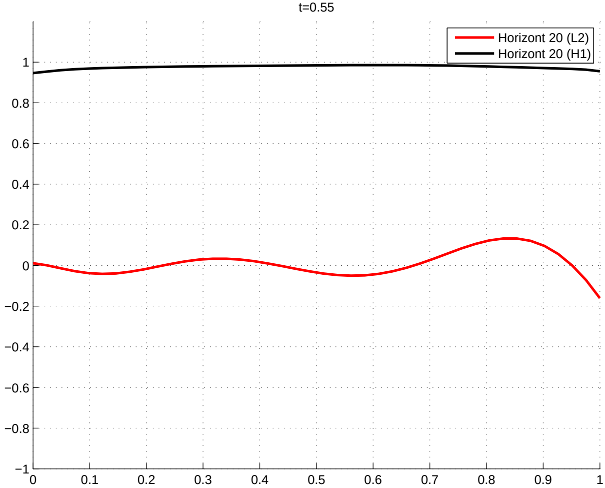

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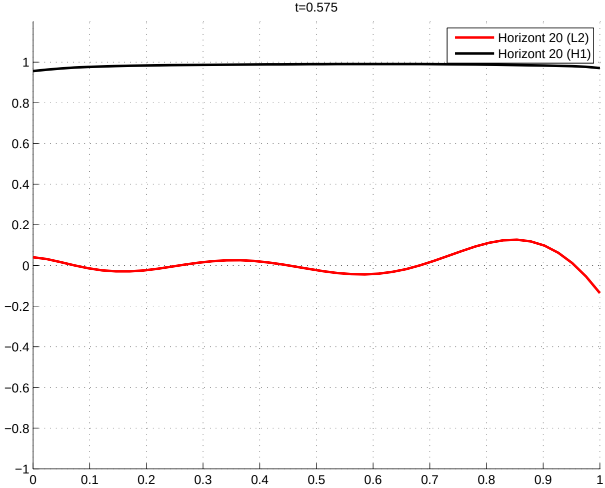

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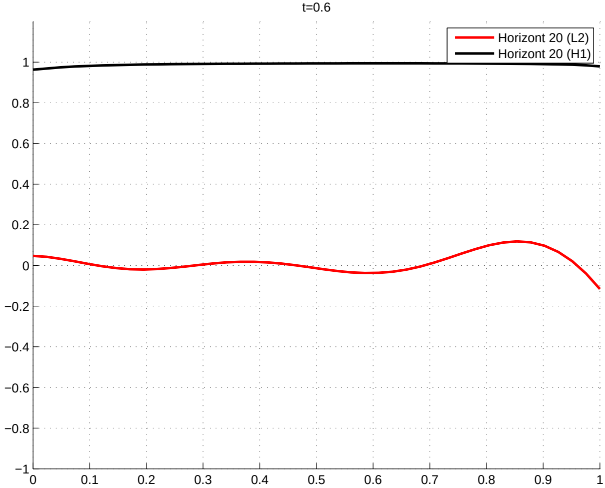

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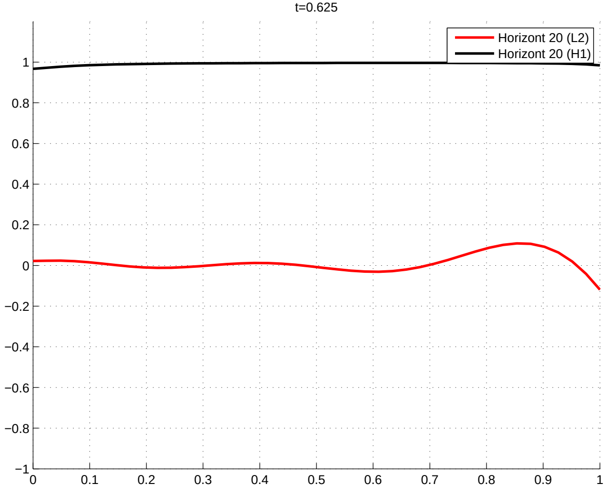

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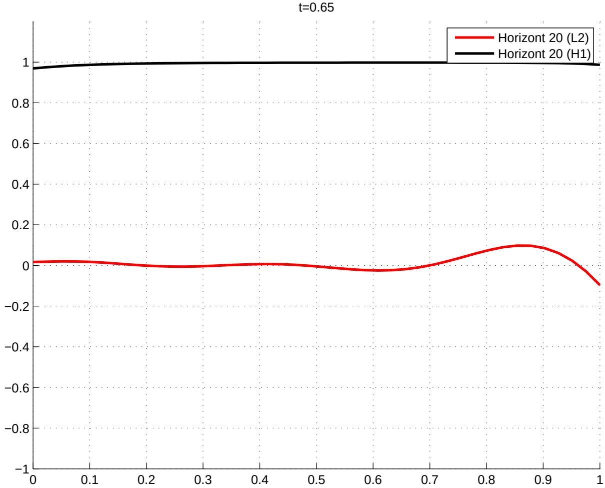

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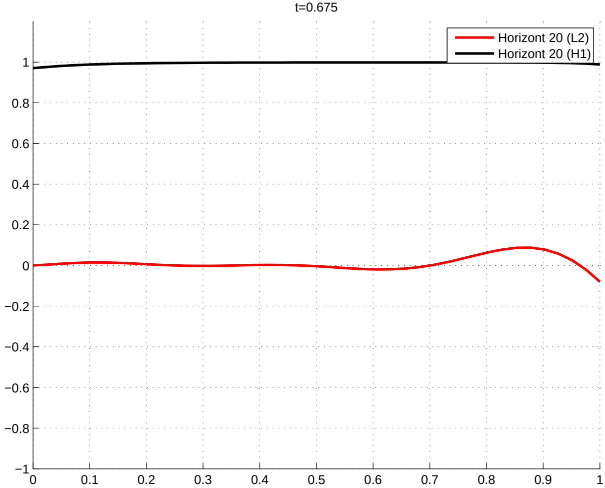

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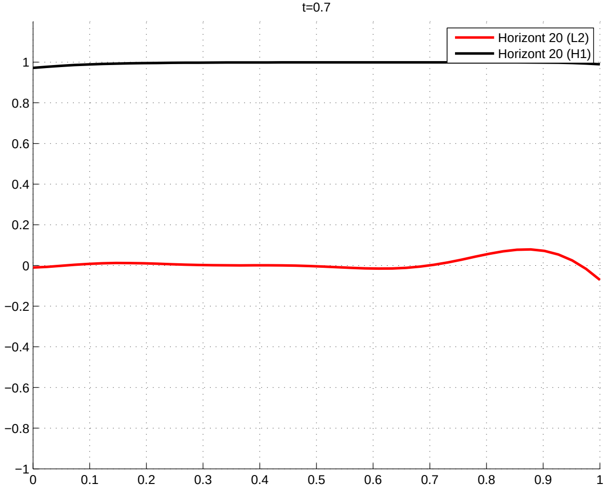

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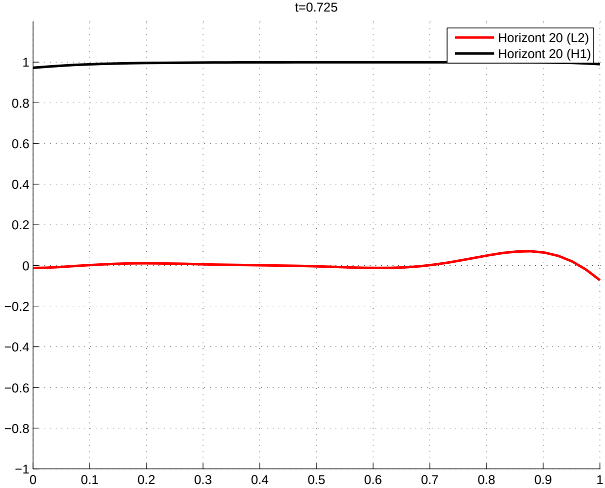

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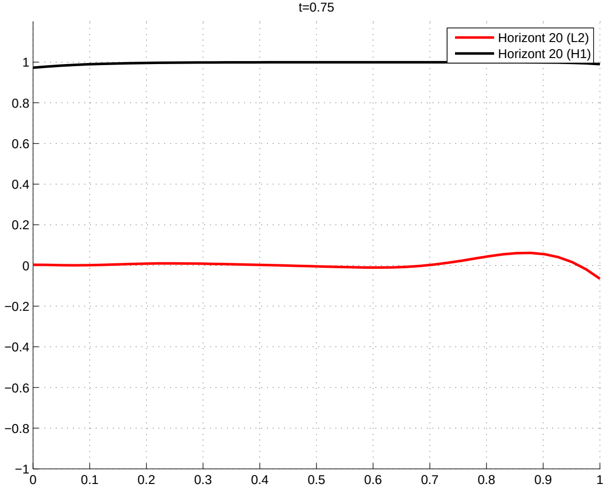

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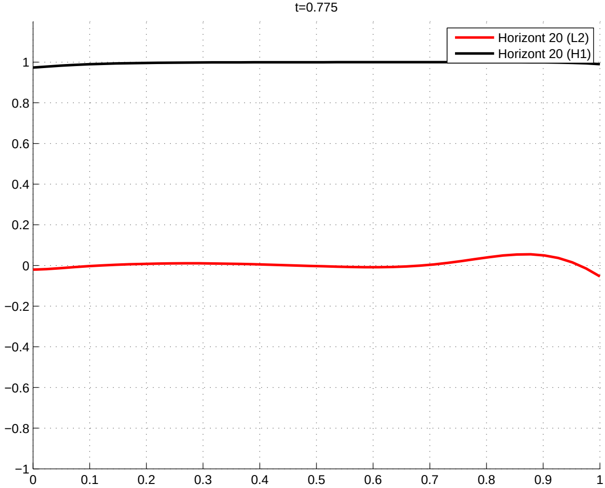

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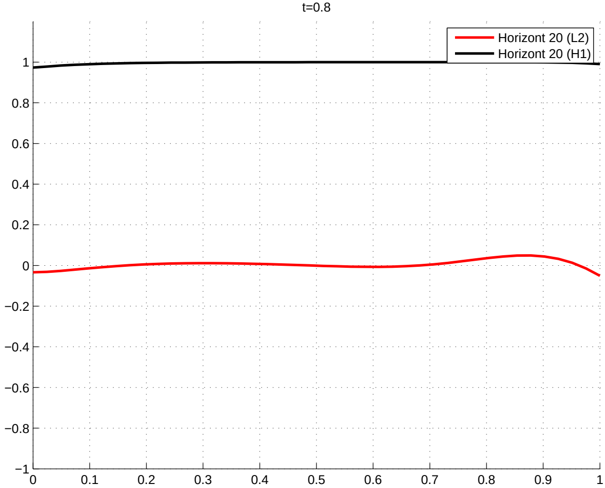

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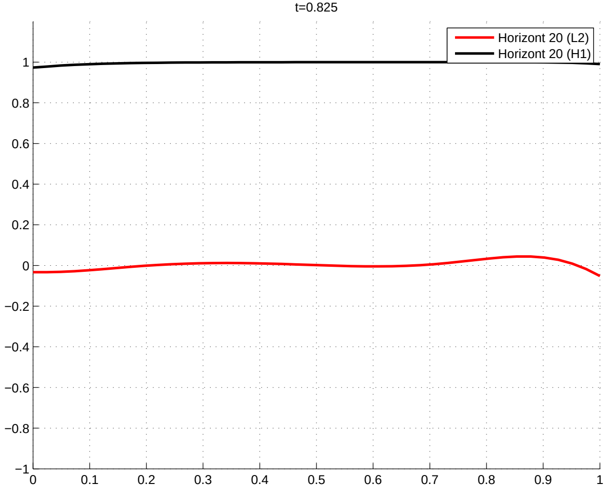

