Input–to–state Stability

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Part IV: Applications

ISS

Consider

$$\dot{x}(t) = f(x(t), w(t))$$

with solutions $\varphi(t, x, w)$

The system is called ISS, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial values x, all perturbation functions w and all times $t \geq 0$ the following inequality holds:

$$\|\varphi(t, x, w)\| \leq \max\{\beta(\|x\|, t),\ \gamma(\|w\|_\infty)\}$$

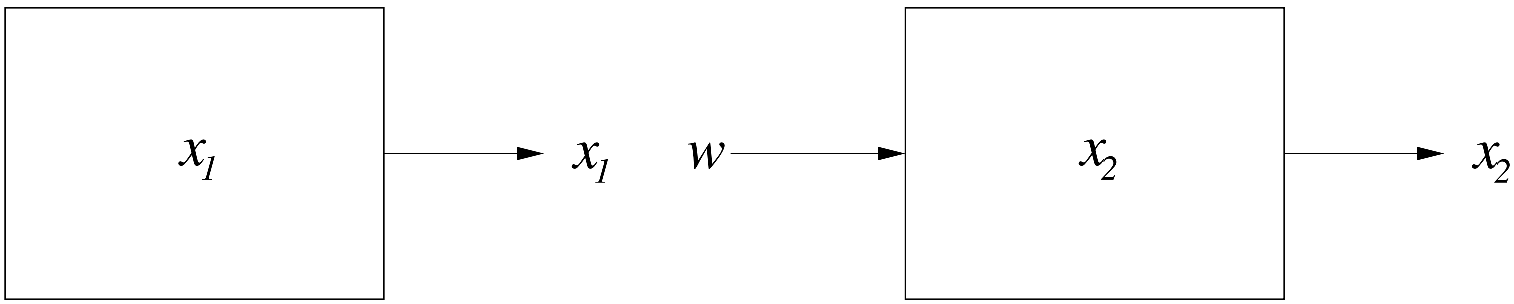
In this part of the course, we will investigate applications and some aspects of ISS controller design

Applications: stability of interconnected and discretized systems

GAS Cascades

Consider

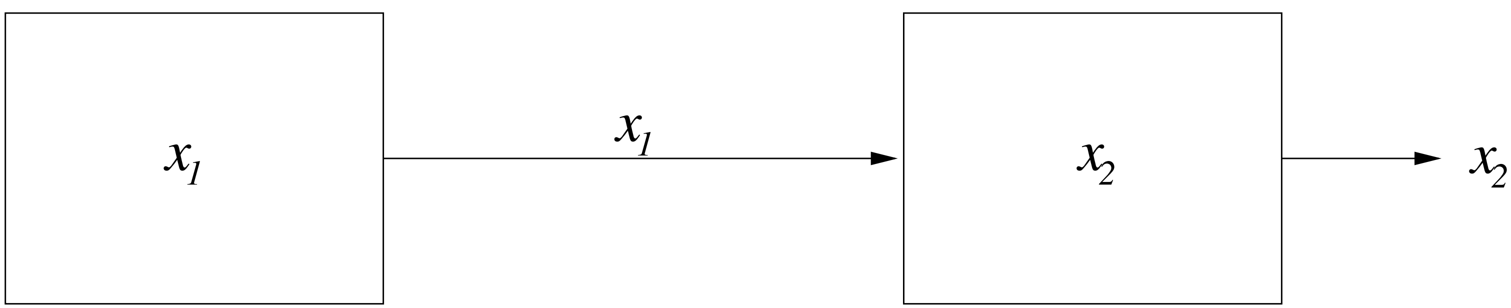
$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t)) \\ \dot{x}_2(t) &= f_2(x_2(t), w(t)) \end{aligned}$$



GAS Cascades

Coupling via $w = x_1$ yields

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t)) \\ \dot{x}_2(t) &= f_2(x_2(t), x_1(t)) \end{aligned}$$



