Input–to–state Stability

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Part II: Relation to Other Stability Concepts

ISS

Consider

$$\dot{x}(t) = f(x(t), w(t))$$

with solutions $\varphi(t, x, w)$

The system is called ISS, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\}$$

In this part of the course, we will investigate how ISS is related to other stability concepts

Local ISS

We start with the property which is easiest to treat:

The system is called locally ISS, if there exist a neighborhood $B \subset \mathbb{R}^n$ of 0, a value $R > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values $x \in B$, all perturbation functions w with $\|w\|_\infty \leq R$ and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t),\, \gamma(\|w\|_\infty)\}$$

Local 0–AS

Setting $w \equiv 0$, local ISS

$$\|\varphi(t,x,w)\| \le \max\{\beta(\|x\|,t),\, \gamma(\|w\|_\infty)\}$$

implies local asymptotic stability for $w \equiv 0$ (local 0–AS):

The system is called locally 0–AS, if there exist a neighborhood $B \subset \mathbb{R}^n$ of 0 and a function $\beta \in \mathcal{KL}$ such that for all initial values $x \in B$ and all times $t \ge 0$ the following inequality holds:

$$\|\varphi(t,x,0)\| \le \beta(\|x\|,t)$$

Theorem: local ISS $\Leftrightarrow$ local 0–AS

Local 0–AS

local ISS $\Leftrightarrow$ local 0–AS

Idea of Proof: local ISS $\Rightarrow$ local 0–AS immediate for $w \equiv 0$

Conversely, local 0–AS implies existence of Lyapunov function

$$DV(x)f(x,0) \leq -\alpha_3(\|x\|)$$

Choosing $\chi \in \mathcal{K}_\infty$ with

$$\max_{\|x\|=r,\, \|w\| \leq \chi(r)} \|DV(x)\| \|f(x,w) - f(x,0)\| \leq \alpha_3(r)/2$$

for all $r \in [0,R]$, $x \in B$, we obtain

$$\|w\| \leq \chi(\|x\|) \;\;\Rightarrow\;\; DV(x)f(x,w) \leq -\alpha_3(\|x\|)/2$$

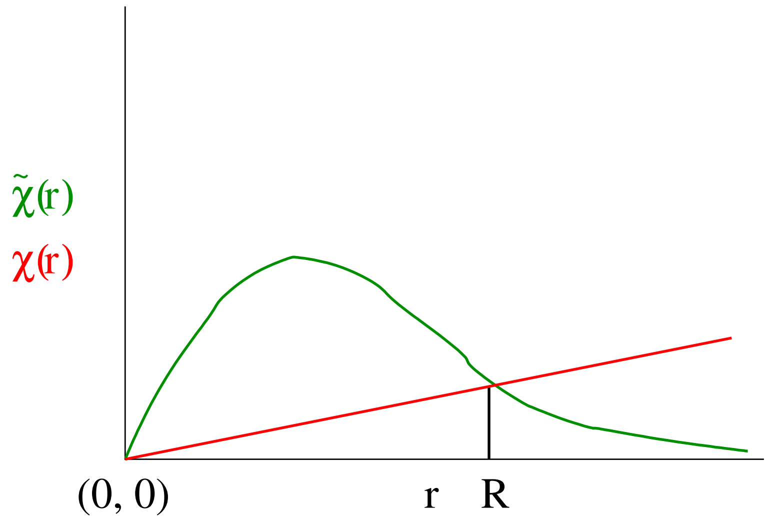
$\Rightarrow$ ISS Lyapunov function $\Rightarrow$ ISS

Local vs. global 0–AS

Note: the maximal function $\tilde{\chi}$ satisfying

$$\max_{\|x\|=r,\, \|w\|\le\tilde{\chi}(r)} \|DV(x)\| \|f(x,w) - f(x,0)\| \le \alpha_3(r)/2$$

can always be underbounded by $\chi \in \mathcal{K}_\infty$ for $r \in [0, R]$



This may not be possible for $r \in [0, \infty)$. This is why

0–GAS $\not\Rightarrow$ global ISS

0–GAS & asymptotic gain

there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ with

$$\|\varphi(t,x,w)\| \leq \max\{\beta(\|x\|,t),\ \gamma(\|w\|_\infty)\}$$

0–GAS: $$\|\varphi(t,x,0)\| \leq \beta(\|x\|,t)$$

Recall the asymptotic gain property:

$$\limsup_{t\to\infty} \|\varphi(t,x,w)\| \leq \gamma(\|w\|_\infty)$$

Theorem (ISS superposition principle):

