Model Predictive Control

Why Model Predictive Control?
Model Predictive Control (MPC) “[…] is the only advanced control technique—that is, more advanced than standard PID control—to have had a significant and widespread impact on industrial process control.”


Industrial applications of MPC include

Historic origins

Predictive control for nonlinear systems
Lee and Markus, Foundations of Optimal Control Theory, 1967, p. 423:
One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to measure the current control process state and then compute very rapidly for the open-loop control function. The first portion of this function is then used during a short time interval, after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated.

Main objective of control design
In] attempting to synthesize a feedback optimizing control structure, our main objective is to translate the economic objective into process control objectives.
Classical tracking NMPC

- Control task (e.g. stabilization / setpoint tracking)
- Task-specific Optimal Control Problem
- Implementation
- Efficient numerical algorithms
- NMPC controller
- State estimation
- Setpoints

Main idea of economic MPC

- Performance specification (i.e. cost function)
- Task-specific Optimal Control Problem
- Implementation
- Efficient numerical algorithms
- Economic NMPC controller
- State estimation

Control hierarchy in nowadays process systems

- Plant-wide economic considerations
  - Real Time Optimization
    - Setpoints for control
    - Static models
  - Advanced Control
    - Enforces setpoints
    - Often MPC or NMPC

Outline of the workshop

<table>
<thead>
<tr>
<th>Part</th>
<th>Topic</th>
<th>Speaker</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>Welcome / Introduction Revisiting stabilizing NMPC</td>
<td>Timm Faulwasser</td>
</tr>
</tbody>
</table>
| II.  | Economic MPC with terminal constraints
  - Optimal operation at steady state
  - Stability using dissipativity and terminal constraints | Matthias Müller |
| III. | Economic MPC without terminal constraints
  - Dissipativity and turnpike properties
  - Recursive feasibility and stability | Timm Faulwasser |
| IV.  | Economic MPC without dissipativity
  - Lyapunov-based EMPC
  - Multi-objective EMPC | Lars Grüne |
| V.   | Advanced topics and open problems
  - Extension to periodic solutions
  - Discounted problems
  - Time-varying problems
  - Economic MPC for uncertain systems | Matthias Müller |
| VI.  | Summary and wrap up | Matthias Müller |

NMPC in discrete and continuous time

Discrete-Time NMPC
1. State measurement $x(t)$ at $t \in \mathbb{N}$

2. Solve discrete-time OCP
   $$\min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N))$$
   subject to
   $$x(k+1) = f(x(k), u(k)), \quad x(0) = x(t)$$
   $$(x(k), u(k))^T \in X \times U$$
   $$x(N) \in X_f$$

3. Apply $u_N(x(t)) = u^*(0|t)$

Sampled-Data Continuous-Time NMPC
1. State measurement $x(t_k)$ at $t_k \in \mathbb{R}_0^+$

2. Solve OCP
   $$\min_{u(t)} \int_0^T \ell(x(\tau), u(\tau))d\tau + V_f(x(T))$$
   subject to
   $$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(0) = x(t_k)$$
   $$(x(\tau), u(\tau))^T \in X \times U$$
   $$x(T) \in X_f$$

3. Apply $u(t) = u^*(\tau|t), \tau \in [t_k, t_k + \delta)$

Revisiting Tracking/Stabilizing NMPC

Here: Focus on discrete-time setting.

Considered control problem
Setpoint Stabilization
- Reference = setpoint $x_s \in X \subseteq \mathbb{R}^n$
- Constraint satisfaction: $\forall t \in \mathbb{N}: u(t) \in U$ and $x(t; x_0, u(\cdot)) \in X$
- Stability: $\forall \varepsilon > 0 \exists \delta > 0$ such that
  $$\|x(0) - x_s\| > \delta \Rightarrow \|x(t; x_0, u(\cdot))\| < \varepsilon \quad \forall t \geq 0$$

Model Predictive Control – Main idea
Model predictive control = repeated optimal control

1. State measurement $x(t)$ at $t \in \mathbb{N}$

2. Solve
   $$\min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N))$$
   subject to
   $$x(k+1) = f(x(k), u(k)), \quad k = 0, \ldots, N-1$$
   $$x(0) = x(t)$$
   $$(x(k), u(k))^T \in X \times U, \quad k = 0, \ldots, N-1$$
   $$x(N) \in X_f$$

3. Apply $u_N(x(t)) = u^*(0|t)$

Notation
- State trajectory predicted at time $t$: $x(\cdot|t)$
- Input trajectory predicted at time $t$: $u(\cdot|t)$
Main ingredients for design of tracking NMPC

**Discrete-Time NMPC**

1. State measurement \( x(t) \) at \( t \in \mathbb{N} \)
2. Solve discrete-time OCP
   \[
   \min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k+1), u(k)) + V_f(x(N))
   \]
   subject to
   \[
   x(k+1) = f(x(k), u(k)), \quad x(0) = x(t) \]
   \( (x(k), u(k))^T \in X \times U \)
   \( x(N) \in X_f \)
3. Apply \( u_N(x(t)) = u^*(0|t) \)

Ingredients

- System model: \( f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x} \)
- State constraints: \( X \subseteq \mathbb{R}^{n_x} \)
- Input constraints: \( U \subseteq \mathbb{R}^{n_u} \)
- State feedback / state estimate \( x(t) \)
  → Assumed to be exactly known.
- Stage cost \( \ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R} \)
- Terminal penalty \( V_f : \mathbb{R}^{n_x} \to \mathbb{R}_+^\dagger \)
- Terminal constraint \( X_f \subseteq X \subseteq \mathbb{R}^{n_x} \)
- Prediction horizon \( N \in \mathbb{N} \)
  → To be designed/chosen!