Theorem: x_1 GAS + x_2 ISS $\Rightarrow$ coupled system GAS

GAS Cascades

Proof: The proof is easier with ISDS formulation

$$\begin{aligned} \|\varphi_1(t, x_1^0)\| &\leq \mu_1(\sigma_1(\|x_1^0\|), t)) \\ \|\varphi_2(t, x_2^0, w)\| &\leq \max\{\mu_2(\sigma_2(\|x_2^0\|), t), \nu_2(w, t)\} \end{aligned}$$

$$\nu_2(w, t) := \operatorname*{ess\,sup}_{\tau \in [0,t]} \mu_2(\gamma_2(\|w(\tau)\|), t - \tau)$$

For $w = \varphi_1$ we obtain

$$\nu_2(\varphi_1, t) \leq \max_{\tau \in [0,t]} \mu_2(\gamma_2(\mu_1(\sigma_1(\|x_1^0\|), \tau), t - \tau) =: \beta_2(\|x_1^0\|, t) \in \mathcal{KL}$$

Thus $\varphi = (\varphi_1, \varphi_2)$ with $\|\varphi\| = \max\{\|\varphi_1\|, \|\varphi_2\|\}$ satisfies

$$\|\varphi(t, x^0)\| \leq \max\{\mu_1(\sigma_1(\|x_1^0\|), t)), \mu_2(\sigma_2(\|x_2^0\|), t), \beta_2(\|x_1^0\|, t)\}$$

GAS Cascades

Note: The function

$$\beta_2(\|x_1^0\|, t) := \max_{\tau\in[0,t]} \mu_2(\gamma_2(\mu_1(\sigma_1(\|x_1^0\|), \tau), t-\tau)$$

takes care of the coupling

For

$$\dot\mu_1 = -g_1(\mu_1) \quad \text{and} \quad \dot\mu_2 = -g_2(\mu_2)$$

it is bounded by

$$\beta_2(r, t) \le \eta_2(\gamma_2 \circ \sigma_1(r), t)$$

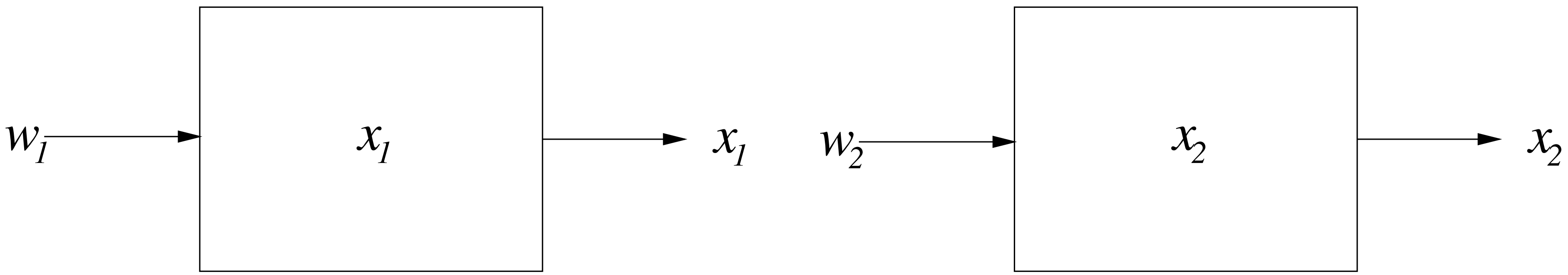
with $\eta_2 \in \mathcal{KLD}$ given by

$$\dot\eta_2 = \max\{-g_2(\eta_2),\, -\gamma_2'(\gamma_2^{-1}(\eta_2))\, g_1(\gamma_2^{-1}(\eta_2))\}$$

