Input–to–state Stability

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Part III: Lyapunov functions and quantitative aspects

ISS

Consider

$$\dot{x}(t) = f(x(t), w(t))$$

with solutions $\varphi(t, x, w)$

The system is called ISS, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\}$$

In this part of the course, we will investigate ISS Lyapunov functions and quantitative aspects of ISS

ISS Lyapunov functions

Theorem: A system is ISS if and only if there exists an ISS Lyapunov function, i.e. a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and functions α_1, α_2, $\chi \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

and

$$\|w\| \leq \chi(\|x\|) \;\; \Rightarrow \;\; DV(x)f(x,w) \leq -\alpha_3(\|x\|)$$

hold for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

ISS Lyapunov functions

Proof: "existence of $V \quad \Rightarrow \quad$ ISS"

1. For $g = \alpha_3 \circ \alpha_2^{-1}$ and $\kappa = \chi \circ \alpha_2^{-1}$ we obtain

$$\|w\| \leq \kappa(V(x)) \quad \Rightarrow \quad DV(x)f(x,w) \leq -g(V(x))$$

2. Integration yields $V(\varphi(t,x,w)) \leq \max\{\mu(V(x),t),\, \kappa^{-1}(\|w\|_\infty)\}$

   for $\mu \in \mathcal{KL}$ given by $\frac{d}{dt}\mu(r,t) = -g(\mu(r,t))$, $\mu(r,0) = r$

3. $$\begin{aligned} \|\varphi(t,x,w)\| &\leq \alpha_1^{-1}(V(\varphi(t,x,w))) \\ &\leq \max\{\alpha_1^{-1}(\mu(\alpha_2(\|x\|),t)),\, \alpha_1^{-1}(\kappa^{-1}(\|w\|_\infty))\} \end{aligned}$$

$\Rightarrow$ ISS with $\beta(r,t) = \alpha_1^{-1}(\mu(\alpha_2(r),t))$ and $\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \chi^{-1}$

ISS Lyapunov functions

Proof: "ISS $\Rightarrow$ existence of V"

1. ISS $\Rightarrow$ existence of stability margin, i.e., $\rho \in \mathcal{K}_\infty$ such that for any Lipschitz feedback map $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ with $\|k(t,x)\| \leq \rho(\|x\|)$ the closed loop system

$$\dot{x}(t) = f(x(t), k(t, x(t)))$$

is globally asymptotically stable, i.e., its solutions $\varphi_k(t,x)$ satisfy

$$\|\varphi_k(t,x)\| \leq \tilde{\beta}(\|x\|, t)$$

for some suitable $\tilde{\beta} \in \mathcal{KL}$ independent of k

In particular, this holds for $k(t,x) = \rho(\|x\|)d(t)$, $\|d(t)\| \leq 1$

ISS Lyapunov functions

Proof: "ISS $\Rightarrow$ existence of V"

1. The solutions $\varphi_\rho(t,x,d)$ of

$$\dot{x}(t) = f(x(t), \rho(\|x\|)d(t))$$

satisfy $\|\varphi_\rho(t,x,d)\| \leq \tilde{\beta}(\|x\|, t)$ for each $d(t)$ with $\|d(t)\| \leq 1$

2. Converse Lyapunov theorem for perturbed GAS systems:

there exists a smooth Lyapunov function V, i.e.,

$$\|d\| \leq 1 \quad \Rightarrow \quad DV(x)f(x, \rho(\|x\|)d) \leq -\alpha_3(\|x\|)$$

3. This implies

$$\|w\| \leq \rho(\|x\|) \quad \Rightarrow \quad DV(x)f(x, w) \leq -\alpha_3(\|x\|)$$