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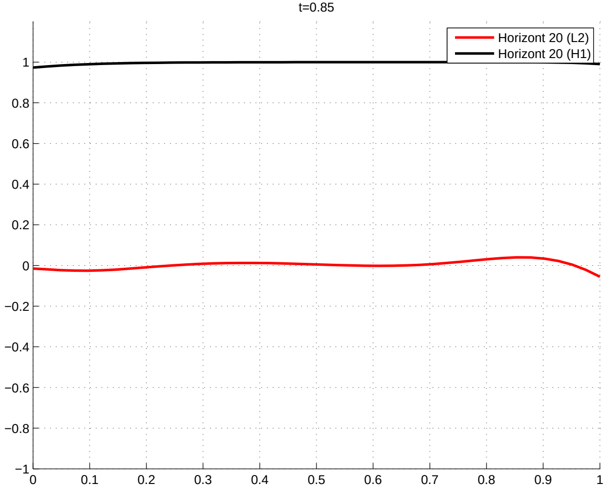

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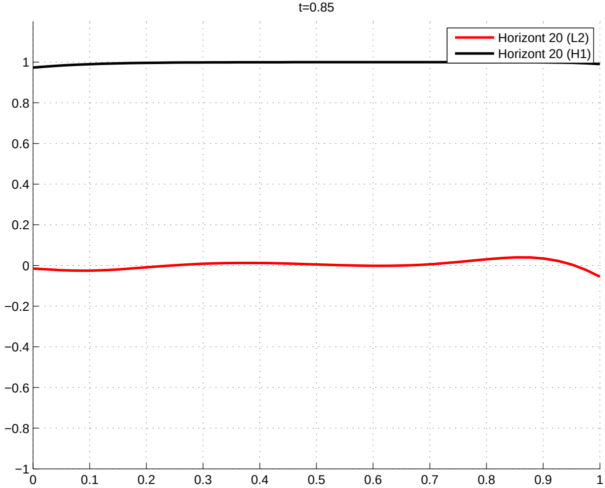

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Can be made rigorous for many PDEs [Altmüller et al. '10ff]

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- Computing tight estimates for C is in general a difficult if not impossible task
- But structural knowledge of the system behavior can be sufficient for choosing a “good” ℓ

(5) Economic Model Predictive Control

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Idea: Use a stage cost ℓ which does not penalize the distance to some x_* but directly encodes the desired economic criterion

Mathematical difference of stabilizing and economic MPC

In stabilizing MPC, the stage cost $\ell(x, u)$ penalizes the distance to some equilibrium $(x_*, u_*) \in \mathbb{X} \times \mathbb{U}$. In particular, we required

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We still consider equilibria, but they are now implicitly defined via the optimization criterion. In order to distinguish them from (x_*, u_*) in stabilizing MPC, they are denoted by (x^e, u^e)

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Only in special cases $K \to \infty$ makes sense

Example: minimum energy control

Example: Keep the state of the system inside the admissible set $\mathbb{X}$ minimizing the quadratic control effort

$$\ell(x,u) = u^2$$

with dynamics

$$x(n+1) = 2x(n) + \mathbf{u}(n)$$

and constraints $\mathbb{X} = [-2,2]$, $\mathbb{U} = [-3,3]$

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(recall: (x^e, u^e) equilibrium $\Leftrightarrow$ $f(x^e, u^e) = x^e$)

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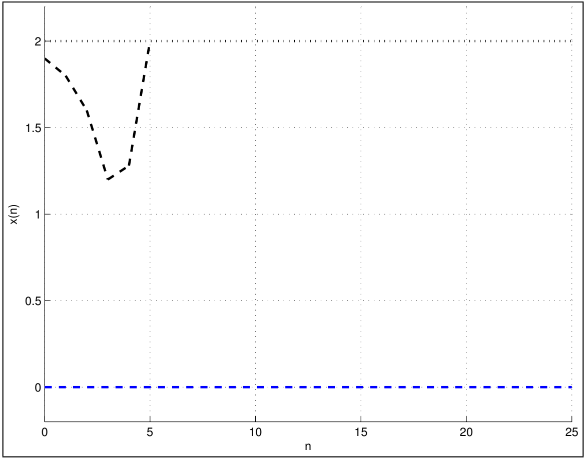

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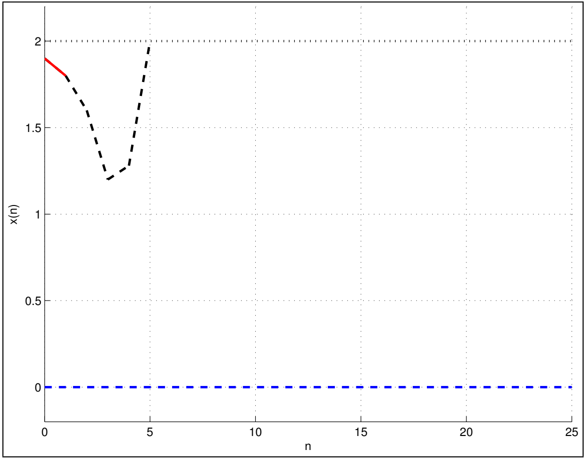

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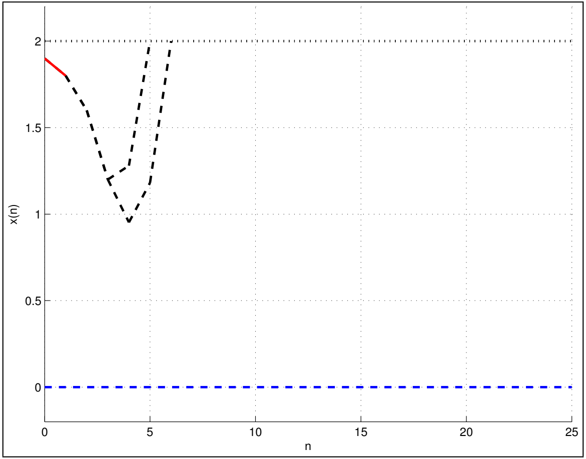

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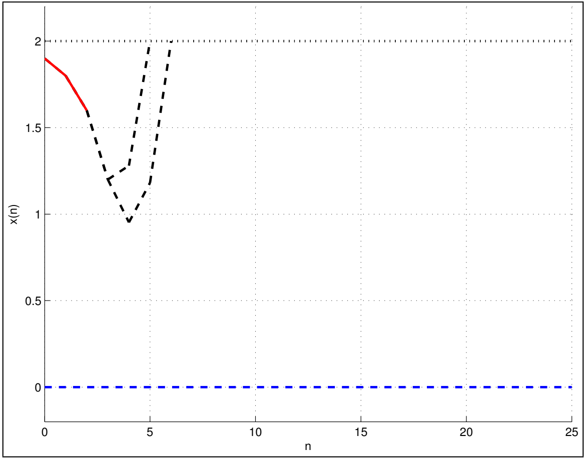

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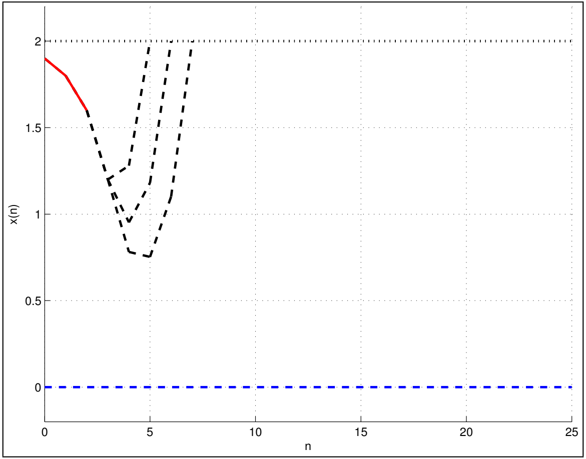

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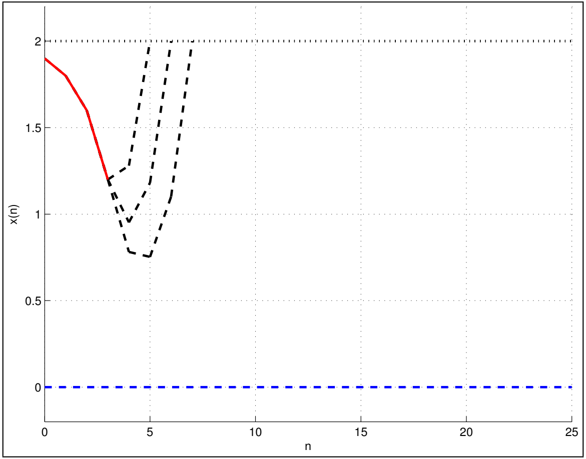

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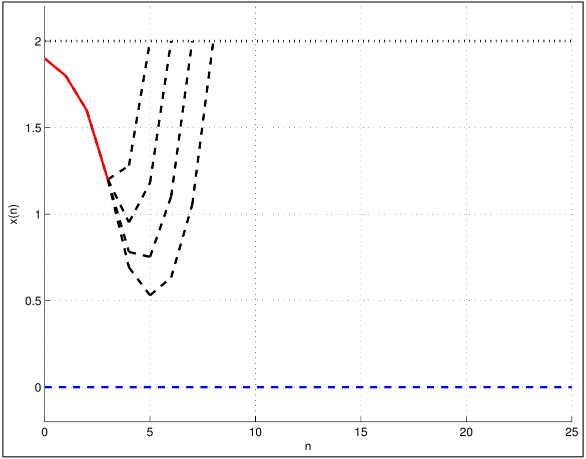

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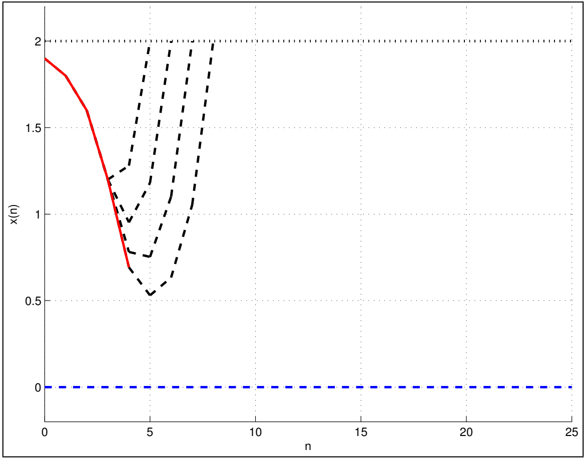

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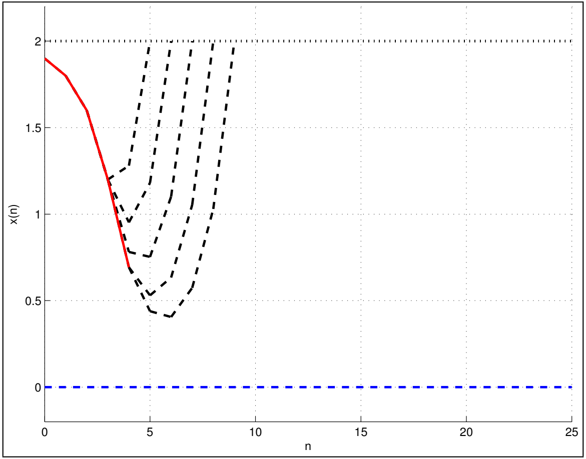