Small Gain Theorem

Consider

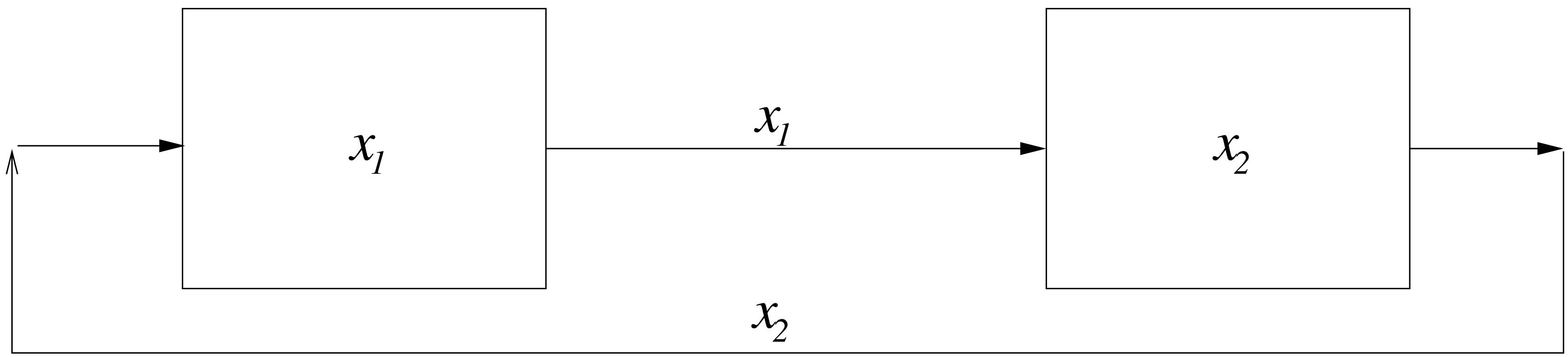
$$\dot{x}_1(t) = f_1(x_1(t), w_1(t))$$
$$\dot{x}_2(t) = f_2(x_2(t), w_2(t))$$



Small Gain Theorem

Coupling via $w_1 = x_2$, $w_2 = x_1$ yields

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), x_2(t)) \\ \dot{x}_2(t) &= f_2(x_2(t), x_1(t)) \end{aligned}$$



Theorem: x_1 ISS + x_2 ISS + $\gamma_1(\gamma_2(r)) < r$

$\Rightarrow$ coupled system GAS

Small Gain Theorem

Proof: For $\theta < 1$ we use ISDS in order to analyze

$$\dot{x}_1(t) = f_1(x_1(t), \theta x_2(t)), \qquad \dot{x}_2(t) = f_2(x_2(t), \theta x_1(t))$$

Induction over a suitable time sequence yields GAS with

$$\|\varphi_i(t, x^0)\| \le \eta_i(\alpha_i(\|x^0\|), t)$$

where

$$\alpha_i(x) \le \max\{\sigma_i(\|x_i\|),\ \gamma_i \circ \sigma_j(\|x_j\|)\}$$

with $\eta_i \in \mathcal{KLD}$ determined by

$$\dot{\eta}_i = \max\{-g_i(\eta_i),\ -\gamma_i'(\gamma_i^{-1}(\eta_i))\, g_j(\gamma_i^{-1}(\eta_i))\}$$

$i = 1, 2$, $j = 2, 1$

By continuity, the estimate also holds for $\theta = 1$

Example

$$\dot{x}_1 = -x_1 + w_1^3/2 \quad \text{and} \quad \dot{x}_2 = -x_2^3 + w_2$$

Using $V_1(x) = V_2(x) = |x|$ we obtain

$$\begin{aligned} \mu_1(r,t) &= e^{-t/4}r, & \gamma_1(r) &= 2r^3/2 \\ \mu_2(r,t) &= \frac{\sqrt{2t+4/r^2}}{t+2/r^2}, & \gamma_2(r) &= \sqrt[3]{4r/3} \end{aligned}$$

⇝ the coupled system is asymptotically stable with $|\varphi_i(t,x^0)| \le \eta_i(\alpha_i(x^0),t)$, where

$$\alpha_1(x) = \max\left\{|x_1|, \tfrac{2}{3}|x_2|^3\right\}, \quad \dot{\eta}_1 = \max\{-c_1\eta_1, -c_2\eta_1^{\frac{5}{3}}\}$$

$$\alpha_2(x) = \max\left\{|x_2|, \sqrt[3]{\tfrac{4}{3}|x_1|}\right\}, \quad \dot{\eta}_2 = \max\{-c_3\eta_2, -c_4\eta_2^3\}$$