ISS $\Leftrightarrow$ 0–GAS & asymptotic gain

Sketch of Proof: "$\Rightarrow$" immediate

0–GAS & asymptotic gain

"ISS $\Leftarrow$ 0–GAS & asymptotic gain"

Step 1: 0–GAS implies local ISS

Step 2: local ISS and asymptotic gain imply uniform as. gain:

For all $\varepsilon > 0$, $\Delta > 0$ there exists $T = T(\varepsilon, \Delta) > 0$ such that

$$\|\varphi(t,x,w)\| \leq \gamma(\|w\|_\infty) + \varepsilon$$

whenever $\|x\| \leq \Delta$, $t \geq T$

Step 3: local ISS and uniform as. gain imply ISS, where the β for large $\|x\|$ is built from $T(\varepsilon, \Delta)$

Stability margins

We say that the system admits a stability margin $\rho \in \mathcal{K}_\infty$ if for any feedback map $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ Lipschitz in x with $\|k(t,x)\| \leq \rho(\|x\|)$ the closed loop system

$$\dot{x}(t) = f(x(t), k(t, x(t)))$$

is globally asymptotically stable, i.e., its solutions $\varphi_k(t,x)$ satisfy

$$\|\varphi_k(t,x)\| \leq \tilde{\beta}(\|x\|, t)$$

for some suitable $\tilde{\beta} \in \mathcal{KL}$ independent of k

Theorem: ISS $\Leftrightarrow$ existence of a stability margin

Stability margins

Proof: ISS $\Rightarrow$ existence of a stability margin

We have $\varphi_k(t,x) = \varphi(t,x,w)$ for $w(t) = k(t,\varphi_k(t,x))$.

For $T > 0$ and $t \in [0,T]$, ISS implies

$$\|\varphi_k(t,x)\| \le \max\left\{\beta(\|x\|,t),\ \max_{t\in[0,T]} \gamma(\rho(\|\varphi_k(t,x)\|))\right\}$$

Now $\rho(r) \le \gamma^{-1}(\beta(\|x\|,T))$ for $r \in [\beta(\|x\|,T), \beta(\|x\|,0)]$ implies

$$\|\varphi_k(t,x)\| \le \beta(\|x\|,t), \quad t \in [0,T]$$

For sufficiently large $T = T(\|x\|)$ we can now proceed inductively in order to construct $\tilde\beta$ from β

Note: $\tilde\beta$ and ρ are only implicitly given

Stability margins

Proof: ISS $\Leftarrow$ existence of a stability margin

Given w and $t > 0$ set $k(\cdot, x) = w(\cdot)$ and consider three cases

1. $\|\varphi(s, x, w)\| \geq \rho^{-1}(\|w\|_\infty)$ for all $s \in [0, t]$

$$\Rightarrow \quad \|\varphi(t, x, w)\| \leq \tilde{\beta}(\|x\|, t)$$

2. $\|\varphi(t, x, w)\| \geq \rho^{-1}(\|w\|_\infty)$ and $\|\varphi(s, x, w)\| \leq \rho^{-1}(\|w\|_\infty)$ for some (maximal) $s \in [0, t)$

$$\Rightarrow \quad \|\varphi(t, x, w)\| \leq \tilde{\beta}(\rho^{-1}(\|\varphi(s, x, w)\|), t - s) \leq \tilde{\beta}(\rho^{-1}(\|w\|_\infty), 0)$$

3. $\|\varphi(t, x, w)\| \leq \rho^{-1}(\|w\|_\infty) \leq \tilde{\beta}(\rho^{-1}(r), 0)$