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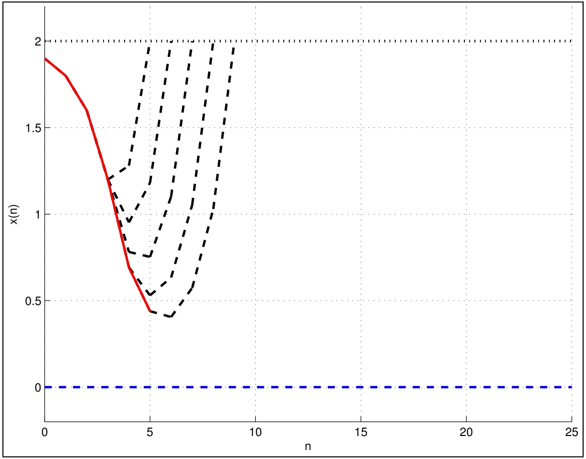

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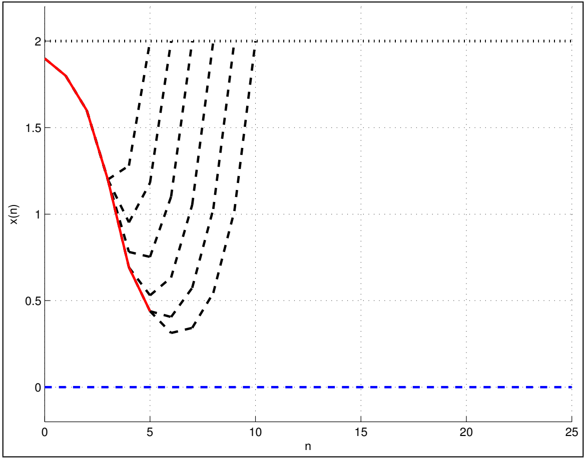

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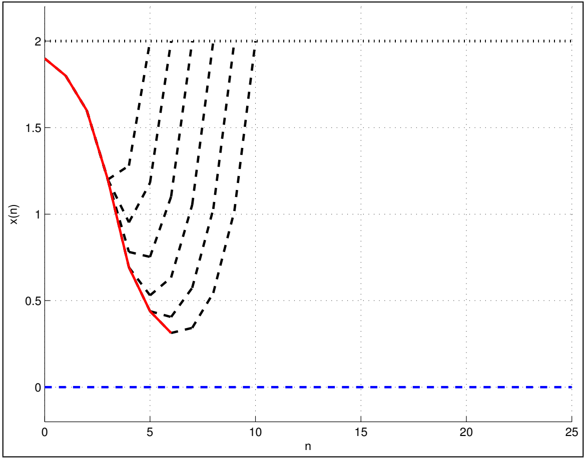

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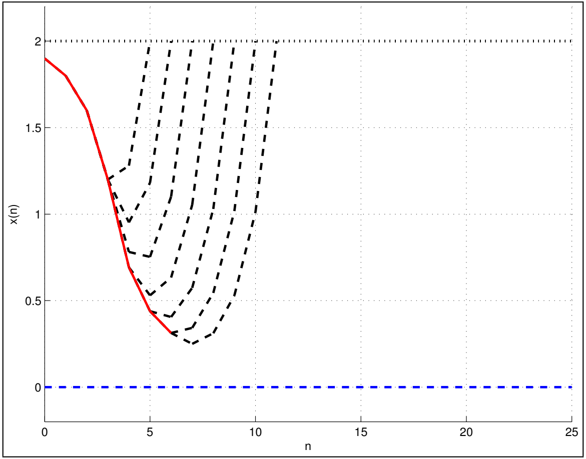

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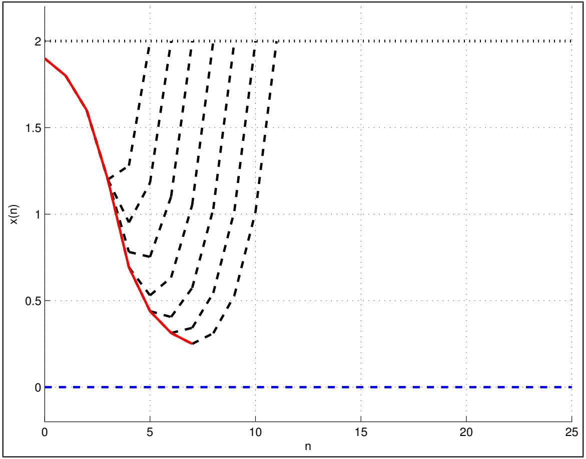

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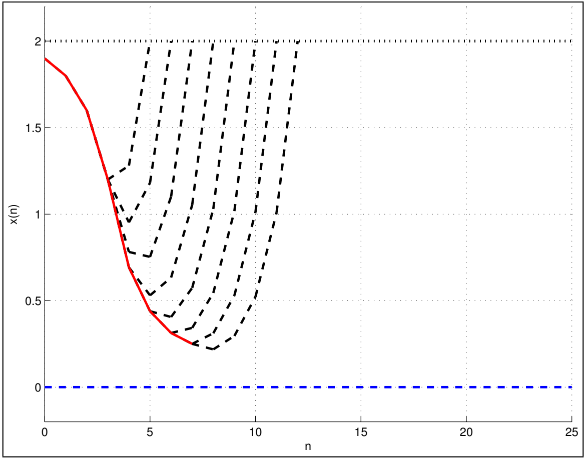

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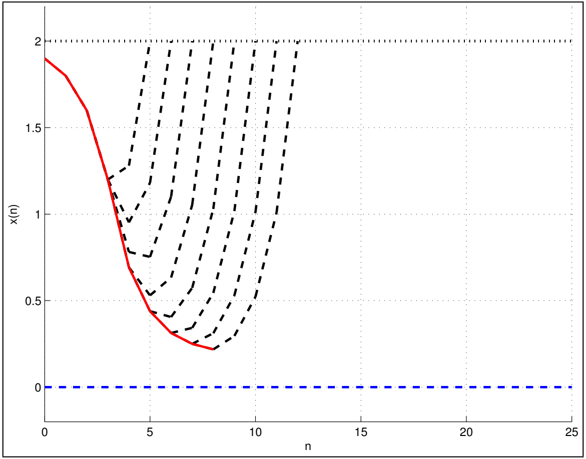

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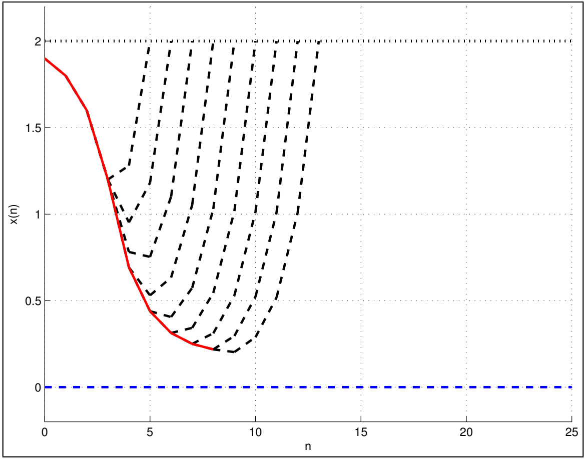

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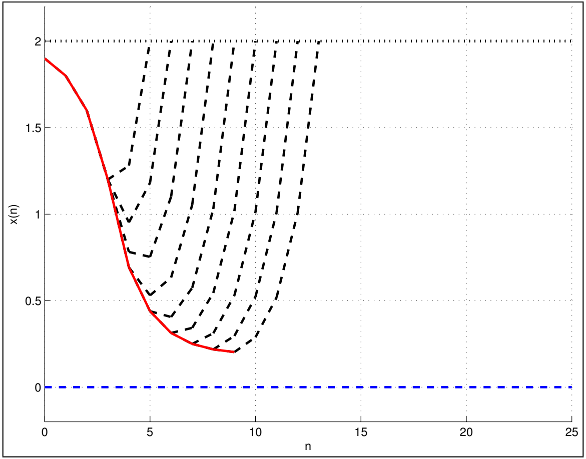

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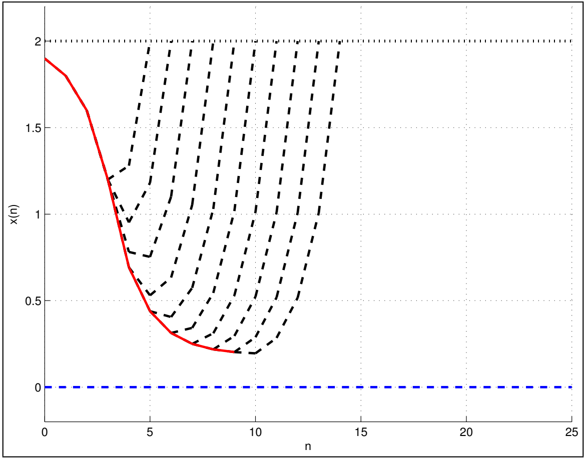

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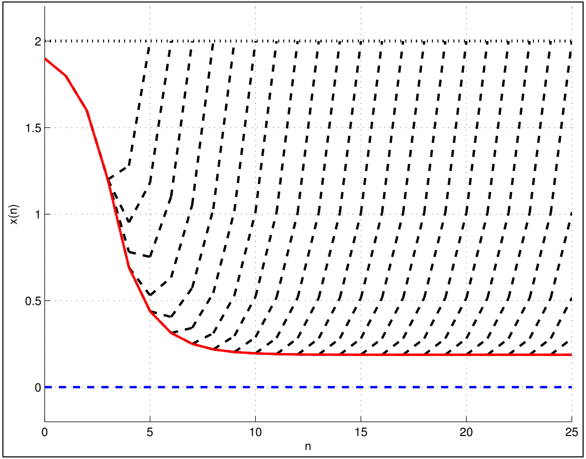

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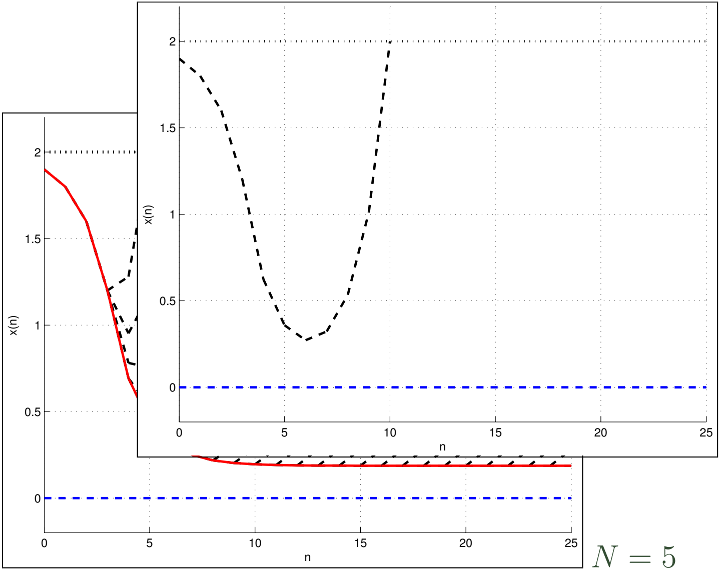

