Input–to–state Stability

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Lars Grüne, Mathematisches Institut, Universität Bayreuth

Part I: Introduction and Basic Definitions

Outline of the course

Part I: Introduction, definitions and basic properties

Part II: Relation to other stability concepts (GAS, H_∞, stability margins)

Part III: ISS Lyapunov functions and quantitative aspects

Part IV: Applications and ISS controller design

Asymptotic stability

Consider a nonlinear ordinary differential equation (ODE)

$$\dot{x}(t) = f(x(t))$$

with state $x \in \mathbb{R}^n$

For initial value $x \in \mathbb{R}^n$ at initial time $t = 0$ we denote the solution by $\varphi(t, x)$, i.e.,

- $x(t) = \varphi(t, x)$ solves the ODE
- $\varphi(0, x) = x$

we assume uniqueness of $\varphi(t, x)$ on its existence interval

Asymptotic stability

Assume that 0 is an equilibrium for the ODE, i.e., $f(0) = 0$

The ODE is called globally asymptotically stable (GAS) if for all initial values $x \in \mathbb{R}^n$ we have

Stability: for all $\varepsilon > 0$ there exists $\delta > 0$ with

$$\|x\| < \delta \;\;\Rightarrow\;\; \|\varphi(t,x)\| < \varepsilon \text{ for all } t \geq 0$$

Global attractivity: for all $\varepsilon > 0$ and $r > 0$ there is $T > 0$ with

$$\|x\| < r \;\;\Rightarrow\;\; \|\varphi(t,x)\| < \varepsilon \text{ for all } t \geq T$$

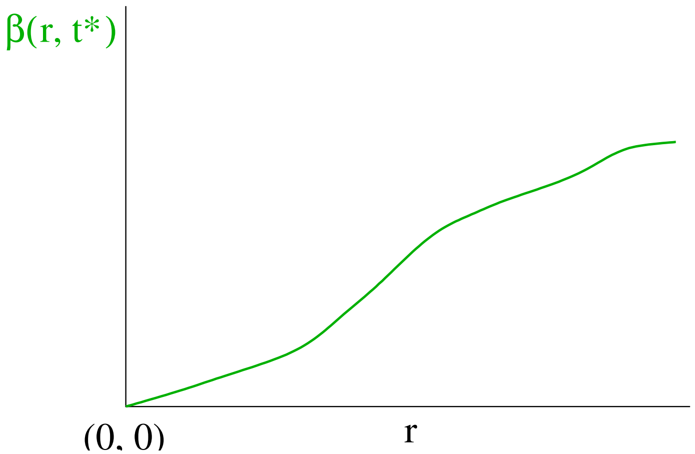
more convenient: define GAS using comparison functions

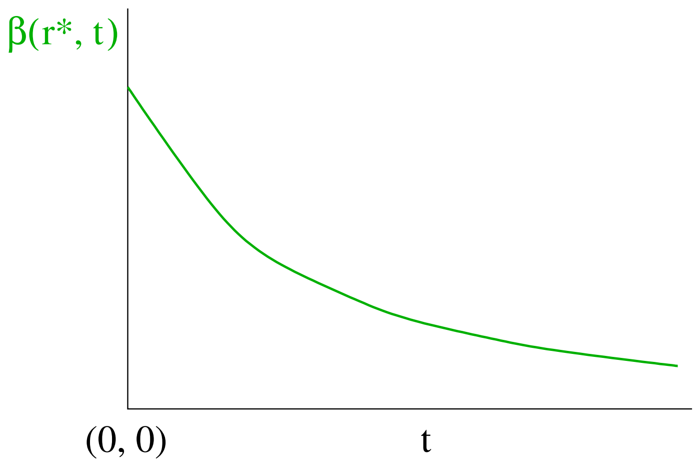
Comparison Functions [Hahn 67]

$\mathcal{K} := \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|$ continuous, strictly increasing, $\alpha(0) = 0\}$

$\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \,|$ unbounded$\}$

$\mathcal{KL} := \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|$ continuous, $\beta(\cdot, r) \in \mathcal{K}$ and β strictly converging to 0 in the 2nd argument$\}$