$\Rightarrow$ ISS with $\beta(r, t) = \tilde{\beta}(r, t)$ and $\gamma(r) = \tilde{\beta}(\rho^{-1}(r), 0)$

Coordinate changes

Coordinate changes formalize the idea of deforming the space:

A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a nonlinear change of coordinates if

- $T(0) = 0$
- T is a homeomorphism, i.e., T is continuous, invertible and T^{-1} is continuous
- T is a diffeomorhpism on $\mathbb{R}^n \setminus \{0\}$, i.e., T and T^{-1} are differentiable for $x \neq 0$

Solutions in new coordinates $y = T(x)$ are $\psi(t, y, w) = T(\varphi(t, T^{-1}(y), w))$ and satisfy

$$\dot{y}(t) = \tilde{f}(y(t), w(t)), \quad \tilde{f}(y, w) = DT(T^{-1}(y))\, f(T^{-1}(y), w)$$

Coordinate changes

E.D. Sontag's "mantra": Nonlinear stability concepts should be invariant under nonlinear coordinate changes, i.e.,

if φ has a certain stability property then ψ should have the same property

This is true for ISS, since from $\psi(t,y,w) = T(\varphi(t,T^{-1}(y),w))$:

$$\psi(t,y,w) \le \max\{\underbrace{\rho_1(\beta(t,\rho_2(\|y\|)))}_{\in\mathcal{KL}}, \underbrace{\rho_1(\gamma(\|w\|_\infty))}_{\in\mathcal{K}_\infty}\}$$

for

$$\rho_1(r) = \max_{\|x\|=r} \|T(x)\| \in \mathcal{K}_\infty \quad \text{and} \quad \rho_2(r) = \max_{\|y\|=r} \|T^{-1}(y)\| \in \mathcal{K}_\infty$$

Coordinate changes and H_∞

E.D. Sontag's "mantra": Nonlinear stability concepts should be invariant under nonlinear coordinate changes

This is not true for H_∞, because the quadratic H_∞ inequality

$$\int_0^t \|\varphi(t,x,w)\|^2 ds \le c\|x\|^2 + c\int_0^t \|w(s)\|^2 ds$$

becomes the — in general weaker — nonlinear inequality

$$\int_0^t \alpha(\psi(t,y,w))ds \le \kappa(\|y\|) + \int_0^t \gamma(\|w(s)\|)ds$$

Theorem: this inequality is equivalent to ISS

Proof: "$\Leftarrow$" by integration, "$\Rightarrow$" via 0–GAS and asymptotic gain

ISS and H_∞

Summary: For any system φ we have

$$H_\infty \text{ for } \varphi \quad \begin{matrix}\Rightarrow \\ \not\Leftarrow\end{matrix} \quad \text{ISS for } \varphi$$

For any coordinate change T and the corresponding ψ we have

$$H_\infty \text{ for } \varphi \quad \Rightarrow \quad \text{ISS for } \psi$$

Question: Can we reverse this coordinate change, i.e., given an ISS system φ, can we find T such that

$$\text{ISS for } \varphi \quad \Rightarrow \quad H_\infty \text{ for } \psi$$

holds? — Yes, but we need coordinate changes for x and w

ISS and H_∞

Construction of coordinate change:

ISS $\Rightarrow$ there exists ISS Lyapunov function

$$\|w\| \le \chi(\|x\|) \;\;\Rightarrow\;\; DV(x)f(x,w) \le -\alpha_3(\|x\|)$$

Replacing V by $\rho(V)$ for suitable $\rho \in \mathcal{K}_\infty$ yields

$$\|w\| \le \chi(\|x\|) \;\;\Rightarrow\;\; DV(x)f(x,w) \le -V(x)$$

Suppose we can find an x–coordinate change T with

$$\widetilde{V}(y) := V(T^{-1}(y)) = \|y\|^2$$

(details later)

ISS and H_∞

$\Rightarrow$ for the transformed system $\tilde f$ and $\tilde V(y) = \|y\|^2$ we obtain

$$\|w\| \le \tilde\chi(\|y\|) \;\Rightarrow\; D\tilde V(y)\tilde f(y,w) = DV(T^{-1}(y))f(T^{-1}(y),w)$$
$$\le -V(T^{-1}(y)) = -\tilde V(y)$$