Closed-loop system

**Discrete-Time NMPC**

1. State measurement \( x(t) \) at \( t \in \mathbb{N} \)
   \[
   u = \mu_N(x(t))
   \]
   \( \Sigma : x^+ = f(x, u) \)
2. Solve discrete-time OCP
   \[
   \min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N))
   \]
   subject to
   \[
   x(k+1) = f(x(k), u(k)), \quad x(0) = x(t) \]
   \( (x(k), u(k))^T \in X \times U \)
   \( x(N) \in X_f \)
3. Apply \( u_N(x(t)) = u^*(0|t) \)

Recursive feasibility

Considered NMPC scheme

\[
\min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k), u(k)) + V_f(x(N)) \tag{1}
\]
subject to
\[
x(k+1) = f(x(k), u(k)), \quad x(k) = x(t)
\]
\( (x(k), u(k))^T \in X \times U \)
\( x(N) \in X_f \)

Definition (Recursive feasibility).
Let \( X_0 \subseteq X \) denote a set of initial conditions \( x(0) = x_0 \) for which OCP (1) admits a feasible solution. OCP (1) is said to be recursively feasible with respect to \( X_0 \) if for all \( x(0) = x_0 \in X_0 \) the inclusion

\[
f(x_0, \mu_N(x_0)) \in X_0
\]
holds.

Comparison functions

- \( \mathcal{L} := \left\{ \gamma : \mathbb{R}_+^\dagger \to \mathbb{R}_+^\dagger \mid \gamma \text{ continuous and decreasing with } \lim_{s \to \infty} \gamma(s) = 0 \right\} \)
- \( \mathcal{K} := \left\{ \alpha : \mathbb{R}_+^\dagger \to \mathbb{R}_+^\dagger \mid \alpha \text{ continuous and strictly increasing with } \alpha(0) = 0 \right\} \)
- \( \mathcal{K}_\infty := \left\{ \alpha \in \mathcal{K} \mid \alpha \text{ unbounded} \right\} \)
- \( \mathcal{K}_\mathcal{L} := \left\{ \beta : \mathbb{R}_+^\dagger \times \mathbb{R}_+^\dagger \to \mathbb{R}_+^\dagger \mid \beta(s, k) \in \mathcal{K}, \beta(r, i) \in \mathcal{L} \right\} \)

Main assumptions for stabilizing NMPC with terminal constraints

Considered NMPC scheme

\[
\min_{u(k)} \sum_{k=0}^{N-1} \ell(u(k), u(k+1)) + V_f(u(N))
\]

subject to

\[
x(k+1) = f(x(k), u(k)), \quad x(0) = x(t)
\]

\[
(x(k), u(k)) \in X \times U
\]

\[
x(N) \in X_f
\]

(1)

Assumption 1 (Lower boundedness of \( \ell \)).
The stage cost satisfies \( \ell(0,0) = 0 \). Furthermore, there exists \( \alpha_1 \in \mathcal{K}_\infty \) such that for all \( (x,u) \in X \times U \)

\[
\alpha_1(\|x\|) \leq \ell(x,u).
\]

Assumption 2 (Local bound on the cost-to-go).
For all \( x \in X_f \), there exist an input \( u = \kappa_f(x) \in U \) s.t. \( f(x, \kappa_f(x)) \in X_f \) holds and

\[
V_f(f(x, \kappa_f(x))) + \ell(x, \kappa_f(x)) \leq V_f(x).
\]

Furthermore, \( V_f(0) = 0 \) and \( V_f(x) \geq 0 \) for all \( x \in X_f \).

Stability of NMPC with terminal constraints

Theorem (Stability of tracking NMPC with terminal constraints).
Let Assumptions 1 and 2 hold. Suppose that \( 0 \in \text{int}(X_f) \) and that there exists \( \alpha_3 \in \mathcal{K}_\infty \) such that, for all \( x \in X_f \), \( V_f(x) \leq \alpha_3(\|x\|) \).

Then the closed-loop system \( x^+ = f(x, \mu_N(x)) \) arising from the NMPC scheme has the following properties:

1. If OCP (1) is feasible for \( t = 0 \), then it is feasible for all \( t \in \mathbb{N} \).
2. The origin \( x = 0 \) is an asymptotically stable equilibrium of \( x^+ = f(x, \mu_N(x)) \).
3. The region of attraction of \( x = 0 \) is given by the set of all initial conditions \( x_0 \) for which OCP (1) is feasible.

References


Blue print for NMPC stability proofs with terminal constraints

- Step 1: Recursive feasibility: append terminal control law

\[
u(k+1) = \begin{cases} u^*(k+1), & k = 0, \ldots, N - 2 \\ \kappa_f(x^*(N)), & k = N - 1 \end{cases}
\]

No plant-model mismatch:

- \( x(t+1) = f(x(t), u^*(1)) = x^*(1) \)
- \( x^*(N) \in X_f \)

Assumption 2:

- \( f(x^*(N), \kappa_f(x^*(N))) \in X_f \)

Blue print for NMPC stability proofs with terminal constraints

- Step 2: Consider the optimal value function as a Lyapunov function

\[
V_N(x(t)) := \sum_{k=0}^{N-1} \ell(x(k), u^*(k)) + V_f(x^*(N))
\]

- Performance of feasible input \( u(\cdot) \) applied at \( x(t+1) = x^*(1) \)

\[
J_N(x(t+1), u(\cdot)) := \sum_{k=0}^{N-1} \ell(x(k+1), u(k+1)) + V_f(x(N+1))
\]

- Decrease of \( V_N(x) \)

\[
V_N(x(t+1)) - V_N(x(t)) \leq J_N(x(t+1), u(\cdot) + 1) - V_N(x(t))
\]
Blue print for NMPC stability proofs with terminal constraints

- Step 2: Consider the optimal value function as a Lyapunov function

  \[ V_N(x(t)) := \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t)) \]

- Performance of feasible input \( u(\cdot|t+1) \) applied at \( x(t+1) = x^*(1|t) \)

  \[ J_N(x(t+1), u(\cdot|t+1)) := \sum_{k=0}^{N-1} \ell(x(k|t+1), u(k|t+1)) + V_f(x(N|t+1)) \]

- Decrease of \( V_N(x) \)

  \[ V_N(x(t+1)) - V_N(x(t)) \leq J_N(x(t+1), u(\cdot|t+1)) - V_N(x(t)) \]

- Assumption 2 \( \leq 0 \)

Tracking NMPC without terminal constraints?