$\Rightarrow$ V is ISS Lyapunov function

Equivalent ISS Lf characterizations

(1) Implication form:

$$\|w\| \leq \chi(\|x\|) \;\;\Rightarrow\;\; DV(x)f(x,w) \leq -\alpha_3(\|x\|)$$

(2) Supremum form:

$$\sup_{\|w\| \leq \chi(\|x\|)} DV(x)f(x,w) \leq -\alpha_3(\|x\|)$$

(3) Dissipation Form:

$$DV(x)f(x,w) \leq -\alpha_3(\|x\|) + \alpha_4(\|w\|)$$

with $\alpha_3 \in \mathcal{K}_\infty$

Note: in general V from (1) or (2) needs to be transformed to satisfy (3)

iISS Lyapunov functions

The existence of a Lyapunov function in dissipation form

$$DV(x)f(x,w) \le -\alpha_3(\|x\|) + \alpha_4(\|w\|)$$

with $\alpha_3 \in \mathcal{K}$ (instead of $\mathcal{K}_\infty$) is equivalent to integral ISS (iISS):

$$\|\varphi(t,x,w)\| \le \beta(\|x\|,t) + \gamma_1\left(\int_0^t \gamma_2(\|w(s)\|)ds\right)$$

Computing ISS Lyapunov functions

It is in general a hard task to find ISS Lyapunov functions, only few constructive techniques are known:

- optimal control and set valued approaches (only feasible numerically in low dimensions)
- backstepping approaches (under suitable structural assumptions, typically strict feedback form)

Both approaches are linked via the inverse optimality formalism

References: Freeman/Kokotović, Krstić/Kannelakopoulos/Kokotović, Krstić/Deng

Quantitative "Problems" of ISS

$$\|\varphi(t,x,w)\| \le \max\{\beta(\|x\|,t),\gamma(\|w\|_\infty)\}$$

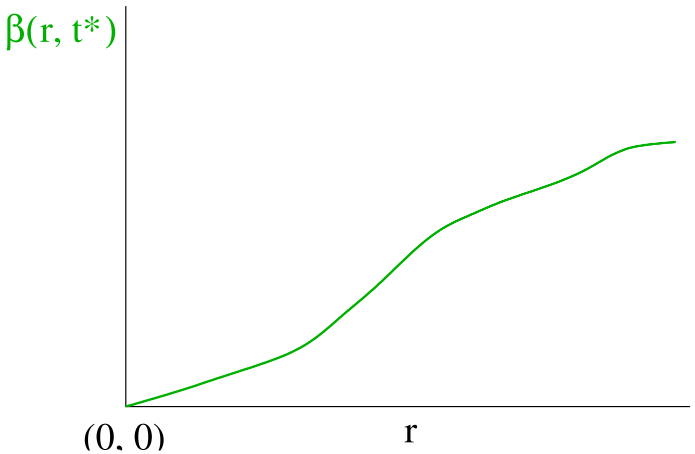
- No explicit estimate if $w(t) \to 0$
- No representation of β and γ in the ISS Lyapunov function V
- No easy method to compute (or estimate) the stability margin ρ (= gain for Lyapunov function V)

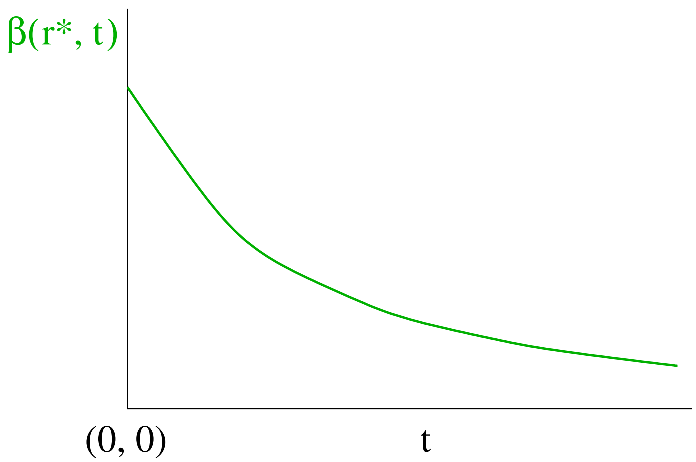
Facts about comparison functions

$\mathcal{K} := \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid$ continuous, strictly increasing, $\alpha(0) = 0\}$

$\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \mid$ unbounded$\}$

$\mathcal{KL} := \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \mid$ continuous, $\beta(\cdot, r) \in \mathcal{K}$ and β strictly converging to 0 in the 2nd argument$\}$