For $\alpha(r) = \sup\limits_{\tilde\chi(\|y\|)\le r,\, \|w\|\le r} D\tilde V(y)\tilde f(y,w)$ we obtain

$$D\tilde V(y)\tilde f(y,w) \le -\tilde V(y) + \alpha(\|w\|)$$

Now, using the w–coordinate change $v = R(w) = \alpha(\|w\|)^2 w/\|w\|$:

$$D\tilde V(y)\tilde f(y, R^{-1}(v)) \le -\tilde V(y) + \|v\|^2$$

$\rightsquigarrow$ integration yields H_∞ inequality for the system $\dot y = \tilde f(y, R^{-1}(v))$

Transformation of V

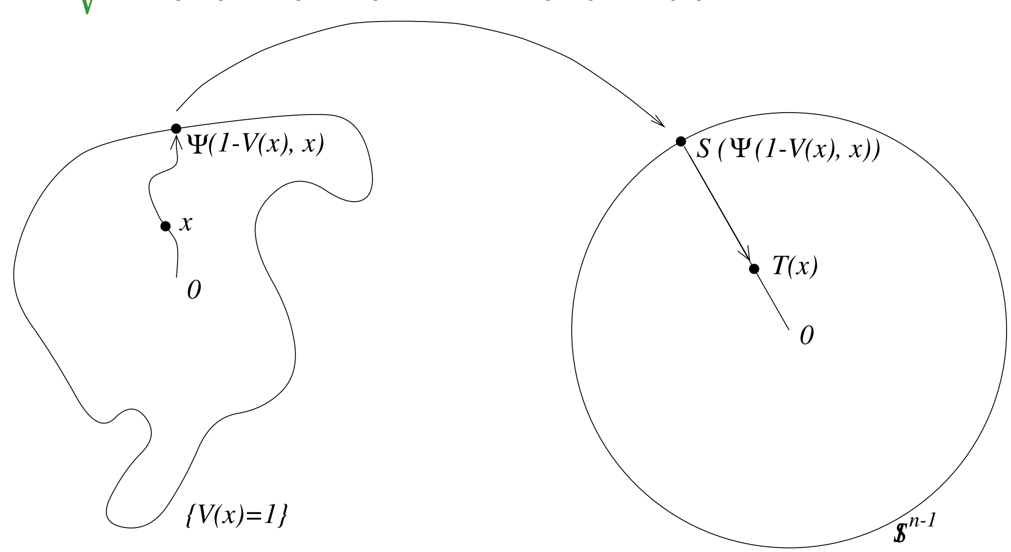
It remains to construct T with $V(T^{-1}(y)) = \|y\|^2 \;\Leftrightarrow\; \|T(x)\| = \sqrt{V(x)}$

$n \neq 4, 5$: there exists diffeo. $S : \{V(x) = 1\} \to \mathbb{S}^{n-1}$ [Milnor '65]

($n = 4, 5$: there exists at least a homeomorphism [Poincaré conjecture, Perelman (2002, 2003)], diffeomorphism open)

Ψ solution of the normed gradient flow $\dot{x} = \nabla V(x) / \|\nabla V(x)\|^2$

Define $T(x) := \sqrt{V(x)}\, S(\Psi(1 - V(x), x))$



ISS and H_∞

This leads to the following

Theorem: If $n \neq 4, 5$, then for any ISS system there exists a coordinate change T for the state x and a coordinate change R for the input w such that the transformed system has the H_∞ property

Short version: ISS and H_∞ are equivalent under coordinate changes

Distinct feature: ISS is invariant under coordinate changes, H_∞ is not!

Summary of Part II

- local ISS $\Leftrightarrow$ local 0–AS
- ISS $\Leftrightarrow$ 0–GAS & asymptotic gain
- ISS $\Leftrightarrow$ existence of a stability margin
- ISS is invariant under nonlinear coordinate changes in space
- ISS $\Leftrightarrow$ H_∞ — in appropriate coordinates