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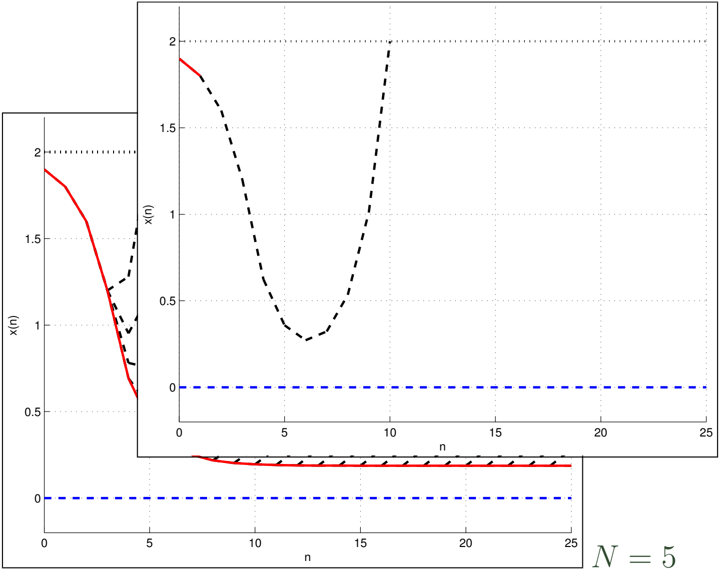

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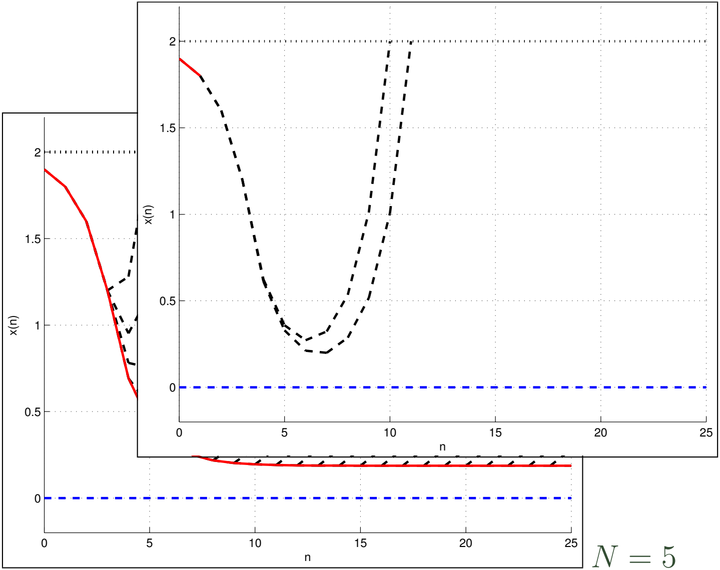

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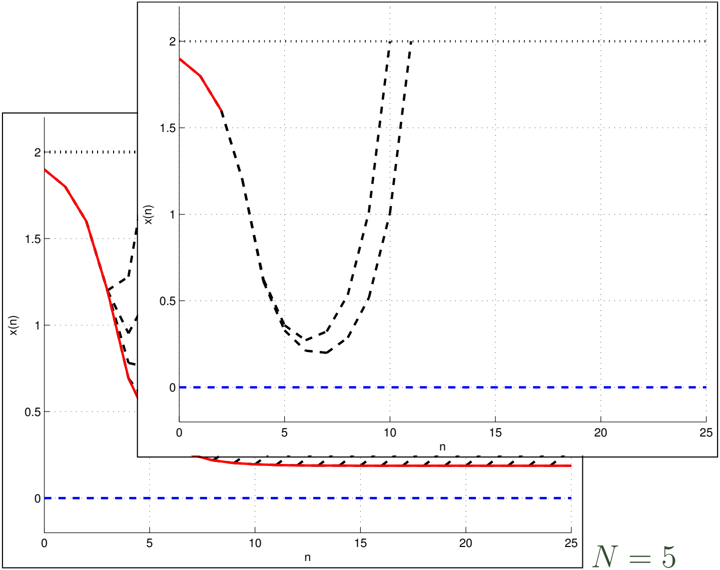

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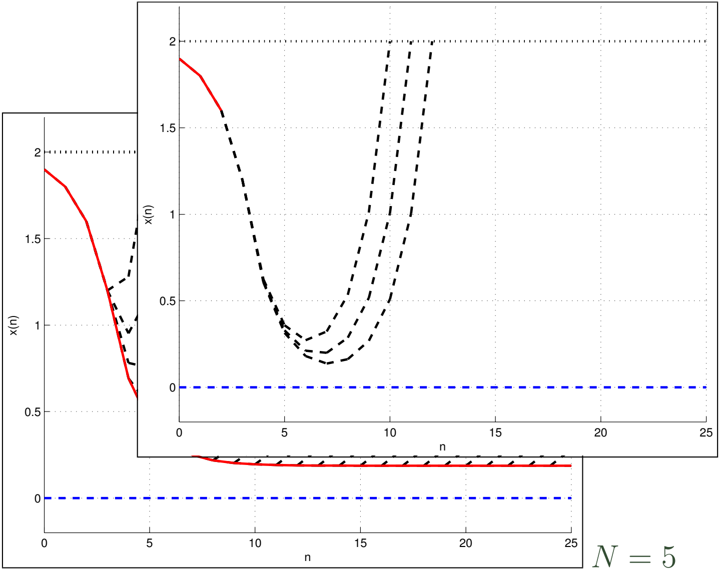

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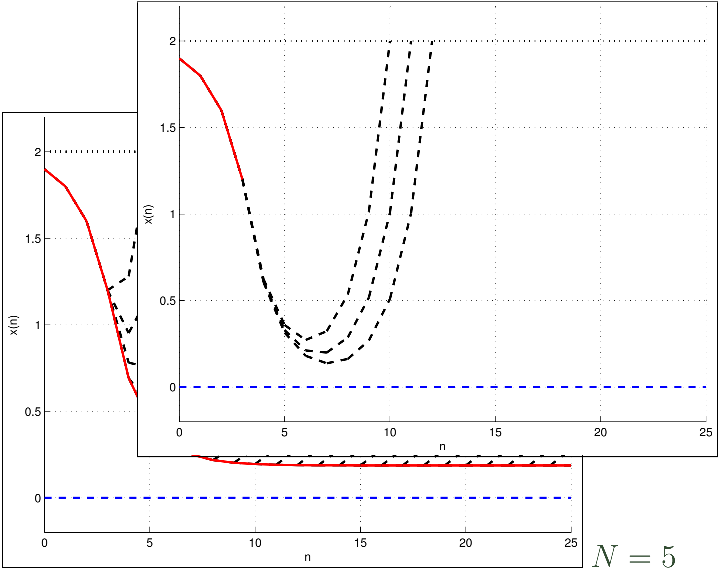

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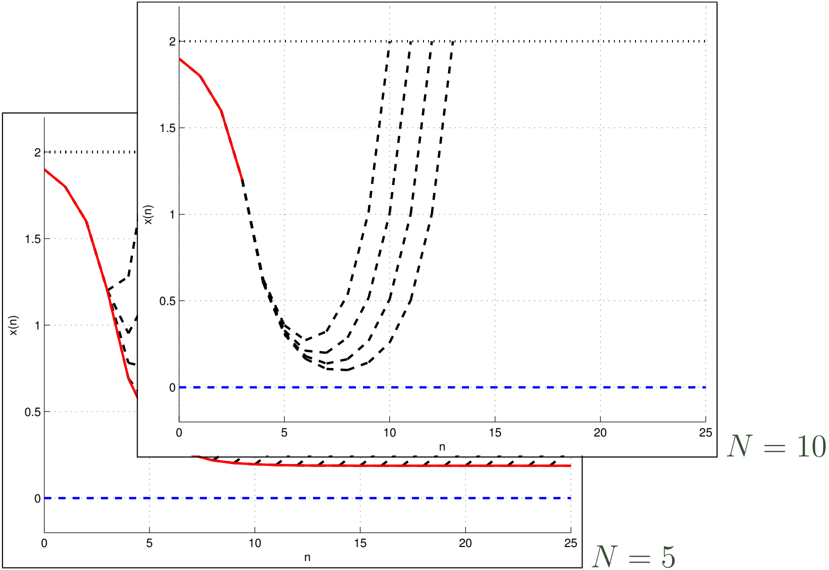

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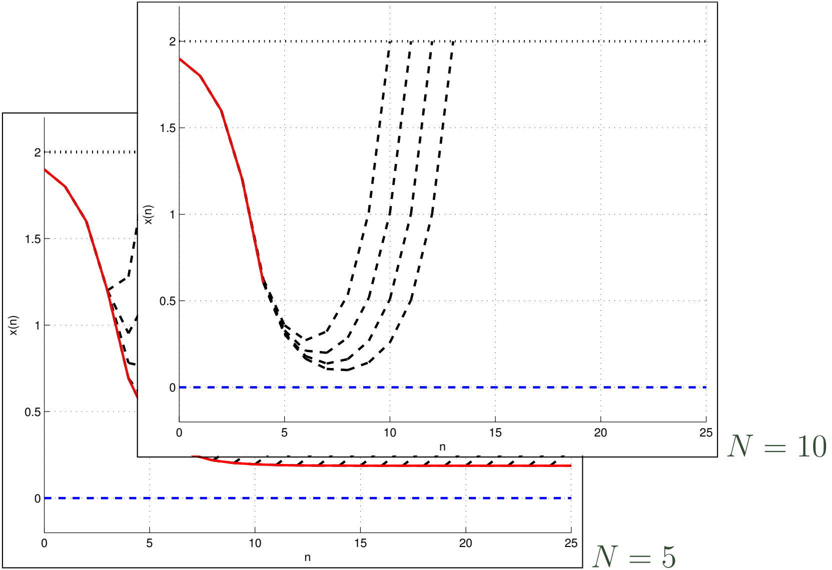

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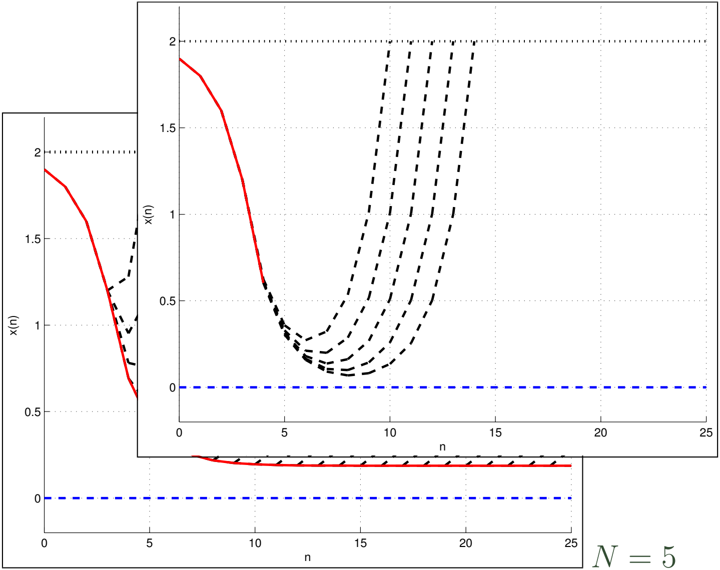