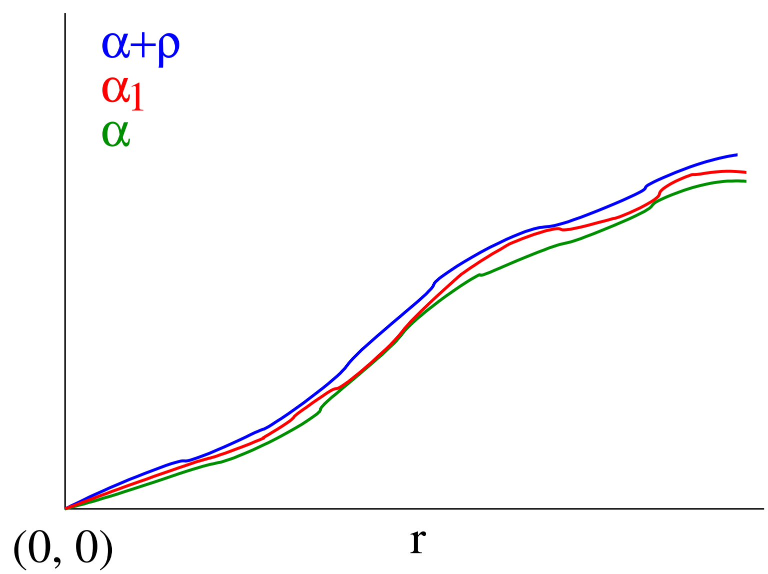
Facts about comparison functions

For any $\alpha \in \mathcal{K}_\infty$ and any function $\rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\rho(r) > 0$ for $r > 0$ there is a smooth $\alpha_1 \in \mathcal{K}_\infty$ with

$$\alpha(r) \le \alpha_1(r) \le \alpha(r) + \rho(r)$$

and

$$\frac{d}{dr}\alpha_1(r) > 0 \text{ for all } r > 0$$



Facts about comparison functions

Sontag's $\mathcal{KL}$–Lemma: For any $\beta \in \mathcal{KL}$ there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\beta(r,t) \le \alpha_1(\alpha_2(r)e^{-t})$$

Corollary: For smooth α_1 and $\mu(r,t) = \alpha_1(re^{-t})$, $\sigma(r) = \alpha_2(r)$:

$$\beta(r,t) \le \mu(\sigma(r),t)$$

and there exists $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $g(r) > 0$ for $r > 0$ such that

$$\frac{d}{dt}\mu(r,t) = -g(\mu(r,t)), \quad \mu(r,0) = r$$

$$\mathcal{KLD} := \{\mu \in \mathcal{KL} \,|\, \tfrac{d}{dt}\mu(r,t) = -g(\mu(r,t)),\, \mu(r,0) = r\}$$

Gain Preserving Lyapunov Functions

$\rightsquigarrow$ without loss of generality β is of the form

$$\beta(r,t) = \mu(\sigma(r),t), \quad \mu \in \mathcal{KLD},\ \sigma \in \mathcal{K}_\infty$$

For systems without input

$$\dot{x}(t) = f(x(t))$$

this is exactly the form we get from integrating

$$DV(x)f(x) \leq -g(V(x))$$

if V satisfies $\quad \|x\| \leq V(x) \leq \sigma(\|x\|)$:

$$\|\varphi(t,x)\| \leq V(\varphi(t,x)) \leq \mu(V(x),t) \leq \mu(\sigma(\|x\|),t)$$

Converse Gain Preserving Theorem

The converse is "almost true":

An ODE $\dot{x} = f(x)$ is GAS with $\beta(r,t) = \mu(\sigma(r),t)$, $\mu \in \mathcal{KLD}$, $\sigma \in \mathcal{K}_\infty$ if and only if for each $\varepsilon > 0$ there exists V_ε satisfying

$$DV_\varepsilon(x)f(x) \leq -(1-\varepsilon)g(V_\varepsilon(x))$$

and

$$\|x\| \leq V_\varepsilon(x) \leq \sigma(\|x\|)$$

Idea for the construction of V_ε [Yoshizawa 66]:

$$V_\varepsilon(x) := \max_{t \geq 0} \mu(\|\varphi(t,x)\|, -(1-\varepsilon)t)$$

+ subsequent smoothing

(same construction with $\varepsilon = 0$ yields discontinuous V_0)

Input–to–state Stability

Question: can we do the same for ISS?