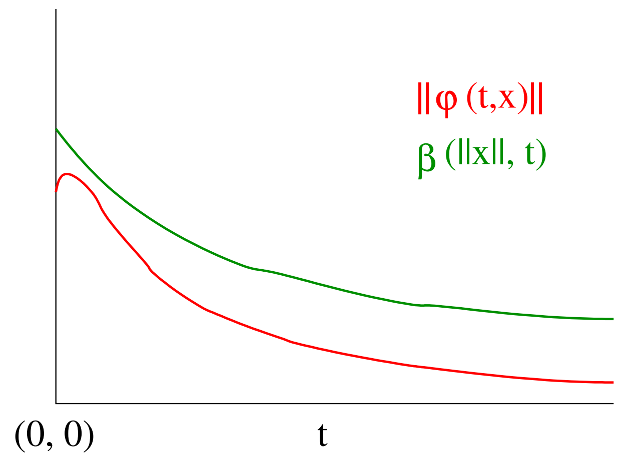
Asymptotic stability

The ODE is globally asymptotically stable (GAS)

$\Leftrightarrow$ there exists $\beta \in \mathcal{KL}$ such that

$$\|\varphi(t,x)\| \le \beta(\|x\|,t)$$

holds for all $x \in \mathbb{R}^n$, $t \ge 0$



Proof: "$\Leftarrow$" follows from the definition of $\mathcal{KL}$ functions

"$\Rightarrow$" follows by setting

$$\beta(r,t) := \max_{\|x\| \le r, s \ge t} \|\varphi(t,x)\| + e^{-t} r$$

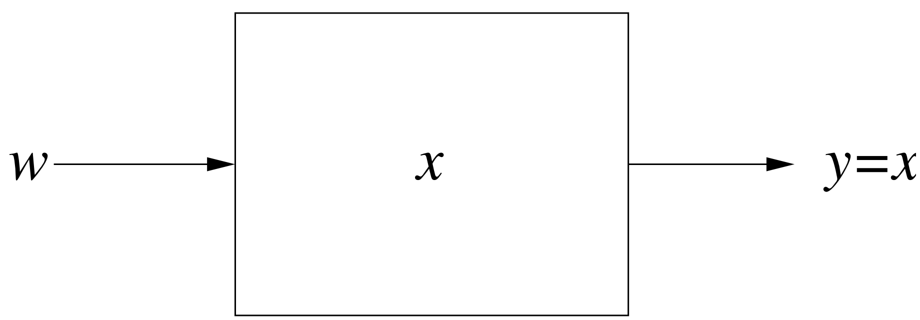
and checking the $\mathcal{KL}$ function properties

What is ISS?

ISS extends the GAS property to nonlinear systems

$$\dot{x}(t) = f(x(t), w(t))$$

with (perturbation) input $w \in \mathbb{R}^m$ and output = state $x \in \mathbb{R}^n$



What is ISS?

For initial value $x \in \mathbb{R}^n$ at initial time $t = 0$

and measurable and essentially bounded $w(t)$ (i.e., $w \in L^\infty$)