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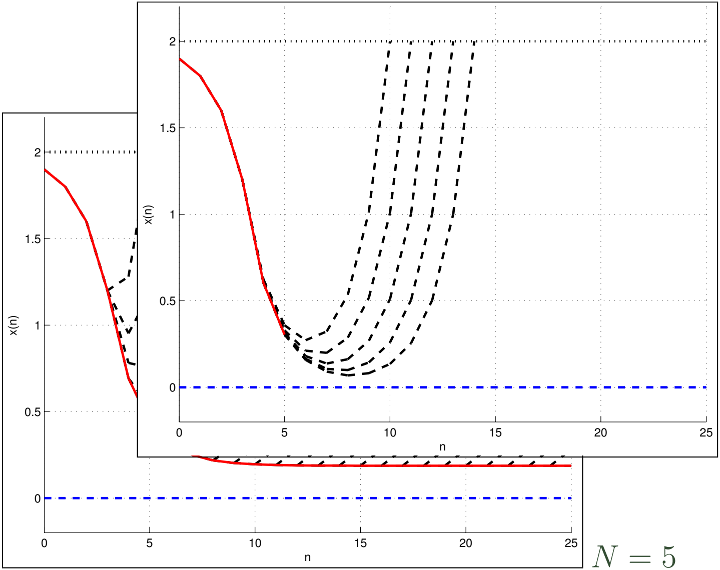

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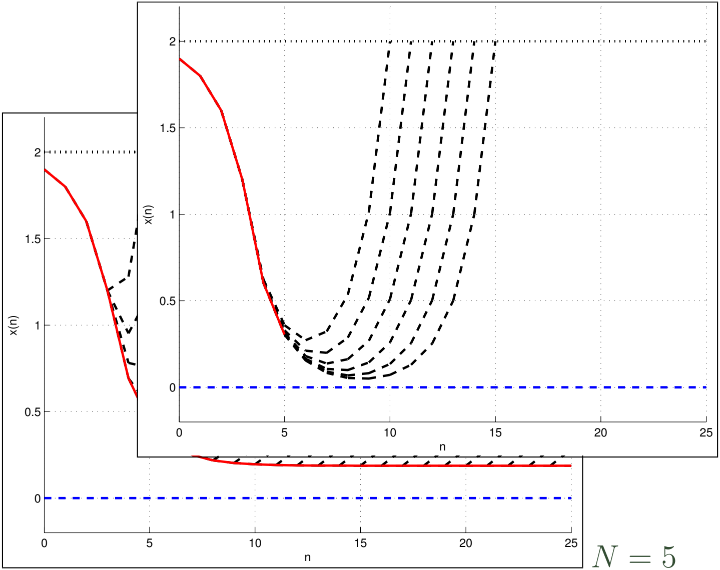

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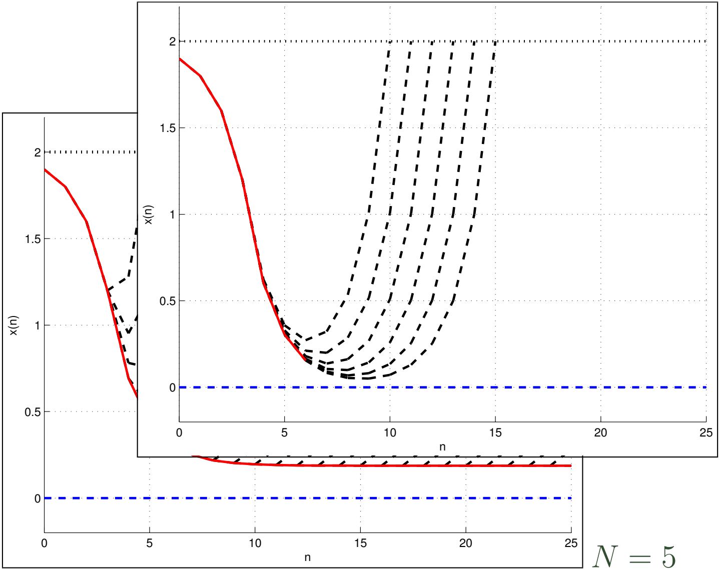

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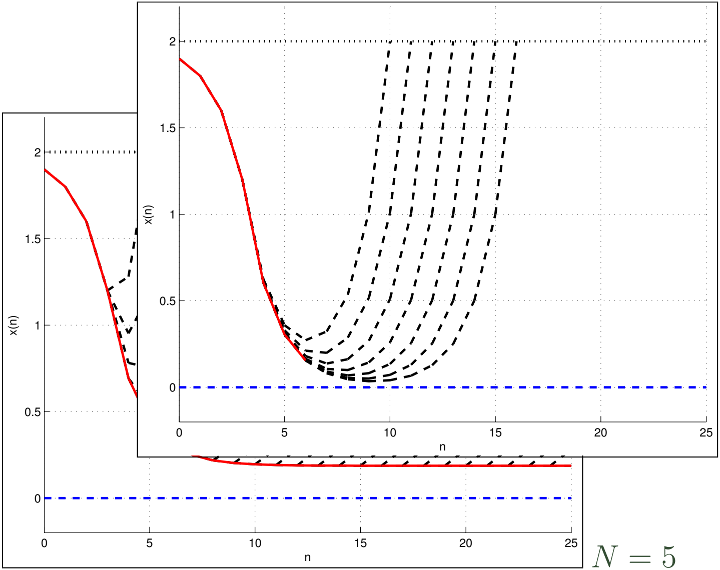

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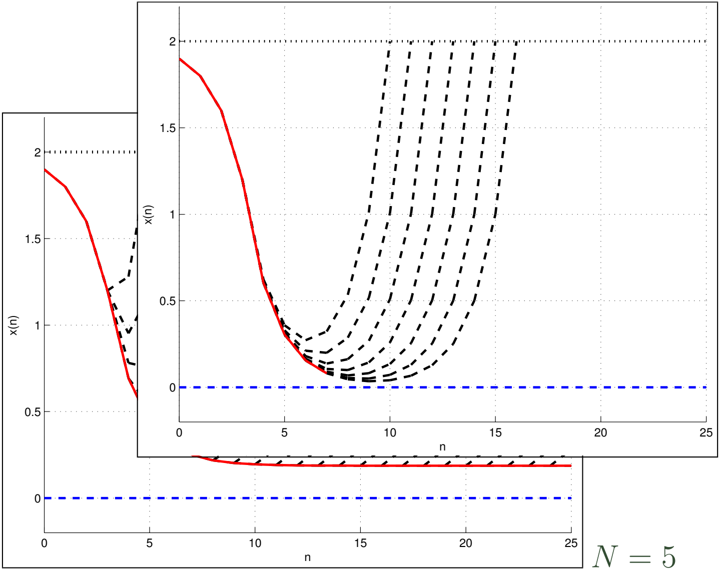

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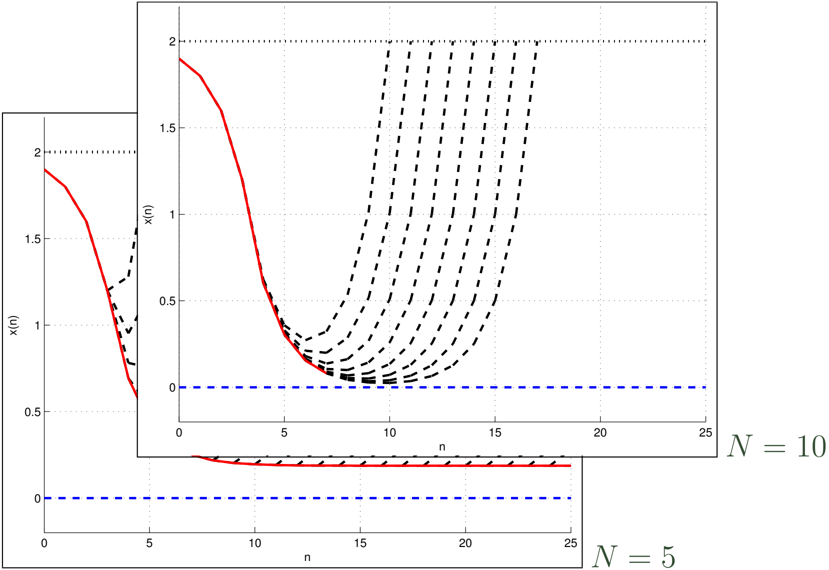

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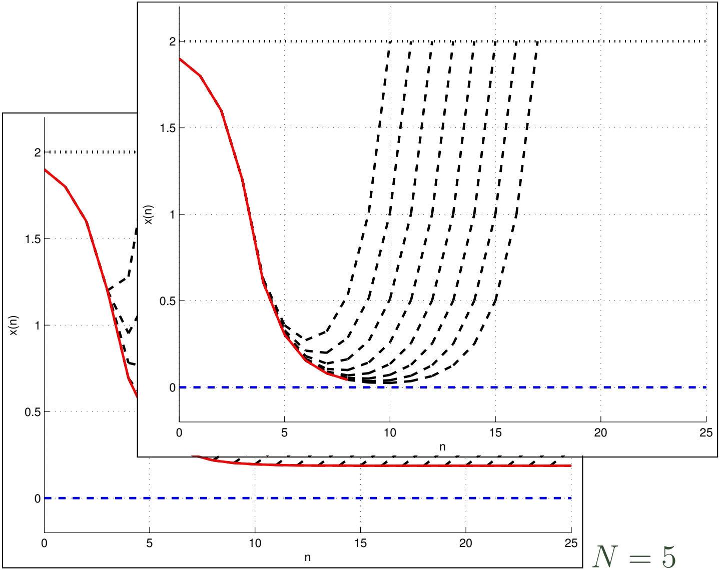

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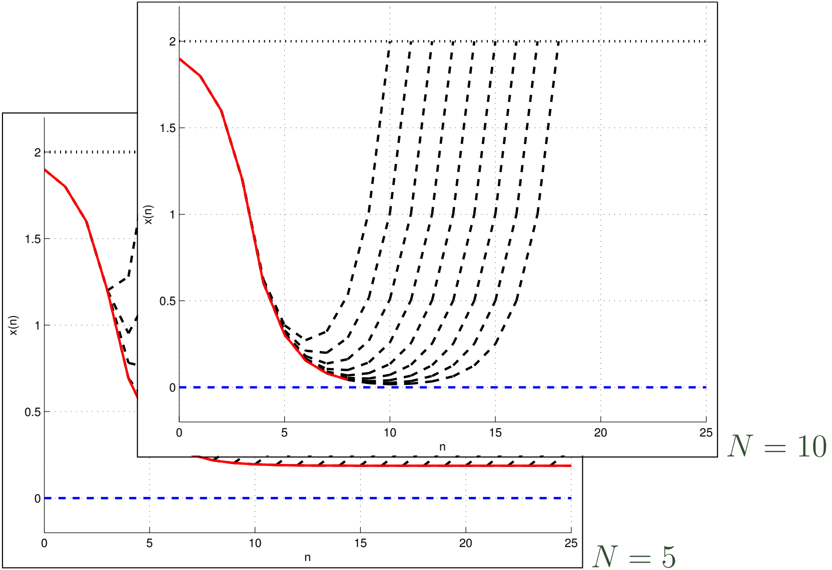