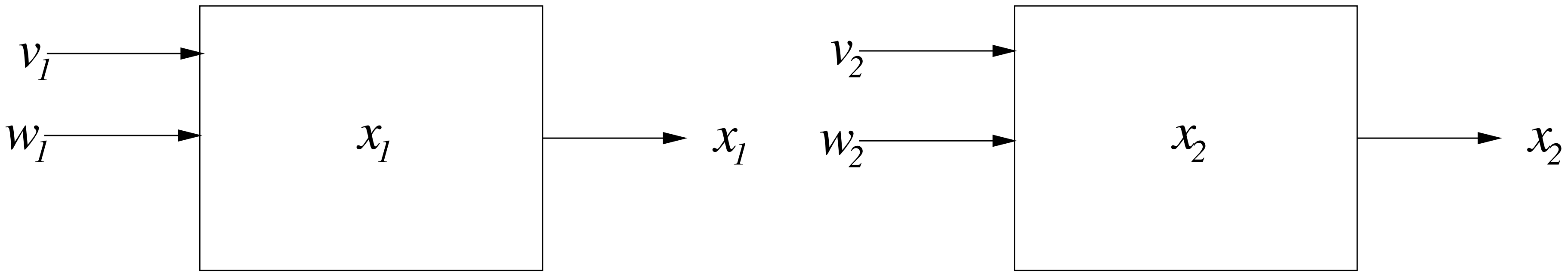
Small Gain Theorem — ISS version

([Jiang/Teel/Praly 94, Teel 95])

Consider

$$\dot{x}_1(t) = f_1(x_1(t), w_1(t), v_1(t))$$

$$\dot{x}_2(t) = f_2(x_2(t), w_2(t), v_2(t))$$



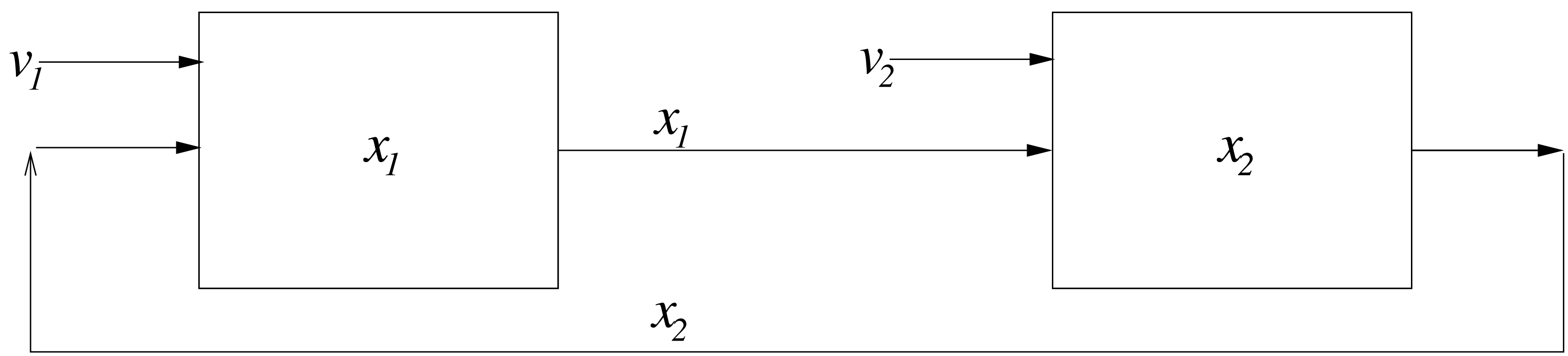
write ISS as $\|\varphi_i(t, x, w_i, v_i)\| \leq \max\{\beta_i(\|x\|, t),\ \gamma_{w_i}(\|w_i\|_\infty),\ \gamma_{v_i}(\|v_i\|_\infty)\}$

Small Gain Theorem — ISS version

([Jiang/Teel/Praly 94, Teel 95])