we denote the solution by $\varphi(t, x, w)$

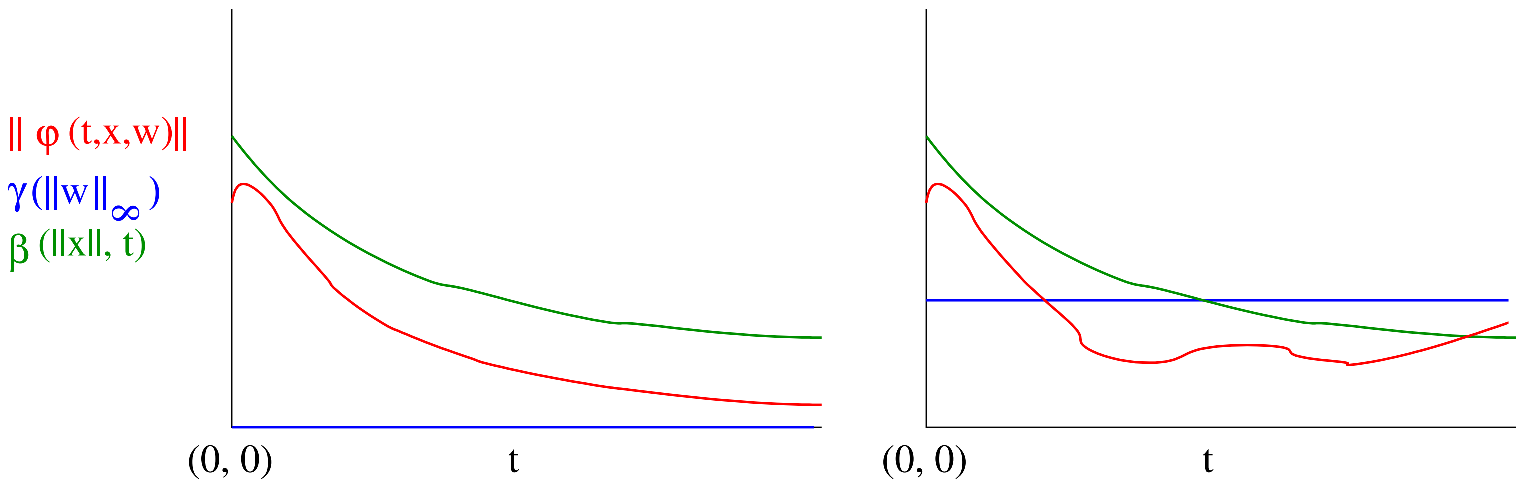
ISS requests that the system with input remains GAS up to an "error term" depending on the size of the perturbation w measured via

$$\|w\|_\infty := \operatorname*{ess\,sup}_{t \geq 0} \|w(t)\|$$

ISS

The system is called ISS, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t,x,w)\| \leq \max\{\beta(\|x\|,t),\ \gamma(\|w\|_\infty)\}$$



ISS

there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ with

$$\|\varphi(t,x,w)\| \le \max\{\beta(\|x\|,t),\, \gamma(\|w\|_\infty)\}$$

Since the solution $\varphi(t,x,w)$ only depends on $w(\tau)$ for $\tau \in [0,t]$, we can deduce the stronger inequality

$$\|\varphi(t,x,w)\| \le \max\{\beta(\|x\|,t),\, \gamma(\|w|_{[0,t]}\|_\infty)\}$$

where

$$w|_{[0,t]}(\tau) := \begin{cases} w(\tau), & \tau \in [0,t] \\ 0, & \tau \notin [0,t] \end{cases}$$

ISS

there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ with

$$\|\varphi(t,x,w)\| \leq \max\{\beta(\|x\|,t),\, \gamma(\|w\|_\infty)\}$$

Equivalent formulation: there exist $\tilde{\beta} \in \mathcal{KL}$, $\tilde{\gamma} \in \mathcal{K}_\infty$ with

$$\|\varphi(t,x,w)\| \leq \tilde{\beta}(\|x\|,t) + \tilde{\gamma}(\|w\|_\infty)$$

"$\Rightarrow$": follows with $\tilde{\gamma} = \gamma$, $\tilde{\beta} = \beta$

"$\Leftarrow$": follows with $\gamma = 2\tilde{\gamma}$, $\beta = 2\tilde{\beta}$

Components of ISS

there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ with

$$\|\varphi(t,x,w)\| \leq \max\{\beta(\|x\|,t),\, \gamma(\|w\|_\infty)\}$$

Looking at β and γ separately, one identifies two components:

0–GAS: global asymptotic stability for $w \equiv 0$

$$\|\varphi(t,x,0)\| \leq \beta(\|x\|,t)$$

asymptotic gain: the solutions are ultimately bounded by $\gamma(\|w\|_\infty)$

$$\limsup_{t\to\infty} \|\varphi(t,x,w)\| \leq \gamma(\|w\|_\infty)$$

fact: ISS $\Leftrightarrow$ 0–GAS and asymptotic gain (“$\Leftarrow$” is nontrivial)

