Input–to–state Stability

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Part II: Relation to Other Stability Concepts
ISS

Consider

\[ \dot{x}(t) = f(x(t), w(t)) \]

with solutions \( \varphi(t, x, w) \)

The system is called ISS, if there exist \( \beta \in \mathcal{KL} \) and \( \gamma \in \mathcal{K}_\infty \) such that for all initial values \( x \), all perturbation functions \( w \) and all times \( t \geq 0 \) the following inequality holds:

\[ \|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\} \]

In this part of the course, we will investigate how ISS is related to other stability concepts.
Local ISS

We start with the property which is easiest to treat:

The system is called locally ISS, if there exist a neighborhood $B \subset \mathbb{R}^n$ of 0, a value $R > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values $x \in B$, all perturbation functions $w$ with $\|w\|_\infty \leq R$ and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\}$$
Setting $w \equiv 0$, local ISS

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_{\infty})\}$$

implies local asymptotic stability for $w \equiv 0$ (local 0–AS):

The system is called locally 0–AS, if there exist a neighborhood $B \subset \mathbb{R}^n$ of 0 and a function $\beta \in KL$ such that for all initial values $x \in B$ and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, 0)\| \leq \beta(\|x\|, t)$$

Theorem: local ISS $\Leftrightarrow$ local 0–AS
Local 0–AS

\[ \text{local ISS } \Leftrightarrow \text{local 0–AS} \]

Idea of Proof: \( \text{local ISS } \Rightarrow \text{local 0–AS} \) immediate for \( w \equiv 0 \)

Conversely, \( \text{local 0–AS} \) implies existence of Lyapunov function

\[ DV(x)f(x, 0) \leq -\alpha_3(\|x\|) \]

Choosing \( \chi \in K_\infty \) with

\[ \max_{\|x\|=r, \|w\|\leq\chi(r)} \|DV(x)\|\|f(x, w) - f(x, 0)\| \leq \alpha_3(r)/2 \]

for all \( r \in [0, R], x \in B \), we obtain

\[ \|w\| \leq \chi(\|x\|) \Rightarrow DV(x)f(x, w) \leq -\alpha_3(\|x\|)/2 \]

\( \Rightarrow \) ISS Lyapunov function \( \Rightarrow \) ISS
Local vs. global 0–AS

Note: the maximal function \( \tilde{\chi} \) satisfying

\[
\max_{\|x\| = r, \|w\| \leq \tilde{\chi}(r)} \|DV(x)\|\|f(x, w) - f(x, 0)\| \leq \alpha_3(r)/2
\]

can always be underbounded by \( \chi \in \mathcal{K}_\infty \) for \( r \in [0, R] \)

This may not be possible for \( r \in [0, \infty) \). This is why

\[0\text{-GAS} \not\Rightarrow \text{global ISS}\]
0–GAS & asymptotic gain

there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ with

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\}$$

0–GAS: $$\|\varphi(t, x, 0)\| \leq \beta(\|x\|, t)$$

Recall the asymptotic gain property:

$$\limsup_{t \to \infty} \|\varphi(t, x, w)\| \leq \gamma(\|w\|_\infty)$$

Theorem (ISS superposition principle):

ISS $\Leftrightarrow$ 0–GAS & asymptotic gain

Sketch of Proof: “$\Rightarrow$” immediate
Step 1: 0–GAS implies local ISS

Step 2: local ISS and asymptotic gain imply uniform as. gain:
For all $\varepsilon > 0$, $\Delta > 0$ there exists $T = T(\varepsilon, \Delta) > 0$ such that
\[ \|\varphi(t, x, w)\| \leq \gamma(\|w\|_{\infty}) + \varepsilon \]
whenever $\|x\| \leq \Delta$, $t \geq T$

Step 3: local ISS and uniform as. gain imply ISS, where the $\beta$ for large $\|x\|$ is built from $T(\varepsilon, \Delta)$
Stability margins

We say that the system admits a **stability margin** $\rho \in \mathcal{K}_\infty$ if for any feedback map $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ Lipschitz in $x$ with $\|k(t, x)\| \leq \rho(\|x\|)$ the closed loop system

$$\dot{x}(t) = f(x(t), k(t, x(t)))$$

is **globally asymptotically stable**, i.e., its solutions $\varphi_k(t, x)$ satisfy

$$\|\varphi_k(t, x)\| \leq \tilde{\beta}(\|x\|, t)$$

for some suitable $\tilde{\beta} \in \mathcal{KL}$ independent of $k$

**Theorem:** **ISS ⇔ existence of a stability margin**
Stability margins

Proof: ISS $\Rightarrow$ existence of a stability margin

We have $\varphi_k(t, x) = \varphi(t, x, w)$ for $w(t) = k(t, \varphi_k(t, x))$.