Coupling via $w_1 = x_2$, $w_2 = x_1$ yields

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1(t), x_2(t), v_1(t)) \\ \dot{x}_2(t) &= f_2(x_2(t), x_1(t), v_2(t))\end{aligned}$$



Theorem: x_1 & x_2 ISS + $\gamma_{w_1}(\gamma_{w_2}(r)) < r$

$\Rightarrow$ coupled system ISS

Small Gain Theorems

Recall the notion of input-to-output stability (IOS) for systems with output $y = h(x)$:

$$\|y(t)\| \leq \max\{\beta(\|x\|, t),\, \gamma(\|w\|_\infty)\}$$

Using IOS, the small gain results are easily extended to systems with output

Numerical Discretization

All stability concepts are easily extended to compact sets $A \subset \mathbb{R}^n$

Denoting the Euclidean distance of $x \in \mathbb{R}^n$ to A by $\|x\|_A$ we can, e.g., define

A is called (locally) asymptotically stable with neighborhood B and attraction rate $\beta \in \mathcal{KL}$, if for all $x \in B$

$$\|\varphi(t,x)\|_A \leq \beta(\|x\|_A, t), \quad t \geq 0$$

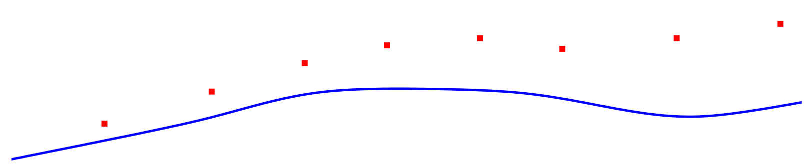
Similarly, all the ISS concepts can be generalized

Numerical Discretization

Goal: find asymptotically stable sets by numerical simulations, e.g., using a one step method

$$x(t+h) = \tilde{\varphi}_h(x(t)),$$

with solution trajectories $\tilde{\varphi}_h(t, x_0)$



Numerical Discretization

Theorem [Kloeden/Lorenz 86] Let A be an asymptotically stable set for $\varphi(t,x)$ and let be $\tilde{\varphi}_h$ an approximation of φ by a numerical one–step method with

$$\|\tilde{\varphi}_h(h,x) - \varphi(h,x)\| \le ch^{q+1}$$

Then $\tilde{\varphi}_h$ has "numerical" asymptotically stable sets $\tilde{A}_h$ with Hausdorff limit $\mathrm{Lim}_{h\to 0}\tilde{A}_h = A$.

But: For arbitrary numerical as. stable sets $\tilde{A}_h$ the limit $A = \mathrm{Lim}_{h\to 0}\tilde{A}_h$ is not asymptotically stable for φ

Question: When is $A = \mathrm{Lim}_{h\to 0}\tilde{A}_h$ as. stable for φ?

Idea: Interpret $\tilde{\varphi}_h$ as perturbed system $\dot{x} = f(x) + w$

Numerical Discretization

Theorem: $A = \operatorname*{Lim}_{h\to 0} \widetilde{A}_h$ asymptotically stable

$$\Leftrightarrow$$

the sets $\widetilde{A}_h$ are locally ISS with $\beta_h \in \mathcal{KL}$, $\gamma_h \in \mathcal{K}_\infty$ such that

$$\beta_h \to \beta \in \mathcal{KL} \text{ and } \gamma_h \to \gamma \in \mathcal{K}_\infty \text{ for } h \to 0$$

$$\Leftrightarrow$$

the sets $\widetilde{A}_h$ have attraction rates $\beta_h \in \mathcal{KL}$ with

$$\beta_h \to \beta \in \mathcal{KL} \text{ for } h \to 0$$

Numerical Discretization: Example

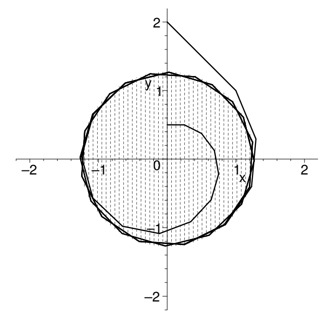
For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ consider