1. Replace \( V_f(x) \) by scaled terminal penalty \( \beta V_f(x) \).


2. Use a control Lyapunov function as terminal penalty.


3. Use a sufficiently long prediction horizon.


4. Consider so-called cost-controllability conditions.


What changes in economic NMPC?

**Tracking NMPC**

- Objective: solve control task

- Stability with & without terminal constraints/penalties

- Stage cost \( \ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R} \)

- Terminal penalty \( V_f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_0^+ \)

- Terminal constraint \( X_f \subseteq X \subseteq \mathbb{R}^{n_x} \)

- Prediction horizon \( N \in \mathbb{N} \)

  \rightarrow To be designed/chosen!

**Economic NMPC**

- Objective: optimize performance; i.e.

  \[ \min_{u(\cdot)} \sum_{t=0}^{\infty} \ell(x(t), u(t)) \quad \text{s.t. } \ldots \]

- Stability?

  \[ \text{Stage cost } \ell \text{ is given} \]

  \[ \text{Terminal penalty } V_f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_0^+ \]

  \[ \text{Terminal constraint } X_f \subseteq X \subseteq \mathbb{R}^{n_x} \]

  \[ \text{Prediction horizon } N \in \mathbb{N} \]

  \rightarrow To be designed/chosen!

Motivating Examples
**Example — Van de Vusse reactor**

Van de Vusse reactor 

\[ \begin{align*} 
A & \xrightarrow{k_A} B \xrightarrow{k_B} C, \\
2A & \xrightarrow{k_D} D 
\end{align*} \]

Dynamics (partial model)

\[ 
\begin{align*} 
\dot{c}_A &= r_A(c_A, \vartheta) + (c_m - c_A)u_1 \\
\dot{c}_B &= r_B(c_A, c_B, \vartheta) - c_B u_1 \\
\dot{\vartheta} &= h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta_1) + (\vartheta_1 - \vartheta)u_1, \\
r_1(c_A, \vartheta) &= k_1(\vartheta) + 2k_3(\vartheta)c_A^3 \\
r_2(c_A, c_B, \vartheta) &= k_2(\vartheta) - k_3(\vartheta) c_B \\
h(c_A, c_B, \vartheta) &= \sigma_1(c_A \vartheta) + \sigma_2(c_B \Delta H_{\text{AB}}) + \sigma_3(c_B \Delta H_{\text{AB}} + 2k_3(\vartheta)c_A^2 \Delta H_{\text{AB}}) \\
\end{align*} \]

Constraints

\[ 
\begin{align*} 
c_A &\in [0, 6\ \text{mol}] \\
c_B &\in [0, 4\ \text{mol}] \\
u_1 &\in [0, 200\ \text{C}] \\
\vartheta &\in [70, 150] \ \text{C}. 
\end{align*} \]

Objective: maximize produced amount of B

\[ J_1(x_0, u^*) = \int_0^T -\beta c_B(t)u_1(t)dt, \quad \beta > 0 \]


**Example — Reactor with parallel reaction**

- Chemical reaction: \[ H \rightarrow H_1, \ H \rightarrow H_2 \]
- States: \( x_1 \approx \) concentration of \( R_1 \), \( x_2 \approx \) concentration of \( P_1 \), \( x_3 \approx \) dimensionless temperature
- Input: \( u \approx \) heat flux through cooling jacket
- Constraints: \( U = [0.049, 0.149] \), \( X = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \)
- Dynamics

\[ \begin{align*} 
\dot{x}_1 &= 1 - r_1(x_1, x_3) - x_1 \\
\dot{x}_2 &= r_2(x_1, x_3) - x_2 \\
\dot{x}_3 &= u - x_3 
\end{align*} \]

- \( r_1 : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( r_2 : \mathbb{R}^2 \rightarrow \mathbb{R} \):

\[ 
\begin{align*} 
r_1(x_1, x_3) &= 10^4 x_1^2 e^{-\frac{x_1}{2}} + 400 x_1 c_0 e^{-\frac{x_3}{10}} \\
r_2(x_1, x_3) &= 10 x_1^2 e^{-\frac{x_1}{2}} - x_3 
\end{align*} \]

- Stage cost \( \ell(x) = -x_2 \)


**Example — Reactor with parallel reaction**

- \( x_1 = \sigma_1 c_A, x_2 = \sigma_2 c_B, x_3 = \sigma_3 \vartheta \)
- Discretized with Runge-Kutta 8(7), \( N = 20 \), sampling rate \( \delta = 0.0033 \)

- Discretized with Runge-Kutta 5(4), \( N = 50 \), sampling rate \( \delta = 0.1 \)
Optimal steady-state operation

Definition -,optimal operation at steady-state
- Optimal steady-state: \((x_s, u_s) = \arg \min_{x \in X, u \in U, x(0) = x} \ell(x, u)\)
- A system is optimally operated at steady-state if for each feasible state and input sequences \(x(t)\) and \(u(t)\) the following holds:
  \[
  \liminf_{T \to \infty} T^{-1} \sum_{t=0}^{T-1} \ell(x(t), u(t)) \geq \ell(x_s, u_s).
  \]

Dissipativity and optimal steady-state operation

A system is strictly dissipative with respect to the supply rate \(s(x, u) = \ell(x, u) - \ell(x_s, u_s)\) if there exists a storage function \(\lambda : X \to \mathbb{R}_+\) such that for all \(x \in X\) and \(u \in U\) it holds that
\[
\lambda(f(x, u)) - \lambda(x) \leq s(x, u) - \alpha \ell(\|x - x_s, u - u_s\|), \quad \alpha \in \mathcal{K}_\infty.
\]

Theorem [Angeli, Amrit, Rawlings '12]
A system is optimally operated at steady-state if it is dissipative with respect to the supply rate \(s(x, u) = \ell(x, u) - \ell(x_s, u_s)\).