ISS for linear systems

Consider a linear system

$$\dot{x}(t) = Ax(t) + Bw(t),$$

with A, B matrices of appropriate dimensions

Fact 1: GAS $\Leftrightarrow$ A Hurwitz, i.e., $\max_{\lambda \text{ eigenvalue of } A} \mathrm{Re}(\lambda) < 0$

$\Leftrightarrow$ $\|e^{At}\| \leq Ce^{-\sigma t}$ for appropriate C, $\sigma > 0$

Fact 2: $\varphi(t, x, w) = e^{At}x + \int_0^t e^{A(t-s)}Bw(s)ds$

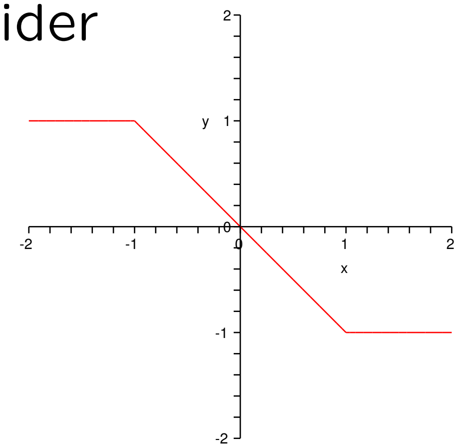
$$\Rightarrow \qquad \|\varphi(t, x, w)\| \leq \beta(\|x\|, t) + \gamma(\|w\|_\infty)$$

for $\beta(r, t) = Ce^{-\sigma t}r$ and $\gamma(r) = \|B\| \int_0^\infty \|e^{As}\| ds\, r$ $\rightsquigarrow$ ISS

ISS for nonlinear systems

For linear systems 0–GAS $\Leftrightarrow$ ISS

For nonlinear systems this is not true: consider



$$\dot{x} = w - \mathrm{sat}(x) = w - \begin{cases} 1, & x > 1 \\ x, & x \in [-1, 1] \\ -1, & x < -1 \end{cases}$$

The system is 0–GAS but for $w(t) \equiv 2$ and $x = 1$ we obtain

$$\varphi(t, x, w) = 1 + t$$

which is unbounded, hence ISS cannot hold

ISS for nonlinear systems

Consequences:

- ISS is strictly stronger than GAS
- ISS generalizes an inherent property of linear GAS systems to nonlinear systems

In Part II we will discuss the relation of ISS to other stability properties, like H_∞

In this context, we will also discuss the role of coordinate changes, which explains why ISS is formulated as it is

Lyapunov functions

Recall that GAS can be characterized via Lyapunov functions:

An ODE is GAS if and only if there exists a Lyapunov function, i.e. a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and functions α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

and

$$DV(x)f(x) \leq -\alpha_3(\|x\|)$$

hold for all $x \in \mathbb{R}^n$

Sufficiency: $\frac{d}{dt}V(\varphi(t,x)) = DV(\varphi(t,x))f(\varphi(t,x)) \leq -\alpha_3(\|\varphi(t,x)\|)$ implies that $V(\varphi(t,x))$ is strictly decreasing and thus tends to 0

ISS Lyapunov functions

A system is ISS if and only if there exists an ISS Lyapunov function, i.e. a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and functions α_1, α_2, $\chi \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

and

$$\|w\| \leq \chi(\|x\|) \quad \Rightarrow \quad DV(x)f(x,w) \leq -\alpha_3(\|x\|)$$

hold for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

ISS Lyapunov functions (and a proof of this result) will be discussed in detail in Part III

Variants of ISS: local ISS

We can restrict ISS to a neighborhood B of 0 and to a restricted set of perturbations $\|w\|_\infty \leq R$

The system is called locally ISS, if there exist a neighborhood $B \subset \mathbb{R}^n$ of 0, a value $R > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values $x \in B$, all perturbation functions w with $\|w\|_\infty \leq R$ and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t),\ \gamma(\|w\|_\infty)\}$$