For $T > 0$ and $t \in [0, T]$, ISS implies

$$\|\varphi_k(t, x)\| \leq \max \left\{ \beta(\|x\|, t), \max_{t \in [0, T]} \gamma(\rho(\|\varphi_k(t, x)\|)) \right\}$$

Now $\rho(r) \leq \gamma^{-1}(\beta(\|x\|, T))$ for $r \in [\beta(\|x\|, T), \beta(\|x\|, 0)]$ implies

$$\|\varphi_k(t, x)\| \leq \beta(\|x\|, t), \quad t \in [0, T]$$

For sufficiently large $T = T(\|x\|)$ we can now proceed inductively in order to construct $\tilde{\beta}$ from $\beta$

Note: $\tilde{\beta}$ and $\rho$ are only implicitly given
Stability margins

Proof: ISS ⇔ existence of a stability margin

Given \( w \) and \( t > 0 \) set \( k(\cdot, x) = w(\cdot) \) and consider three cases

1. \( \| \phi(s, x, w) \| \geq \rho^{-1}(\| w \|_\infty) \) for all \( s \in [0, t] \)
   \[ \Rightarrow \| \phi(t, x, w) \| \leq \tilde{\beta}(\| x \|, t) \]

2. \( \| \phi(t, x, w) \| \geq \rho^{-1}(\| w \|_\infty) \) and \( \| \phi(s, x, w) \| \leq \rho^{-1}(\| w \|_\infty) \) for some (maximal) \( s \in [0, t] \)
   \[ \Rightarrow \| \phi(t, x, w) \| \leq \tilde{\beta}(\rho^{-1}(\| \phi(s, x, w) \|), t - s) \leq \tilde{\beta}(\rho^{-1}(\| w \|_\infty), 0) \]

3. \( \| \phi(t, x, w) \| \leq \rho^{-1}(\| w \|_\infty) \leq \tilde{\beta}(\rho^{-1}(r), 0) \)
   \[ \Rightarrow \text{ISS with } \beta(r, t) = \tilde{\beta}(r, t) \text{ and } \gamma(r) = \tilde{\beta}(\rho^{-1}(r), 0) \]
Coordinate changes

Coordinate changes formalize the idea of deforming the space:

A map \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called a nonlinear change of coordinates if

- \( T(0) = 0 \)
- \( T \) is a homeomorphism, i.e., \( T \) is continuous, invertible and \( T^{-1} \) is continuous
- \( T \) is a diffeomorphism on \( \mathbb{R}^n \setminus \{0\} \), i.e., \( T \) and \( T^{-1} \) are differentiable for \( x \neq 0 \)

Solutions in new coordinates \( y = T(x) \) are \( \psi(t, y, w) = T(\varphi(t, T^{-1}(y), w)) \) and satisfy

\[
\dot{y}(t) = \tilde{f}(y(t), w(t)), \quad \tilde{f}(y, w) = DT(T^{-1}(y)) f(T^{-1}(y), w)
\]
E.D. Sontag's “mantra”: Nonlinear stability concepts should be invariant under nonlinear coordinate changes, i.e.,

if \( \varphi \) has a certain stability property then \( \psi \) should have the same property

This is true for ISS, since from \( \psi(t, y, w) = T(\varphi(t, T^{-1}(y), w)) \):

\[
\psi(t, y, w) \leq \max \{ \rho_1(\beta(t, \rho_2(\|y\|))), \rho_1(\gamma(\|w\|_\infty)) \} \\
\text{for} \\
\rho_1(r) = \max_{\|x\|=r} \|T(x)\| \in \mathcal{K}_\infty \quad \text{and} \quad \rho_2(r) = \max_{\|y\|=r} \|T^{-1}(y)\| \in \mathcal{K}_\infty
\]
Coordinate changes and $H_\infty$

E.D. Sontag’s “mantra”: Nonlinear stability concepts should be invariant under nonlinear coordinate changes.

This is not true for $H_\infty$, because the quadratic $H_\infty$ inequality

$$\int_0^t \|\varphi(t, x, w)\|^2 ds \leq c\|x\|^2 + c \int_0^t \|w(s)\|^2 ds$$

becomes the — in general weaker — nonlinear inequality

$$\int_0^t \alpha(\psi(t, y, w))ds \leq \kappa(\|y\|) + \int_0^t \gamma(\|w(s)\|)ds$$

Theorem: this inequality is equivalent to ISS.