Sketch of proof: By dissipativity, we have
\[
0 \leq \lim_{T \to \infty} \frac{\lambda(x(T)) - \lambda(x(0))}{T} \leq \liminf_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \left[\ell(x(k), u(k)) - \ell(x_s, u_s)\right]
\]
Optimal steady-state operation

**Theorem [Willems ’72]**

A system is dissipative with respect to the supply rate $s$ if and only if the available storage $S_s$ is bounded for all $x$. Moreover, $S_s$ is a possible storage function.

$$S_s(x) := \sup_{z(0)=x} \sum_{k=0}^{T-1} -s(z(k), v(k))$$

**Definition - Dissipativity [Willems ’72, Byrnes & Lin ’94]**

A system is **dissipative** with respect to the supply rate $s$ if there exists a storage function $\lambda: \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ it holds that

$$\lambda(f(x, u)) - \lambda(x) \leq s(x, u).$$

**Dissipativity and optimal steady-state operation**

Additional controllability condition

[Müller, Angeli, Allgöwer ’15]

Optimal operation at steady-state

Dissipativity w.r.t. supply rate

$$s(x, u) = \ell(x, u) - \ell(x_s, u)$$

**Sketch of proof (by contradiction):**

- For each $r \geq 0$, there exist sequences with $\sum_{k=0}^{T_r-1} \ell(x_r(k), u_r(k)) \leq -r$.
- Can steer the system to $x_s$ and from there to $x_0$ in $N$ steps at a time.
- We have $\sum_{k=0}^{T_r+2N-1} \ell(x_r(k), u_r(k)) < 0$.
- $\Rightarrow \lim_{T \to \infty} \sum_{k=0}^{T-1} \ell(x(k), u(k)) < 0$.
- This contradicts optimal steady-state operation.

**Definitions**

- $\mathcal{X}_N$: set of states which can be controlled to $x_s$ in $N$ steps
- $\mathcal{R}_N$: set of states which can be reached from $x_0$ in $N$ steps
- $Z_N$: set of state/input pairs which are part of a feasible trajectory staying inside $\mathcal{X}_N \cap \mathcal{R}_N$

**Theorem [Müller, Angeli, Allgöwer ’15]**

Suppose that a system is optimally operated at steady-state. Then it is dissipative on $Z_N$ with supply rate $s(x, u) = \ell(x, u) - \ell(x_s, u)$ for each $N \geq 0$.

**Stability using terminal constraints**

If steady-state operation is optimal, does closed-loop system converge to $x_s$?

$$V_N(x(t)) := \min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t))$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k = 0, \ldots, N - 1$$

$$x(0|t) = x(t)$$

$$(x(k|t), u(k|t))^T \in \mathbb{X} \times \mathbb{U}, \quad k = 0, \ldots, N - 1$$

$$x(N|t) = x_s$$

**Remark:** Can be extended to framework including terminal region and cost.
Stability using terminal constraints

**Theorem** [Angeli, Amrit, Rawlings ‘12]

Assume
- strict dissipativity w.r.t. supply rate \( s(x, u) = \ell(x, u) - \ell(x_s, u_s) \),
- \( V_N \) and \( \lambda \) are continuous at \( x_s \).

Then \( x_s \) is an asymptotically stable equilibrium of the resulting closed-loop system.

Main idea for stability proof in stabilizing MPC: use optimal value function as Lyapunov function
\[
V_N(x(t + 1)) - V_N(x(t)) = \tilde{\ell}(x(t), u(t)) + \ell(x_s, u_s) \leq -\alpha(\|x(t) - x_s\|)
\]

In economic MPC: second inequality **not** satisfied!

**Stability using terminal constraints**

**Theorem** [Angeli, Amrit, Rawlings ‘12]

Assume
- strict dissipativity w.r.t. supply rate \( s(x, u) = \ell(x, u) - \ell(x_s, u_s) \),
- \( V_N \) and \( \lambda \) are continuous at \( x_s \).

Then \( x_s \) is an asymptotically stable equilibrium of the resulting closed-loop system.

Define rotated cost function
\[
\tilde{\ell}(x, u) = \ell(x, u) - \ell(x_s, u_s) + \lambda(x) - \lambda(f(x, u))
\]

If system is strictly dissipative: \( \tilde{\ell}(x, u) \geq \alpha_e(\|x - x_s\|) \)

**Modified optimization problem**

\[
\tilde{V}_N(x(t)) = \min_{\{k(t)\}} \sum_{k=0}^{N-1} \tilde{\ell}(x(k|t), u(k|t))
\]

s.t. \( x(0|t) = x(t) \), \( x(k + 1|t) = f(x(k|t), u(k|t)) \), \( k = 0, \ldots, N - 1 \)
\[
(x(k|t), u(k|t)) \in \mathbb{X} \times \mathbb{U}, \quad k = 0, \ldots, N - 1, \quad x(N|t) = x_s
\]
Example - chemical reactor with dissipativity

Van de Vusse reactor:
- Reactions $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ and $2A \xrightarrow{k_3} D$, with $A$: reactant, $B$: desired product, $C, D$: waste products

\[
\begin{align*}
\dot{c}_A &= r_A(c_A, \vartheta) + (c_{in} - c_A)u_1 \\
\dot{c}_B &= r_B(c_A, c_B, \vartheta) - c_Bu_1 \\
\dot{\vartheta} &= h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1.
\end{align*}
\]

\[\dot{c}_A, \dot{c}_B, \dot{\vartheta}:\text{ temperature in the reactor, } u_1: \text{ normalized flow rate of } A, u_2: \text{ temperature in cooling jacket}\]