Variants of ISS: iISS

If we interpret the system as a map $w \mapsto \varphi$, then ISS can be seen as an "$L^\infty \to L^\infty$" stability property

A weaker property is the "$L^2 \to L^\infty$" stability, which leads to the integral ISS (iISS) property:

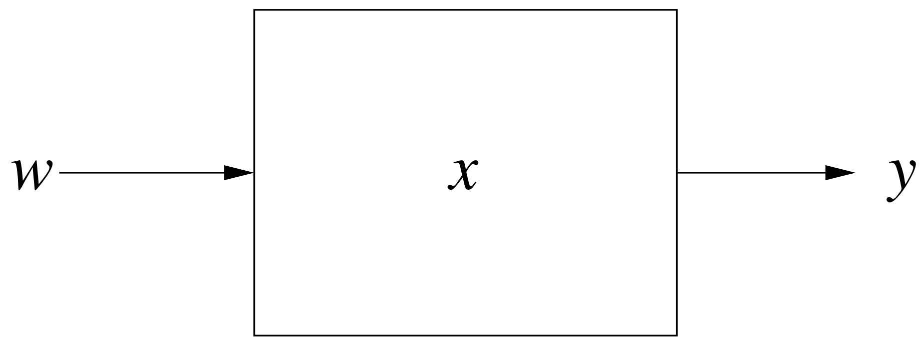
There exist $\beta \in \mathcal{KL}$, $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t,x,w)\| \leq \beta(\|x\|,t) + \gamma_1\left(\int_0^t \gamma_2(\|w(s)\|)ds\right)$$

Variants of ISS: systems with output

Consider a nonlinear system with output $y \in \mathbb{R}^l$

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ y(t) &= h(x(t)) \end{aligned}$$



The output can enter ISS in many different ways, we sketch two of them

Variants of ISS: IOS

The system is called input-to-output-stable (IOS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|y(t)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\}$$

Interpretation: external stability robust w.r.t. w

Motivation: regulator theory, generalized small gain theorem

Variants of ISS: IOSS

The system is called input-output-to-state stable (IOSS), if there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t,x,u)\| \leq \max\{\beta(\|x\|,t),\, \gamma_1(\|w\|_\infty),\, \gamma_2(\|y\|_\infty)\}$$

Interpretation: appropriate nonlinear version of zero detectability

Variants of ISS: ISDS

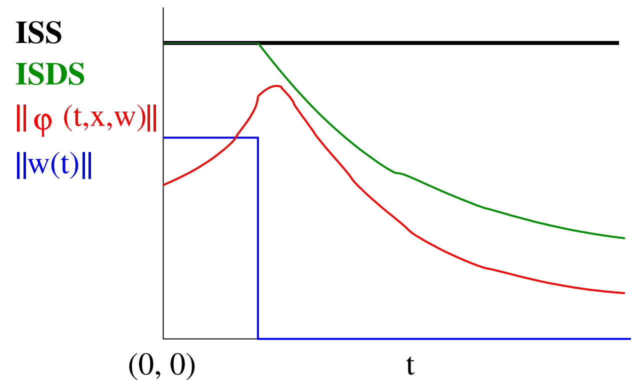
ISS has the disadvantage that the dependence on $w(t)$ is static because the time dependence is not reflected in $\gamma(\|w\|_\infty)$

This is overcome by the input-to-state-dynamical-stability (ISDS), where $\gamma(\|w\|_\infty)$ is replaced by the time–varying term

$$\nu(w,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \; \mu(\gamma(\|w(\tau)\|), t-\tau)$$

for $\mu \in \mathcal{KL}$

$\rightsquigarrow$ fading of past perturbations



ISDS is qualitatively equivalent to ISS but more suitable for quantitative studies, details will be discussed in Part III

Summary of Part I

- ISS is a generalization of GAS for nonlinear perturbed systems
- ISS combines 0–GAS and asymptotic gain
- for linear systems ISS is equivalent to 0–GAS — but not for nonlinear systems
- ISS can be characterized via ISS Lyapunov functions (if and only if)
- ISS variants: local version, integral bounds, output versions