What do we get when we integrate

$$\gamma(\|w\|) \leq V(x) \;\;\Rightarrow\;\; DV(x)f(x) \leq -g(V(x))$$

with V satisfying $\quad \|x\| \leq V(x) \leq \sigma(\|x\|)$?

Input–to–state Stability

Proceeding as before we get

$$\|\varphi(t,x,w)\| \leq \max\{\mu(\sigma(\|x\|),t),\gamma(\|w\|_\infty)\}$$

for γ, $\sigma \in \mathcal{K}_\infty$ and $\mu \in \mathcal{KLD}$

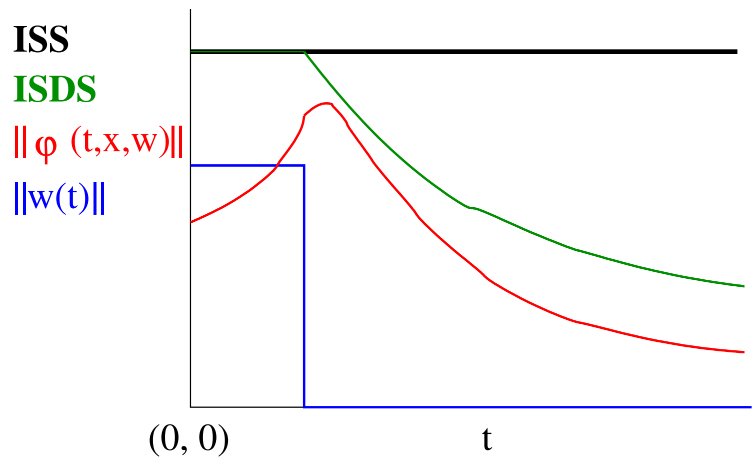
Input–to–state dynamical Stability

In fact we get more: input–to–state dynamical stability (ISDS)

$$\|\varphi(t,x,w)\| \le \max\{\mu(\sigma(\|x\|),t),\ \ \nu(w,t)\ \}$$

for γ, $\sigma \in \mathcal{K}_\infty$ and $\mu \in \mathcal{KLD}$, where

$$\nu(w,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]}\ \mu(\gamma(\|w(\tau)\|),t-\tau)$$



Input–to–state dynamical Stability

Sketch of proof: Fix $\Delta t > 0$ and set $t_i = i\Delta t$

Iterative integration, $i = 1, 2, 3, \ldots$, yields for $t \in [t_{i+1}, t_i]$

$$\|\varphi(t,x,w)\| \le \max\{\mu(\sigma(\|x\|),t), \nu_{\Delta t}(w,t)\}$$

where

$$\nu_{\Delta t}(w,t) := \max_{t_i \le t} \; \mu(\gamma(\|w|_{[t_{i-1},t_i]}\|), t - t_i)$$

For $\Delta t \to 0$ we obtain

$$\nu_{\Delta t}(w,t) \to \nu(w,t) = \operatorname*{ess\,sup}_{\tau\in[0,t]} \; \mu(\gamma(\|w(\tau)\|), t-\tau)$$

Converse Gain Preserving ISDS Theorem

Theorem: A system is ISDS with rate $\mu(\sigma(r),t)$ and robustness gain γ if and only if for each $\varepsilon > 0$ there exists V_ε with

$$\gamma\left(\frac{\|w\|}{1+\varepsilon}\right) \le V_\varepsilon(x) \quad \Rightarrow \quad DV_\varepsilon(x)f(x,w) \le -(1-\varepsilon)g(V_\varepsilon(x))$$

and

$$\frac{\|x\|}{1+\varepsilon} \le V_\varepsilon(x) \le \sigma(\|x\|)$$

Construction of V_ε:

$V_\varepsilon(x) = \sup_w \inf\{\alpha \ge 0 \,|\, \|\varphi(t,x,w)\| \le \rho_\varepsilon(\mu(\gamma(\alpha),t)) \max\{\mu(\gamma(\alpha),(1-\varepsilon)t), \nu(w,t)\}\,\}$

with $\rho_\varepsilon \in [1, 1+\varepsilon]$ strictly increasing + subsequent smoothing

Again, $\varepsilon = 0$ is possible and yields discontinuous V

Converse Gain Preserving ISDS Theorem

Consequence:

For ISDS there is a one–to–one correspondence between

- the rate and gains in the trajectorywise formulation

and

- the rate and gains in the Lyapunov function formulation

This allows to

- compute ISDS rate and gains from Lyapunov functions
- use Lyapunov functions in quantitative statements
  — at least theoretically

Computing ISDS gains

Example 1: $\dot{x} = f(x, w) := -x + w^3$

Set $V(x) = |x|$, then

$$DV(x)f(x, 0) \leq -V(x) =: -g_0(V(x))$$

Choose γ such that the implication

$$\gamma(|w|) \leq V(x) \;\Rightarrow\; DV(x)w^3 \leq g_0(V(x))/2$$

holds $\rightsquigarrow$ $\gamma(r) = 2r^3$.

Then V, γ and $g = g_0/2$ satisfy the theorem, and we obtain ISDS with $\mu(r, t) = e^{-t/2}r$ and $\gamma(r) = 2r^3$

Computing ISDS gains

Example 2: $\dot{x} = f(x,w) := -x^3 + w$

Set $V(x) = |x|$, then

$$DV(x)f(x,0) \leq -V(x)^3 =: -g_0(V(x))$$

Choose γ such that the implication

$$\gamma(|w|) \leq V(x) \;\Rightarrow\; DV(x)w \leq g_0(V(x))/2$$

holds $\rightsquigarrow$ $\gamma(r) = \sqrt[3]{2r}$.

Then V, γ and $g = g_0/2$ satisfy the theorem, and we obtain ISDS with $\mu(r,t) = \frac{r}{\sqrt{2tr^2+1}}$ and $\gamma(r) = \sqrt[3]{2r}$

Stability margins and ISDS

Recall: $\rho \in \mathcal{K}_\infty$ is a stability margin if for any Lipschitz feedback map $k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ with $\|k(t,x)\| \le \rho(\|x\|)$ the closed loop system

$$\dot{x}(t) = f(x(t), k(t, x(t)))$$

is GAS, i.e., $\|\varphi_k(t,x)\| \le \tilde{\beta}(\|x\|, t)$ for some $\tilde{\beta} \in \mathcal{KL}$

Thm: ISDS $\Rightarrow$ $\rho = \gamma^{-1}$ is stability margin with $\tilde{\beta}(r,t) = \mu(\sigma(r), t)$

Proof: for arbitrary $\varepsilon > 0$ take V_ε and $\|k(t,x)\| \le \rho(\|x\|)$

$$\begin{aligned}
& DV_\varepsilon(x) f(x, (1-\varepsilon)k(t,x)) && \le -(1-\varepsilon) g(V_\varepsilon(x)) \\
\Rightarrow \quad & \|\varphi_{(1-\varepsilon)k}(t,x)\| && \le \mu(\sigma(\|x\|), (1-\varepsilon)t) \\
\overset{\varepsilon \to 0}{\Rightarrow} \quad & \|\varphi_k(t,x)\| && \le \mu(\sigma(\|x\|), t)
\end{aligned}$$

ISDS vs. ISS

Obviously, ISDS implies ISS with same γ and $\beta(r,t) = \mu(\sigma(r),t)$

Theorem: Assume ISS with $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$

Then the system is ISDS for each $\tilde{\gamma} \in \mathcal{K}_\infty$ satisfying

$$\tilde{\gamma}(r) > \gamma(r) \text{ for all } r > 0,$$

$\sigma(r) = \beta(r,0)$ and suitable attraction rate $\mu \in \mathcal{KLD}$ depending on the choice of $\tilde{\gamma}$

Idea of Proof: Use stability and asymptotic gain property

Corollary: ISS $\Rightarrow$ any $\rho < \gamma^{-1}$ is a stability margin

Summary of Part III

- ISS $\Leftrightarrow$ existence of ISS Lyapunov function V
- iISS $\Leftrightarrow$ existence of iISS Lyapunov function V in dissipation form
- ISDS $\Leftrightarrow$ existence of ISDS Lyapunov function V_ε maintaining the quantitative information
- ISDS allows to compute gains from Lyapunov function
- ISDS allows to use Lyapunov function in quantitative estimates, e.g., for stability margins