- Control objective: maximize production rate of $B \to \ell(x, u) = -c_Bu_1$
- System is strictly dissipative w.r.t. supply rate $s(x, u) = \ell(x, u) - \ell(x_s, u_s)$

Example - chemical reactor without dissipativity

Continuous flow stirred-tank reactor with parallel reactions
- Reactions $R \to P_1$ and $R \to P_2$, with $R$: reactant, $P_1$: desired product, $P_2$: waste product

\[
\begin{align*}
\dot{x}_1 &= 1 - 10^4x_1^2e^{-1/x_3} - 400x_1e^{-0.55/x_3} - x_1 \\
\dot{x}_2 &= 10^4x_1^2e^{-1/x_3} - x_2 \\
\dot{x}_3 &= u - x_3
\end{align*}
\]

\[x_1: \text{ concentration of } R, x_2: \text{ concentration of } P_1, x_3: \text{ temperature in the reactor, } u: \text{ proportional to heat flux through cooling jacket}\]

- Control objective: maximize product $P_1 \to \ell(x, u) = -x_2$
Example - chemical reactor without dissipativity

Optimal periodic orbit length: \( T^* \approx 11.444 \)

\[
\min_{\theta, \tau} \frac{1}{T} \int_{0}^{T} -x_2(\tau) \, d\tau
\]
subject to \( x(0) = x(T), \ \ T \in [5, 20] \).

\[
\text{Example - chemical reactor without dissipativity}
\]

\[\text{Infinite horizon averaged performance:} \]
\[J_\infty^d(x_0, \mu_N) := \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell(x(t), \mu_N(x(t)))\]

**Theorem** [Angeli, Amrit, Rawlings ’12]

\[J_\infty^d(x_0, \mu_N) \leq \ell(x_0, u_1)\]

**Sketch of proof:**
- \( V_\delta(x(t+1)) - V_\delta(x(t)) \leq -\ell(x(t), u(t)) + \ell(x_1, u_1) \)
- Iterate this inequality, divide by \( T \) and take \( \liminf \)

**Remark:** This bound is valid independent of dissipativity.

---

Example - chemical reactor without dissipativity

Recovering steady-state optimality through regularization:

\[\ell(x, u) = -x_2 + \omega(u - u_1)^2, \ \ \omega > 0\]

---

**Asymptotic average and transient performance**

What can be said about closed-loop performance?

- Infinite horizon averaged performance:
  \[
  J_\infty^d(x_0, \mu_N) := \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell(x(t), \mu_N(x(t)))
  \]
- Finite horizon non-averaged performance:
  \[
  J_T^d(x_0, \mu_N) := \sum_{t=0}^{T-1} \ell(x(t), \mu_N(x(t)))
  \]

**Assumption:** Strict dissipativity plus technical (continuity) assumptions on storage and optimal value function.

\( \Rightarrow \) Closed loop satisfies \( \|x(T) - x_s\| \leq \beta(\|x_0 - x_s\|, t) \) with \( \beta \in \mathcal{K} \).

- Define \( \mathcal{U}_{\ell(x_0)} \) := \{ \( u \in U \) \mid \text{admissible and} \ \|x(T, x_0, u) - x_s\| \leq \kappa \}

**Theorem** [Grüne & Panin ’15]

The following performance bounds hold:

\[ J_\infty^d(x_0, \mu_N) \leq V_\delta(x_0) + \delta(N) \text{ with } \delta \in \mathcal{L} \]
\[ J_T^d(x_0, \mu_N) \leq \inf_{u \in \mathcal{U}_\ell(x_0)} \int T(x_0, u) + \delta_1(N) + \delta_2(T) \text{ with} \]
\[ \kappa = \beta(\|x - x_0\|, T) \text{ and } \delta_1, \delta_2 \in \mathcal{L} \]


Example – Optimal fish harvest

\[ \min_{u(.)} \int_0^T ax(t) + bu(t) - cx(t)u(t)dt \]
subject to
\[ \dot{x} = x(s - x - u), \quad x(0) = x_0 \]
\[ u(t) \in [0, u_{\text{max}}], x(t) \in (0, \infty) \]
- \( x \): fish density
- \( u \): fishing rate
- \( x_0 \): initial sustainable fish density
- \( a = 1, b = c = 2, u_{\text{max}} = 5 \)

Example – Optimal fish harvest (quad. objective cont’d)

\[
\min_{u(\cdot)} \int_0^T \frac{1}{2} q(x(t) - x_C)^2 + \frac{1}{2} r(u(t) - u_C)^2 dt
\]
subject to
\[
\begin{align*}
\dot{x} &= x(x_S - x - u), \quad x(0) = x_0 \\
u(t) &
\in [0, u_{\max}], x(t) \in (0, \infty) \\
u_{\max} &= 5, x_S = 5 \\
q &= 10, r = 1, x_C = 2.75, u_C = 2.25
\end{align*}
\]

\[\to\] Similar behavior for different initial conditions and horizon lengths.
\[\to\] Similarity properties of solutions of parametric OCPs. \[\to\] Turnpike property!

Turnpike properties in OCPs

Proposition (Turnpike in OCP (2)).

Let Assumptions 3 and 4 hold, and suppose that the storage function \( \lambda \) is bounded on \( \mathcal{X} \). Then there exists \( C < \infty \), such that, for all \( x_0 \in \mathcal{X}_D \), we have

\[
\# \mathcal{Q}_C \geq N - \frac{C}{\alpha_\varepsilon(\varepsilon)}
\]

where \( \mathcal{Q}_C := \{k \in \{0, \ldots, N-1\} \mid ||(x^*(k; x_0), u^*(k; x_0)) - (x_s, u_s)|| \leq \varepsilon \} \), \#\mathcal{Q}_C \) is the cardinality of \( \mathcal{Q}_C \), \( \alpha_\varepsilon \) is the value of the dissipation inequality on slide II.3.