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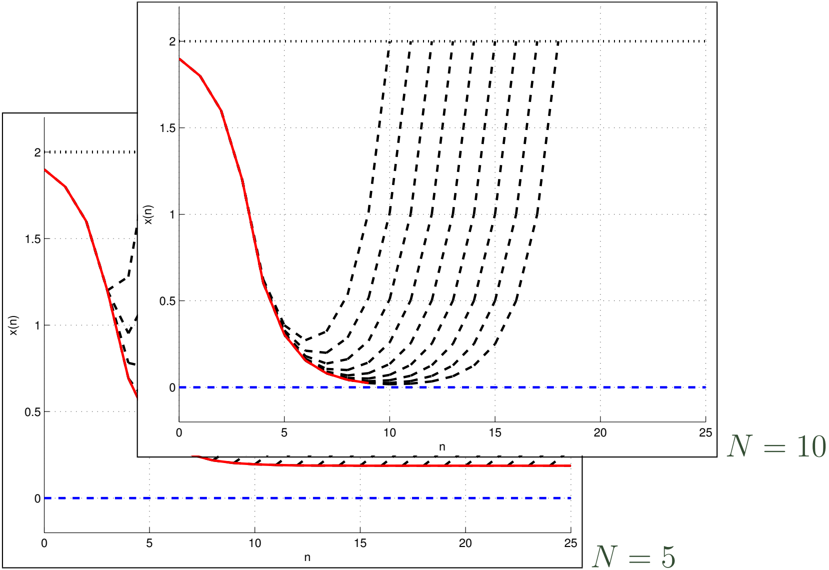

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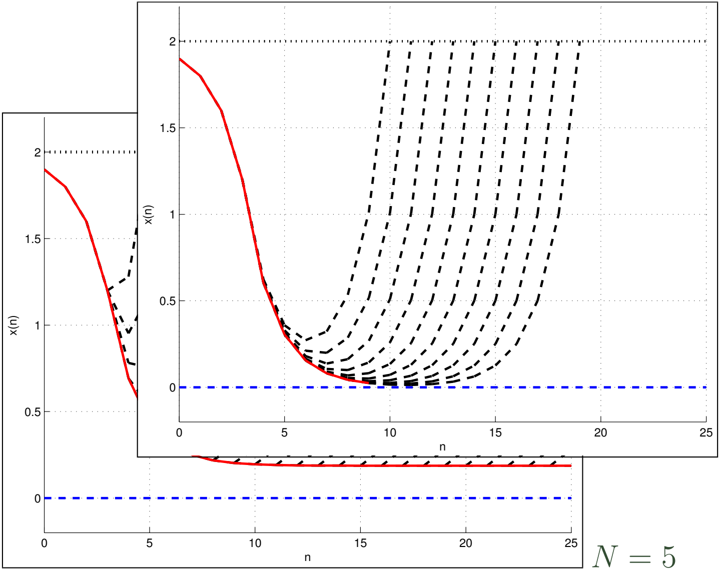

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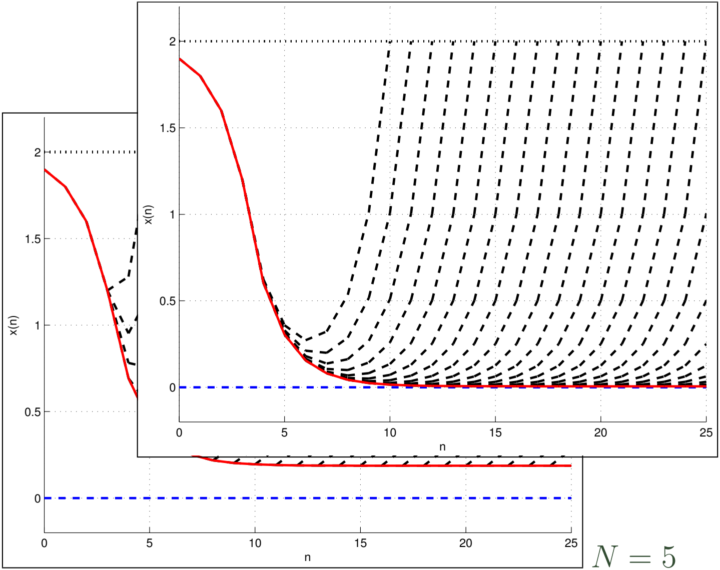

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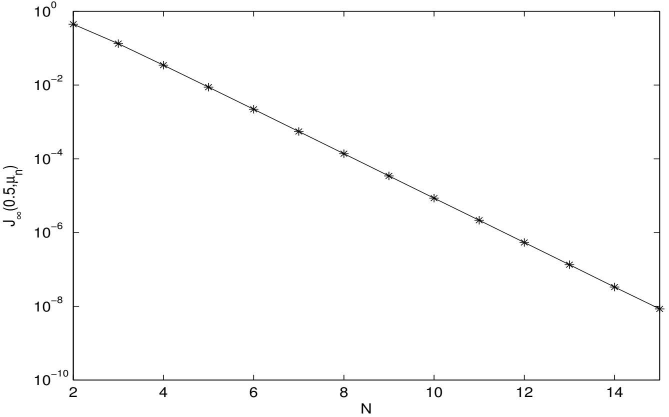
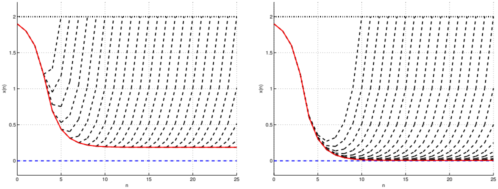
Example: averaged closed loop performance



$\overline{J}^{cl}_\infty(0.5, \mu_N) - \ell(x^e, u^e)$ depending on N, logarithmic scale

Observations



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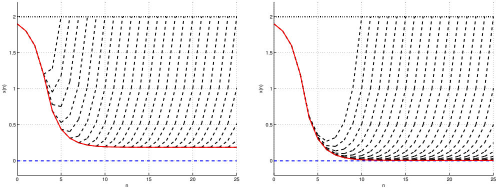

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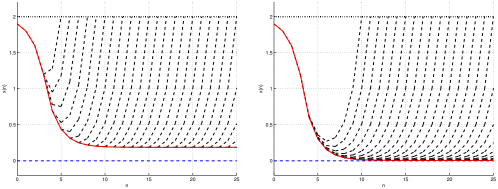

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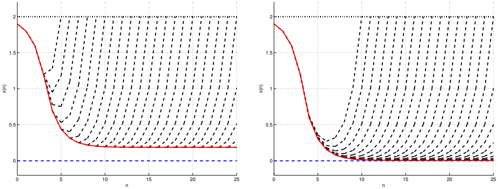

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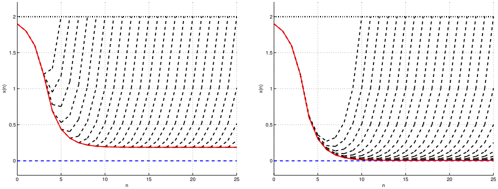

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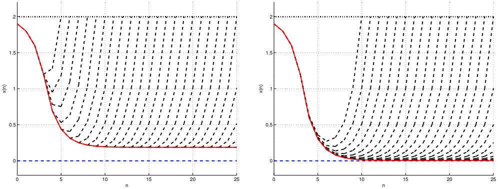

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Can we prove this behavior?

The first property will turn out to be the crucial one

Towards a performance estimate

Defining the optimal value function $V_N(x) := \inf_{\mathbf{u}} J_N(x, \mathbf{u})$,
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In economic MPC, the desired inequality is

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Using this inequality for $x = x_{\mu_N}(0), \ldots, x_{\mu_N}(K-1)$ yields

$$\overline{J}_K^{cl}(x, \mu_N) \quad = \quad \frac{1}{K} \sum_{n=0}^{K-1} \ell(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n)))$$

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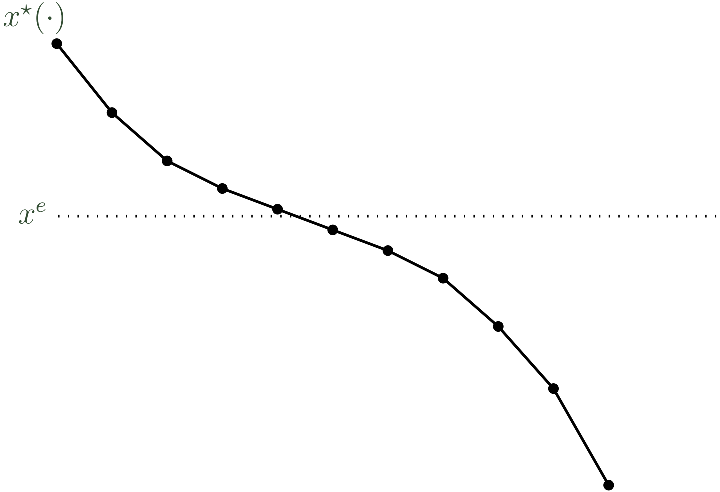
This can be achieved by prolonging the trajectory close to x^e

Prolonging near x^e

Sketch of the idea:



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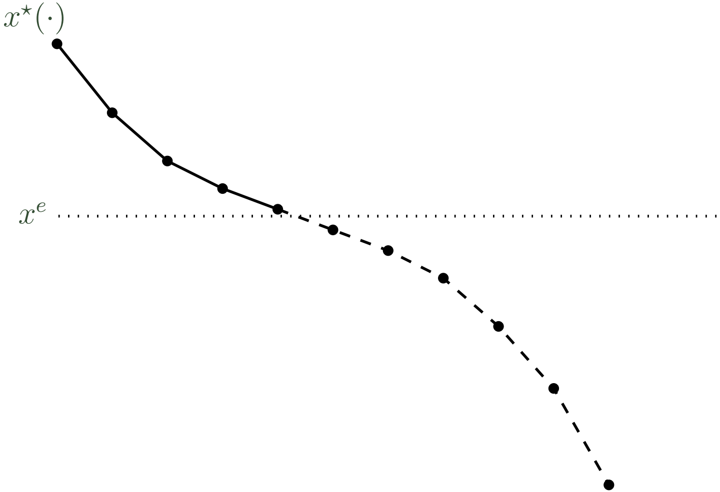

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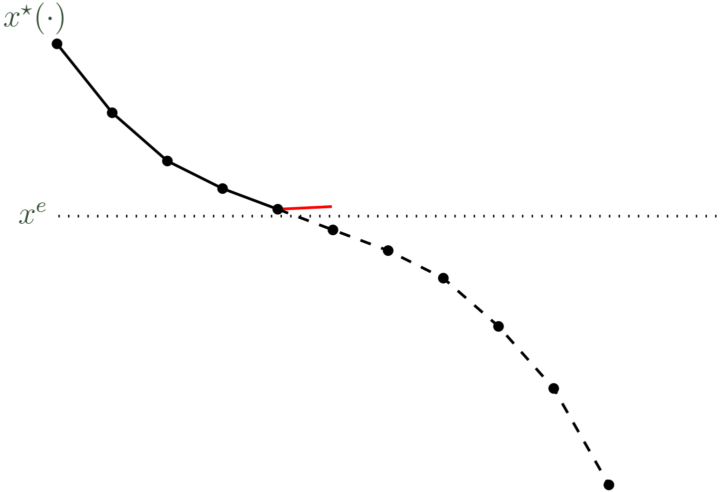