Proof: “$\leftarrow$” by integration, “$\Rightarrow$” via $0$–GAS and asymptotic gain.
ISS and $H_\infty$

**Summary:** For any system $\varphi$ we have

$$H_\infty \text{ for } \varphi \nRightarrow \text{ ISS for } \varphi$$

For any **coordinate change** $T$ and the corresponding $\psi$ we have

$$H_\infty \text{ for } \varphi \Rightarrow \text{ ISS for } \psi$$

**Question:** Can we reverse this **coordinate change**, i.e., given an ISS system $\varphi$, can we find $T$ such that

$$\text{ISS for } \varphi \Rightarrow H_\infty \text{ for } \psi$$

holds? — **Yes**, but we need coordinate changes for $x$ and $w$
ISS and $H_{\infty}$

Construction of coordinate change:

**ISS** $\Rightarrow$ there exists **ISS Lyapunov function**

$$\|w\| \leq \chi(\|x\|) \Rightarrow DV(x)f(x,w) \leq -\alpha_3(\|x\|)$$

Replacing $V$ by $\rho(V)$ for suitable $\rho \in K_{\infty}$ yields

$$\|w\| \leq \chi(\|x\|) \Rightarrow DV(x)f(x,w) \leq -V(x)$$

Suppose we can find an $x-$coordinate change $T$ with

$$\tilde{V}(y) := V(T^{-1}(y)) = \|y\|^2$$

(details later)
ISS and $H_\infty$

⇒ for the transformed system $\tilde{f}$ and $\tilde{V}(y) = \|y\|^2$ we obtain

$$\|w\| \leq \tilde{\chi}(\|y\|) \Rightarrow D\tilde{V}(y)\tilde{f}(y, w) = DV(T^{-1}(y))f(T^{-1}(y), w) \leq -V(T^{-1}(y)) = -\tilde{V}(y)$$

For $\alpha(r) = \sup_{\tilde{\chi}(\|y\|) \leq r, \|w\| \leq r} D\tilde{V}(y)\tilde{f}(y, w)$ we obtain

$$D\tilde{V}(y)\tilde{f}(y, w) \leq -\tilde{V}(y) + \alpha(\|w\|)$$

Now, using the $w$–coordinate change $v = R(w) = \alpha(\|w\|)^2 w/\|w\|:

$$D\tilde{V}(y)\tilde{f}(y, R^{-1}(v)) \leq -\tilde{V}(y) + \|v\|^2$$

⇒ integration yields $H_\infty$ inequality for the system $\dot{y} = \tilde{f}(y, R^{-1}(v))$
Transformation of $V$

It remains to construct $T$ with $V(T^{-1}(y)) = \|y\|^2 \iff \|T(x)\| = \sqrt{V(x)}$

$n \neq 4, 5$: there exists diffeo. $S : \{V(x) = 1\} \to S^{n-1}$ [Milnor ’65]

($n = 4, 5$: there exists at least a homeomorphism [Poincaré conjecture, Perelman (2002, 2003)], diffeomorphism open)

$\Psi$ solution of the normed gradient flow $\dot{x} = \nabla V(x)/\|\nabla V(x)\|^2$

Define $T(x) := \sqrt{V(x)}S(\Psi(1 - V(x), x))$

\[
\begin{array}{c}
\Psi(1-V(x), x) \\
\text{x} \\
\text{0} \\
\{V(x) = 1\}
\end{array}
\begin{array}{c}
S(\Psi(1-V(x), x)) \\
\text{T(x)} \\
\text{0} \\
S^{n-1}
\end{array}
\]
ISS and $H_\infty$

This leads to the following

**Theorem:** If $n \neq 4, 5$, then for any ISS system there exists a coordinate change $T$ for the state $x$ and a coordinate change $R$ for the input $w$ such that the transformed system has the $H_\infty$ property

**Short version:** ISS and $H_\infty$ are equivalent under coordinate changes

**Distinct feature:** ISS is invariant under coordinate changes, $H_\infty$ is not!
Summary of Part II

- local ISS ⇔ local 0–AS
- ISS ⇔ 0–GAS & asymptotic gain
- ISS ⇔ existence of a stability margin
- ISS is invariant under nonlinear coordinate changes in space
- ISS ⇔ $H_\infty$ — in appropriate coordinates