Assumptions for economic NMPC without terminal constraints

Considered OCP

\[
\begin{align*}
\min_{u(\cdot)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) \\
\text{subject to} \\
x(k + 1|t) &= f(x(k|t), u(k|t)), \quad x(0) = x(t) \\
(x(k|t), u(k|t))^1 &\in \mathcal{X} \times \mathcal{U}
\end{align*}
\]

Assumption 3 (Strict dissipativity of OCP (2)).

There exists a bounded non-negative storage function \( \lambda : \mathcal{X} \to \mathbb{R}_+ \) such that OCP (2) is strictly dissipative with respect to \( (x_s, u_s) \in \text{int}(\mathcal{X} \times \mathcal{U}) \) in the sense of the Definition on slide II.3.

Assumption 4 (Exponential reachability of \( x_s \)).

For all \( x_0 \in \mathcal{X}_D \), there exists an infinite-horizon admissible input \( u(\cdot; x_0) \), \( c > 0, \rho \in [0, 1) \), such that

\[
||x(k; x_0, u(\cdot; x_0)) - (x_s, u_s)|| \leq c \rho^k,
\]

i.e. the steady state \( x_s \) is exponentially reachable.

Turnpike properties in OCPs

Proof sketch

- \( V_N(x_0) \) is the optimal value function of OCP (2).
- \( \ell(x_s, u_s) = 0 \)
- The strict dissipation inequality implies

\[
V_N(x_0) \geq \lambda(x^*(N; x_0)) - \lambda(x_0) + \sum_{k=0}^{N-1} c \rho^k ||x^*(k; x_0) - (x_s, u_s)||
\]

\[
\geq -2\lambda := \sup_{x \in \mathcal{X}} \lambda(x)
\]
Turnpike properties in OCPs

Proof sketch

- $N(x_0)$ is the optimal value function of OCP (2).
- $\ell(x, u) = 0$
- The strict dissipation inequality implies
  \[
  V_N(x_0) \geq \lambda(x^*(N, x_0)) - \lambda(x_0) + \sum_{k=0}^{N-1} \alpha_k(\|x^*(k; x_0), u^*(k; x_0) - (x_k, u_k)\|) \\
  \geq -2\lambda := \sup_{x \in X} |\lambda(x)|
  \]
  Exp. reachability implies: $V_N(x_0) \leq \frac{L_N e}{1 - \rho}$
  \[
  \sum_{k=0}^{N-1} \alpha_k(\|x^*(k; x_0), u^*(k; x_0) - (x_k, u_k)\|) \geq (N - \#Q_\epsilon)\alpha_\epsilon(\epsilon)
  \]

$\Rightarrow$

$\#Q_\epsilon \geq N - \frac{L_N e (1 - \rho)^{-1} + 2\lambda}{\alpha_\epsilon(\epsilon)}$

Relation of dissipativity and turnpike properties

Under suitable technical assumptions, additional relations (black arrows) can be established:

The system $x^+ = f(x, u)$ is strictly dissipative w.r.t. to $(x_x, u_x)$.

The system $x^+ = f(x, u)$ is optimally operated at $(x_x, u_x)$.

OCP (2) has a turnpike at $(x_x, u_x)$.

$\Rightarrow$ Turnpike and dissipativity properties of OCPs are essentially, i.e. almost, equivalent.

References

Recursive feasibility

Assumption 5 (Local controllability around $(x_x, u_x)$).

The Jacobian linearization of $x^+ = f(x, u)$ at $(x_x, u_x)$ is $n_x$-step reachable.

Proposition (Recursive feasibility of OCP (2)).

Let Assumptions 3-5 hold. Then, there exists a finite horizon $N \in \mathbb{N}$ such that, for all $x_0 \in X_0$,

OCP (2) is recursively feasible.

Practical Stability without Terminal Constraints
Recursive feasibility

Proof sketch

- The turnpike property of OCP (2) implies that, for any $\varepsilon > 0$, there exists a finite $N$ such that $k_1, k_2$, with $k_1 + 2n_x \leq k \leq N$, such that $x_{k_1} := x^*(k_1; x_0) \in B_r(x_a)$ and $x_{k_2} := x^*(k_2; x_0) \in B_r(x_a)$.

Stability of economic NMPC without terminal constraints

**Theorem** (Practical stability of economic NMPC with terminal constraints).

Let Assumptions 3–5 hold and suppose that $X$ is compact. Then, for sufficiently large horizon $N \in \mathbb{N}$ the closed-loop system $x^+ = f(x, \mu_N(x))$ satisfies:

1. If, for the horizon $N \in \mathbb{N}$, OCP (2) is feasible for $t = 0$ and $x(0) \in X_0$, then it is feasible for all $k \in \mathbb{N}$.
2. There exist $\rho \in \mathbb{R}^+$ and $\beta \in L$ such that, for all $x(0) \in X_0$, the closed-loop trajectories generated by $x^+ = f(x, \mu_N(x))$ satisfy $\|x(t) - x_\ast\| \leq \max\{\beta\|x_0 - x_\ast\|, t, \rho\}$.

Recap – Rotated OCP

- Rotated cost function
  $$\tilde{\ell}(x, u) = \ell(x, u) - \ell(x_a, u_a) + \lambda(x) - \lambda(f(x, u))$$

- Rotated OCP
  $$\tilde{V}_N(x(t)) := \min_{u(t)} \sum_{k=0}^{N-1} \tilde{\ell}(x(k|t), u(k|t))$$
  subject to
  $$\begin{align*}
  x(k+1|t) &= f(x(k|t), u(k|t)), \quad x(0|t) = x(t) \\
  (x(k|t), u(k|t))^T &\in X \times U
  \end{align*}$$
Stability of economic NMPC without terminal constraints

**Theorem (Practical stability of economic NMPC with terminal constraints).**
Let Assumptions 3–5 hold and suppose that $X$ is compact. Then, for sufficiently large horizon $N \in \mathbb{N}$ the closed-loop system $x^+ = f(x, \mu_N(x))$ satisfies:

(i) If, for the horizon $N \in \mathbb{N}$, OCP (2) is feasible for $t = 0$ and $x(0) \in X_0$, then it is feasible for all $k \in \mathbb{N}$.