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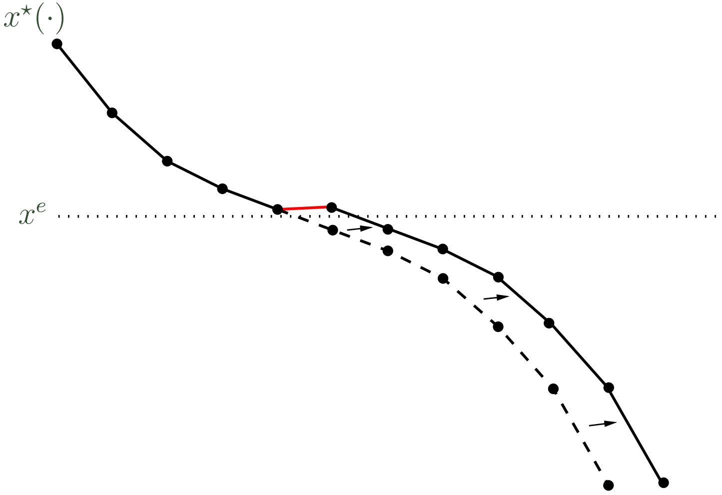

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Instead of the turnpike property, in the MPC literature another property is usually imposed: strict dissipativity

Strict dissipativity [Willems '72]

The optimal control problem is called strictly dissipative if there exists $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below and $\alpha \in \mathcal{K}_\infty$ with

$$\ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)) \geq \alpha(\|x - x^e\|)$$

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While originally introduced as a sufficient condition guaranteeing the turnpike property, a recent result shows:

Theorem [Gr./Müller '16]: Under suitable controllability conditions, strict dissipativity is equivalent to a robust turnpike property plus optimality of the equilibrium (x^e, u^e)

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Stabilizing functionals are strictly dissipative with $\lambda \equiv 0$

Economic MPC theorem

Theorem: [Gr./Stieler '14]

Let f and ℓ be Lipschitz, $\mathbb{X}$ and $\mathbb{U}$ be compact and assume

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(for alternative conditions see also [Porretta/Zuazua '13]
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Under assumptions (i)–(iii), there exist $\varepsilon_1(N), \varepsilon_2(K) \to 0$ as $N \to \infty$ and $K \to \infty$, exponentially fast if additionally (iv) holds, such that the following properties hold

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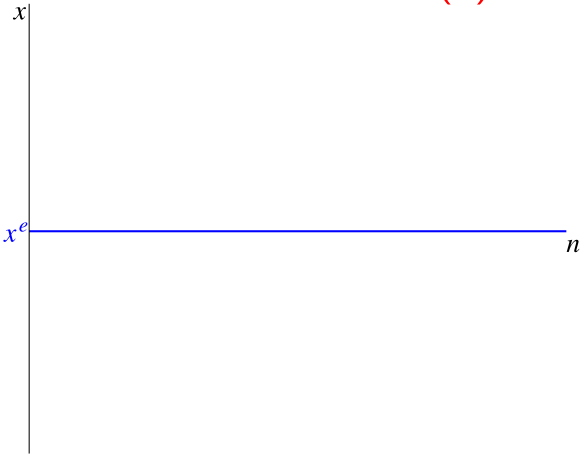
$$J^{cl}_K(x, \mu_N(x)) \le J_K(x, \mathbf{u}) + K\varepsilon_1(N) + \varepsilon_2(K)$$

for all admissible $\mathbf{u}$ with $\|x_{\mathbf{u}}(K,x) - x^e\| \le \beta(\|x - x^e\|, K) + \varepsilon_1(N)$

Illustration of (2) and (3)



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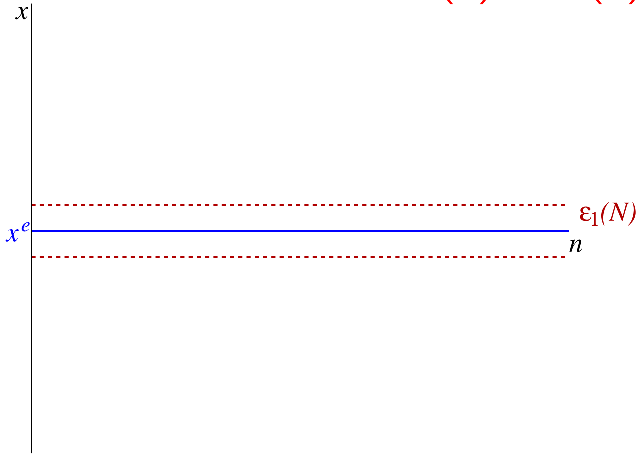

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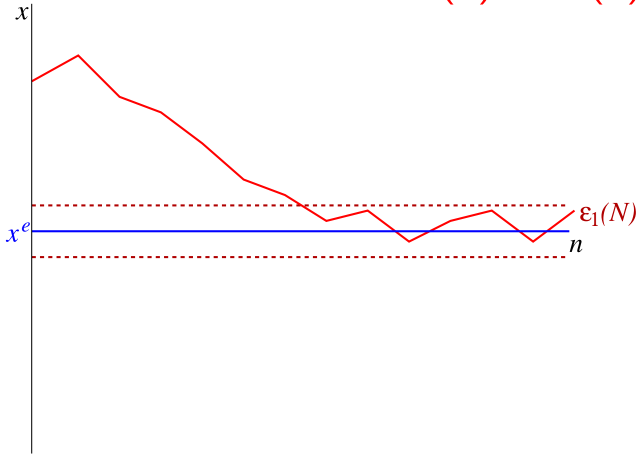

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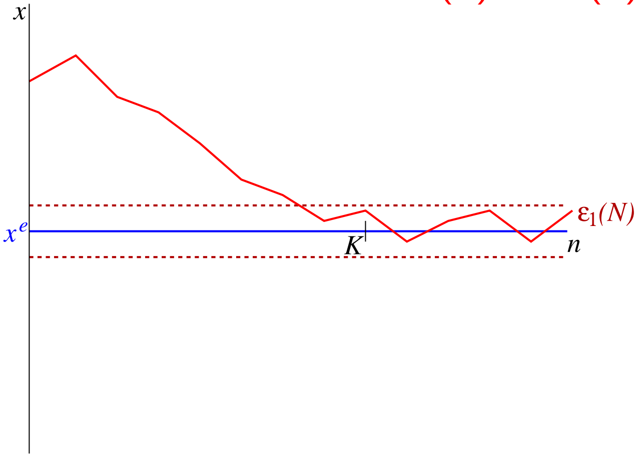

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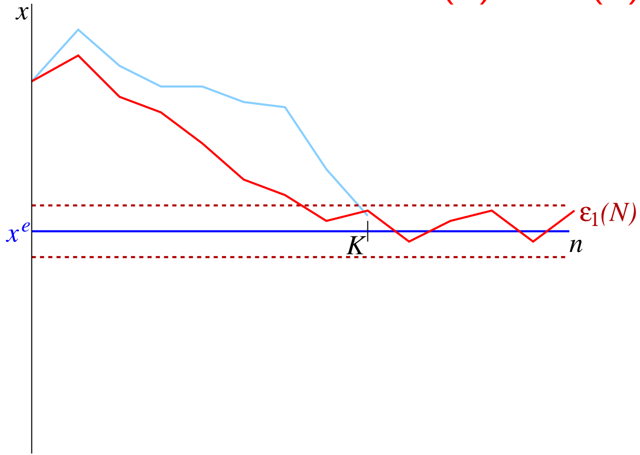

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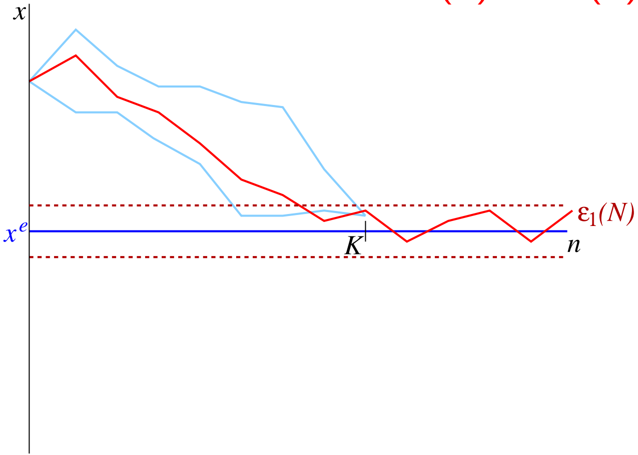

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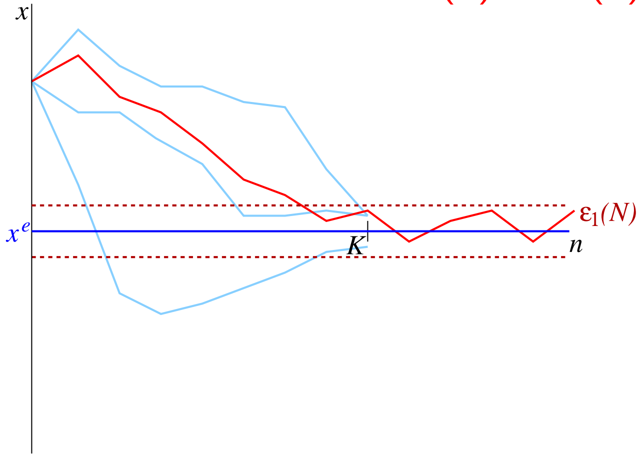

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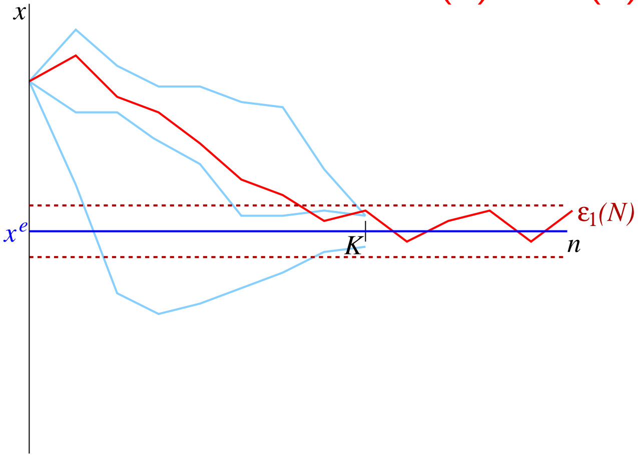

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(3): cost of all other trajectories reaching the ball at time K is higher than that of $x_{\mu_N}(n)$ up to the error $K\varepsilon_1(N) + \varepsilon_2(K)$

Economic MPC with terminal conditions

Like in stabilizing MPC, terminal conditions can be defined

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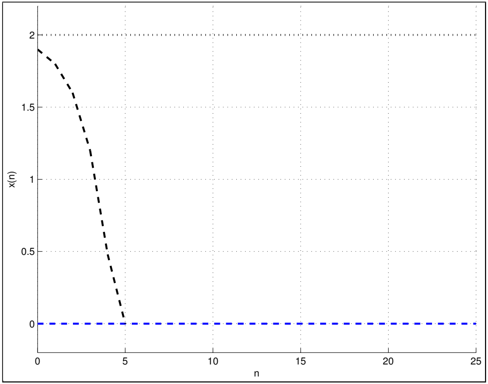
We illustrate the effect of terminal condition for the previous example with the simplest choice of the terminal constraints $\mathbb{X}_0 = \{x^e\}$