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x - \max\{\|x\| - 1, 0\} x$$

Euler approximation suggests that

$$\mathbb{S}_1 = \{x \in \mathbb{R}^2 \,|\, \|x\| = 1\}$$

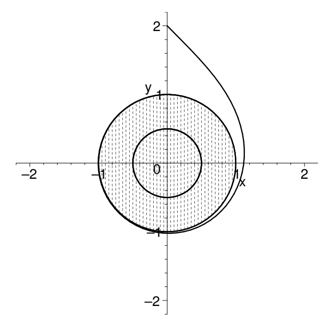
is asymptotically stable



In fact,

$$D_1 = \{x \in \mathbb{R}^2 \,|\, \|x\| \le 1\}$$

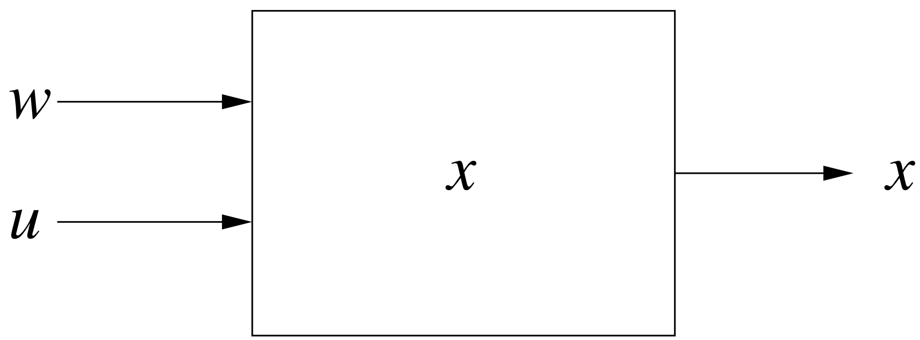
is the only asymptotically stable set



Construction of ISS Feedbacks

Consider

$$\dot{x}(t) = f(x(t), u(t), w(t))$$

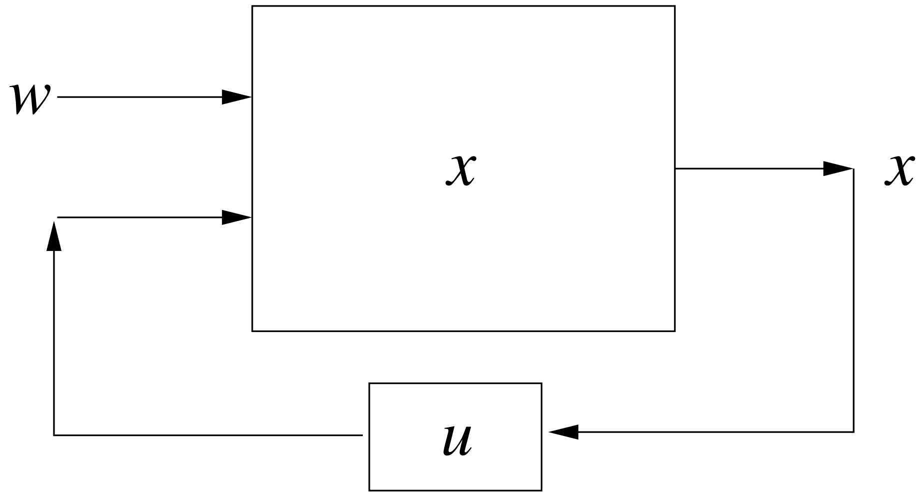


Construction of ISS Feedbacks

Find a feedback law u such that

$$\dot{x}(t) = f(x(t), u(x(t)), w(t))$$

is ISS



Construction of ISS Feedbacks

This is a special case of stabilization via feedback, hence the same obstructions arise:

- continuous static state feedbacks might not exist (Brockett's condition)

- no "universal" design method for the general nonlinear case

Here, we will focus on

- an abstract existence result

- a design procedure based on an ISS Lyapunov function

Sampled solutions

How to define solutions for a discontinuous feedback map $u : \mathbb{R}^n \to U$? — by sampling:

Consider a sampling sequence

$$\pi = (t_i)_{i\in\mathbb{N}_0}, \quad 0 = t_0 < t_1 < t_2 < \ldots \to \infty$$

with maximal sampling rate

$$\Delta(\pi) := \sup_{i\in\mathbb{N}} t_i - t_{i-1} < \infty$$

Define sampled solution $\varphi_\pi(t, x)$ recursively for $i = 0, 1, 2, \ldots$ via

$$x_i := \varphi_\pi(t_i, x), \;\; \varphi_\pi(t, x) := \varphi(t - t_i, x_i, u(x_i)), \;\; t \in [t_i, t_{i+1}]$$

Stabilization by discontinuous feedback

This framework allows for a general abstract stabilization result

Theorem [Clarke/Ledyaev/Sontag/Subbotin 97]:

If the system is asymptotically controllable to 0 then there exists a feedback $u : \mathbb{R}^n \to U$ such that the sampled system is semiglobally practically asymptotically stable, i.e.:

there exists $\beta \in \mathcal{KL}$ such that for all $R, \varepsilon > 0$ there is $\delta > 0$ with

$$\|\varphi_\pi(t,x)\| \leq \beta(\|x\|, t) + \varepsilon$$

if $\|x\| \leq R$ and $\Delta(\pi) \leq \delta$

ISS Stabilization by discontinuous feedback

Consider the input affine system $\dot{x} = f(x) + G(x)u + G(x)w$

Theorem: If the system is asymptotically controllable to 0 for $w \equiv 0$ then there exists a feedback $u : \mathbb{R}^n \to U$ such that the sampled system is semiglobally practically ISS, i.e.:

there exists $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for all $R, \varepsilon > 0$ there is $\delta > 0$ with

$$\|\varphi_\pi(t, x, w)\| \leq \max\{\beta(\|x\|, t), \gamma(\|w\|_\infty)\} + \varepsilon$$

if $\|x\| \leq R$, $\gamma(\|w\|_\infty) \leq R$ and $\Delta(\pi) \leq \delta$

Idea of Proof: Use control Lyapunov function V and nonsmooth analysis techniques to make V an ISS Lyapunov function

ISS Stabilization via universal formula

A more constructive approach is based on ISS control Lyapunov functions (ISS clf) for control affine systems $\dot{x} = f(x, w) + G(x)u$:

A smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS clf (in dissipation form), if there exist $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3, 4$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

and

$$\inf_{u \in U} DV(x)(f(x, w) + G(x)u) \leq -\alpha_3(\|x\|) + \alpha_4(\|w\|)$$

hold for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

ISS Stabilization via universal formula

Given: $\inf_{u \in U} DV(x)(f(x,w) + G(x)u) \le -\alpha_3(\|x\|) + \alpha_4(\|w\|)$

Theorem: Consider $u(x) = K(\bar{\omega}(x), DV(x)G(x)^T)$ with

$$K(a,b) := \begin{cases} -\dfrac{a + \sqrt{a^2 + \|b\|^4}}{\|b\|^2} b, & b \neq 0 \\ 0, & b = 0 \end{cases}$$

and $\bar{\omega}$ being a continuous and outside 0 smooth function with

$$\bar{\omega}(x) \approx \omega(x) := \max_{w} \{DV(x)f(x,w) - \alpha_4(\|w\|)\}$$

and assume the small control property, i.e., for small $\|x\|$ there exists small $\|u\|$ with $\omega(x) + DV(x)G(x)u \le -\alpha_3(\|x\|)$

Then u is continuous and smooth outside 0 and the closed loop system $\dot{x} = f(x,w) + G(x)u(x)$ is ISS (in the classical sense)

ISS Stabilization via universal formula

A similar result is available for integral ISS

Proof: Show that V is an ISS Lyapunov function for the closed loop system

Summary of Part IV

- ISS can be used for the stability analysis of cascades and fully interconnected systems

- ISS can be used for the analysis of numerical discretizations

- ISS controller design: abstract result and universal formula