(ii) There exist $\rho \in \mathbb{R}^+$ and $\beta \in X \times \mathbb{R}^+$ such that, for all $x(0) \in X_0$, the closed-loop trajectories generated by $x^+ = f(x, \mu_N(x))$ satisfy

$$||x(t) - x_a|| \leq \max(\beta(||x_0 - x_a||, t), \rho).$$

(iii) If additionally
(a) there exist $\gamma \in X$ such that for each $N \in \mathbb{N}$ and all $x \in X_0$, $V_N(x) - V_N(x_0) \leq \gamma \|x - x_0\|$, 
(b) and the storage function $\lambda$ is continuous at $x = x_a$,

then (ii) holds with $\rho = \rho(N)$ where $\rho(N) \to 0$ for $N \to \infty$.

**Proof sketch**

Part (i): already shown.

Part (ii): w.l.o.g. $\ell(x_v, u) = 0$

- Consider shifted value function $V_N(x) := \lambda(x) + V_N(x) - V_N(x_a)$
- Decrease condition:

$$\Delta(t) = \lambda(x(t+1)) - \ell(x(t)) + \ell(x(t), u^*(0|t))$$

Proof sketch

**Part (ii) (cont’d):**

- $\Delta(t) = \lambda(x(t+1)) - \lambda(x(t)) - \ell(x(t), u^*(0|t))$
- $\geq (\sum_{k=0}^{k_1-1} \ell(x(k|t+1), u(k|t+1)) - (\sum_{k=0}^{k_1} \ell(x^*(k|t), u^*(k|t)))$
- $\sum_{k=k_1}^{k_2} \ell(x(k|t+1), u(k|t+1)) - (\sum_{k=k_1}^{k_2} \ell(x^*(k|t), u^*(k|t)))$
- $\sum_{k=k_2}^{N} \ell(x(k|t+1), u(k|t+1)) - (\sum_{k=k_2}^{N} \ell(x^*(k|t), u^*(k|t)))$

Grüne, L. Economic receding horizon control without terminal constraints. *Automatica*, 2013, 49, 725-734


Proof sketch

Part (ii) (cont’d):

\[ \Delta(t) = \lambda(x(t) + 1) - \lambda(x(t)) - \ell(x(t), u^*(0,t)) \]
\[ + \sum_{k=0}^{k_1-1} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=1}^{k_1} \ell(x^*(k|t), u^*(k|t)) \]
\[ + \sum_{k=1}^{k_1} \ell(x^*(k|t), u^*(k|t)) \]
\[ + \sum_{k=k_1+1}^{k_1+2n_x} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+1}^{k_1+2n_x} \ell(x^*(k|t), u^*(k|t)) \]
\[ + \sum_{k=k_1+2n_x}^{N-1} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+2n_x}^{N-1} \ell(x^*(k|t), u^*(k|t)) \]
\[ \Rightarrow \tilde{V}_N(x(t+1)) - \tilde{V}_N(x(t)) \leq \Delta(t) \leq -\alpha_\ell(\|x(t) - x_s\|) + (2n_x + 1)L_\ell(\varepsilon) \]

Example – Van de Vusse reactor (revisited)

Van de Vusse reactor \[ A \xrightarrow{\alpha} B \xrightarrow{\beta} C, \quad 2A \xrightarrow{\delta} D \]

Dynamics (partial model)
\[ \hat{c}_A = r(A, c_A, \theta) + \left(c_m - c_A\right)u_1 \]
\[ \hat{c}_B = r_B(c_A, c_B, \theta) - c_B u_1 \]
\[ \dot{\theta} = (c_A, c_B, \theta) + \alpha(u_2 - \theta) + (\theta_{in} - \theta)u_1, \]
\[ r(A, c_A, c_B, \theta) = k_1 \theta c_A c_B^2 - k_2 \theta c_A \]
\[ r_B(c_A, c_B, \theta) = k_3 (\theta c_A c_B^2 - E_0 \theta) \]
\[ h(c_{AI}, c_{IA}) = -k_4 \theta c_A c_B + k_5 \theta c_B + k_6 \theta c_A \theta + k_7 \theta c_B \theta \]

Constraints
\[ c_A \in [0, 6] \text{ mol}, \quad c_B \in [0, 4] \text{ mol}, \quad \theta \in [70, 150] \text{°C} \]
\[ u_1 \in [3, 25] \text{ K}, \quad u_2 \in [0, 200] \text{ K} \]

Objective = maximize produced amount of B
\[ J_p(x_0, u(\cdot)) = \int_0^T -\beta c_B(t) u_1(t) dt, \quad \beta > 0 \]

- As predicted by the last theorem, for increasing horizon $N$, the closed-loop system converges to smaller neighborhoods of the turnpike $x_0$. 
EMPC without strict dissipativity

In this section we discuss a selection of schemes which use relaxed terminal conditions or yield stability without imposing strict dissipativity.