Example with terminal conditions

Example:
$N = 5$



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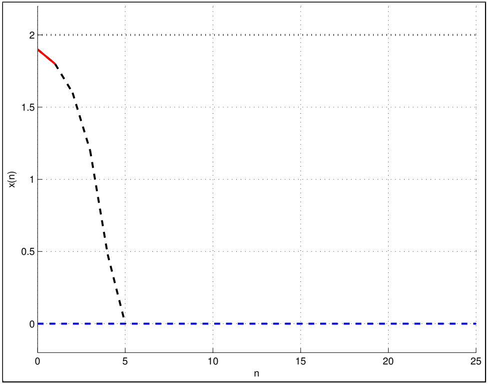

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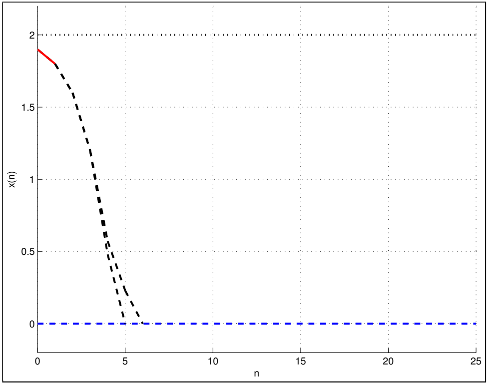

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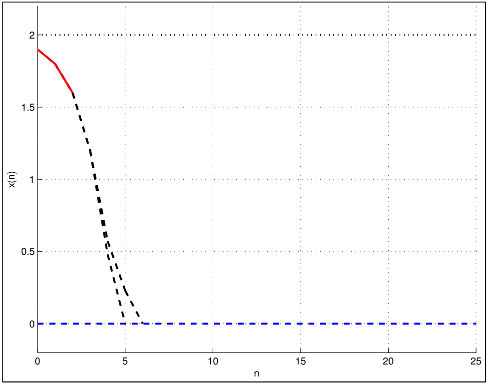

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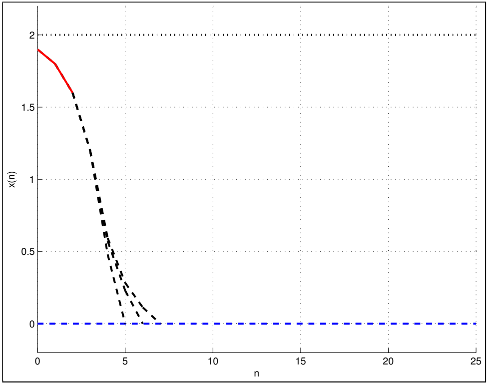

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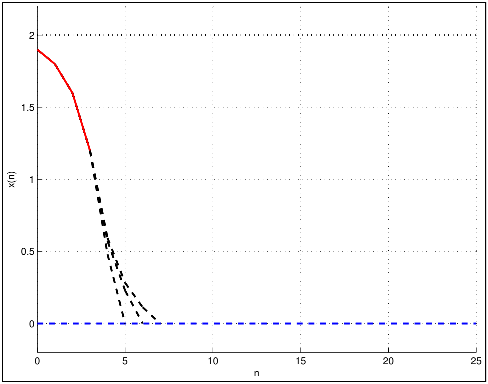

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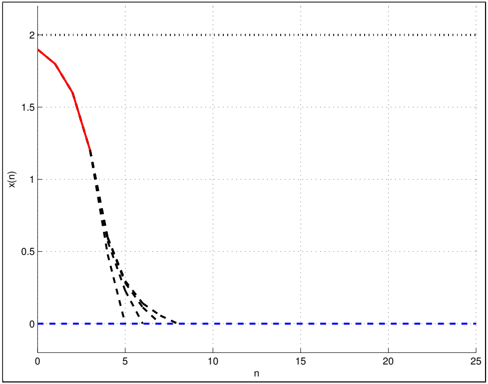

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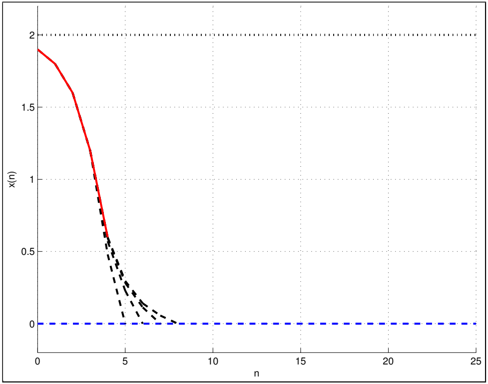

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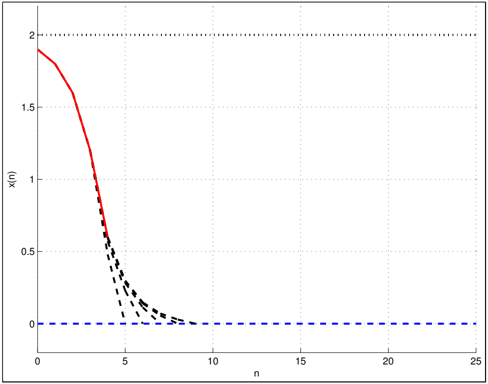

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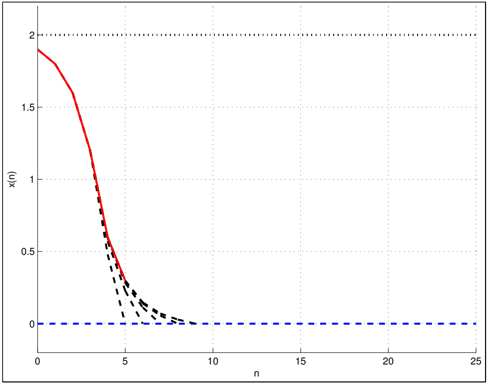

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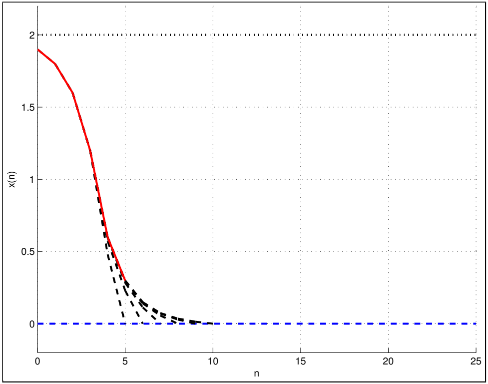

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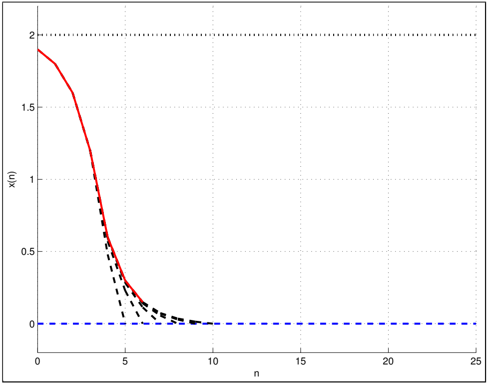

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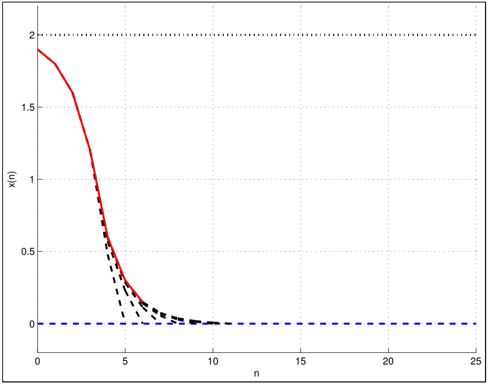

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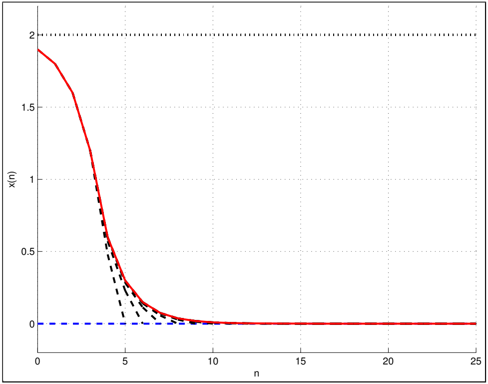

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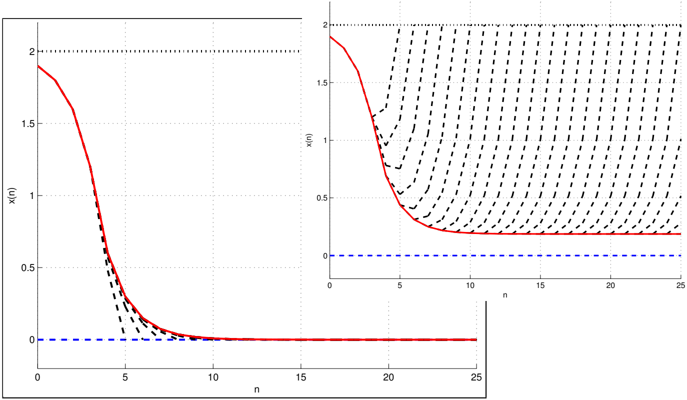

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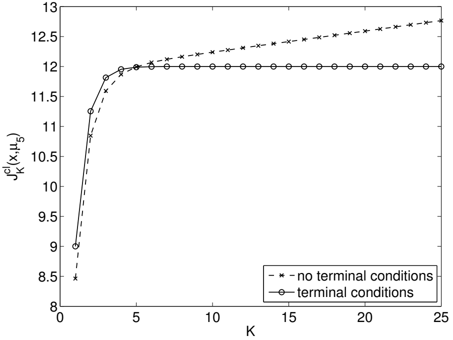

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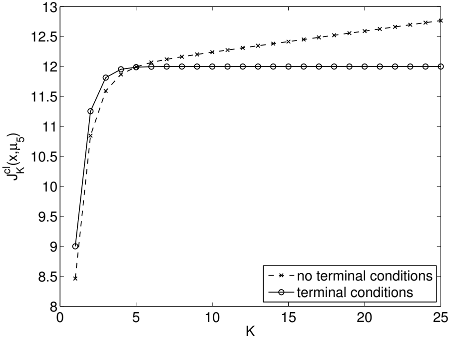

But: terminal constraints can cause infeasibility and severe numerical problems

Extensions, further results

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- First results for discounted optimal control problems [Gr./Semmler/Stieler '15, Gr./Kellett/Weller '16]

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- Strict dissipativity is essentially equivalent to this property and may be used as a checkable condition
- The required uniform continuity may pose a problem for control systems governed by PDEs $\rightsquigarrow$ work in progress

Selected literature

D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, Automatica, 36(2000), 789–814 ("The" classical reference for the results from (3))

L. Grüne and J. Pannek, Nonlinear Model Predictive Control, Springer, 2011 (contains most of the material from (2)–(4))

L. Grüne, NMPC without terminal constraints, Proceedings of the IFAC Conference on Nonlinear Model Predictive Control, 2012, 1–13 (survey with some results from (4) and (5))

D. Angeli, R. Amrit, J.B. Rawlings, On average performance and stability of economic model predictive control, IEEE Trans. Autom. Control, 57 (2012), 1615–1626 (some results from (5))

L. Grüne and M. Stieler, Asymptotic stability and transient optimality of economic MPC without terminal conditions, Journal of Process Control, 24 (2014), 1187–1196 (some results from (5))

L. Grüne, Approximation properties of receding horizon optimal control, DMV Nachrichten, 118 (2016), 3–38 (survey with results from Section (5) and variants)