Outline

- Generalized terminal constraints
- Lyapunov-based approach
- Multi-objective approach

Generalized terminal constraints

It may happen that

- EMPC with equilibrium terminal constraints $x(t|N) = x_s$ is too restrictive / numerically infeasible
- the terminal cost $V_f$ for EMPC with regional terminal constraints $x(t|N) \in X_f$ is too difficult to compute

In these cases, other types of constraints may be useful

Idea: Require that $x(t|N)$ is an equilibrium, but not necessarily equal to $x_s$

[Fagiano/Teel '13, Müller/Angeli/Allgöwer '13, Ferramosca/Limon/Camacho '14]

(based on earlier ideas from stabilizing MPC)
Scheme with generalized terminal constraints

\[
\min_{u(t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + \beta \ell(x(N|t), u(N|t))
\]

subject to

\[
\begin{align*}
  x(k+1|t) &= f(x(k|t), u(k|t)), & k = 0, \ldots, N-1 \\
  x(0|t) &= x(t) \\
  (x(k|t), u(k|t))^T &\in X \times U, & k = 0, \ldots, N \\
  x(N|t) &= f(x(N|t), u(N|t)), \\
  \ell(x(N|t), u(N|t)) &\leq \kappa(t),
\end{align*}
\]

where \( \kappa(t+1) = \ell(x(N|t), u(N|t)), \kappa(0) \) “large”, \( \beta > 0 \)

- Always end in an equilibrium that is at least as good as the previous one
- A large \( \beta \) provides incentive to select a good equilibrium

Properties

Using this \( \beta \) and the assumption that from each steady state \((x, u)\) a better steady state \((x', u')\), i.e.,

\[
\ell(x', u') \leq \max\{\ell(x_s, u_s), \ell(x, u) - \varepsilon\}
\]

can be reached in \( N \) steps, [Fagiano/Teel ’13] propose an EMPC scheme which eventually reaches \( \ell(x_s, u_s) \) up to \( \varepsilon \)

Problems:
- The scheme discards recent optimization results if the terminal equilibrium value does not improve
- The appropriate \( \beta \) may be difficult to find

The second point can be addressed by the adaptive choice

\[
\beta(t+1) = B(\beta(t), x(t), \kappa(t)), \quad \beta(0) = \beta_0 \geq 0
\]

where \( \beta \) increases as long as the terminal equilibrium value can be improved [Müller/Angeli/Allgöwer ’13f]

Properties

Theorem [Fagiano/Teel ’13] Given \( \varepsilon > 0 \), there exists \( \beta(\varepsilon) > 0 \) such that

\[
\ell(x^*(N|t), u^*(N|t)) \leq \ell_{\min}(x(t)) + \varepsilon
\]

where \( \ell_{\min}(x(t)) \) is the cost of the best equilibrium that is reachable from initial condition \( x(t) \) in \( N \) steps

Problem: \( \ell_{\min} \) may be significantly larger than \( \ell(x_s, u_s) \)

Discussion

Discussion of generalized equilibrium terminal constraints

- Averaged performance is bounded by “eventual” terminal equilibrium
- No transient performance estimates known (problem: influence of \( \beta \))
- Asymptotic stability of the optimal steady state can be shown under additional (so far still rather restrictive) conditions, including strict dissipativity [Ferramosca/Limon/Camacho ’14]
- Results can be extended to generalized regional terminal constraints [Müller/Angeli/Allgöwer ’14] and to periodic constraints [Limon/Pereira/Muñoz de la Peña/Alamo/Grosso ’14, Houska/Müller ’17]
Lyapunov based EMPC

Lyapunov based EMPC combines the goals of stabilizing and economic MPC.

- Stabilize a given set \( \Omega \) (\( \Omega = \{x_s\} \) or a larger set)
- While at the same time minimizing an economic objective

The algorithmic ideas described in the next slides go back to [Heidarinejad/Liu/Christofides '12]. They rely on the knowledge of a stabilizing controller and a corresponding Lyapunov function for the system.

Lyapunov function

Let \( x_s \in X \) be an equilibrium with open neighborhood \( O \).

Let \( h : O \to U \) a controller with \( f(x, h(x)) \in O \) for all \( x \in O \).

\( W : O \to \mathbb{R} \) is a Lyapunov function with respect to \( h \) if there are \( \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty \) such that for all \( x \in O \) we have

\[
\alpha_1(|x - x_s|) \leq W(x) \leq \alpha_2(|x - x_s|)
\]

and

\[
W(f(x, h(x))) \leq W(x) - \alpha_3(|x - x_s|)
\]

Note: decrease of \( W \) ensures asymptotic stability of any level set \( \Omega := \{x \in \mathbb{R}^n \mid W(x) \leq \rho\} \), \( \rho \geq 0 \), for \( x^+ = f(x, h(x)) \).

Idea: impose decrease of \( W \) as additional constraint in the EMPC scheme, until \( \Omega \) is reached.

Lyapunov based EMPC scheme

\[
\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t))
\]

subject to

\[
x(k+1|t) = f(x(k|t), u(k|t)), \quad k = 0, \ldots, N-1
\]

\[
x(0|t) = x(t)
\]

\[
(x(k|t), u(k|t))^\top \in X \times U, \quad k = 0, \ldots, N
\]

\[
W(x(1|t)) \leq W(f(x(t), h(x(t)))) \text{ if } W(x(t)) > \rho
\]

\[
W(x(k|t)) \leq \rho, \quad k = 0, \ldots, N \text{ if } W(x(t)) \leq \rho
\]

Idea: enforce decrease of \( W \) until \( \Omega \) is reached, afterwards stay in \( \Omega \) by ensuring \( W(x(k|t)) \leq \rho \).
**Properties**

**Theorem:** The Lyapunov-based EMPC scheme has the following properties for all $x(0) \in O$ and $\tilde{\rho} = W(x(0))$

(i) The scheme is recursively feasible and $W(x(t)) \leq \max\{\rho, \tilde{\rho}\}$ for all $t \geq 0$

(ii) If $\rho > 0$ then there is $\tilde{\ell} > 0$ with $x(t) \in \Omega$ for all $t \geq \tilde{\ell}$

(iii) If $\rho = 0$ then $x(t) \to x_s$ as $t \to \infty$

**Note:** It is also possible to change $\rho$ with time (already present in the original reference [Heidarinejad/Liu/Christofides '12])

---

**Discussion**

**Discussion of Lyapunov-based EMPC**

- Theorem does not require strict dissipativity
- No performance estimates known so far, except average performance in case $\rho = 0$
- Under strict dissipativity, other performance estimates could possibly be achieved (open question!)
- Many variants available, see the monograph [Ellis/Liu/Christofides, Economic Model Predictive Control, Springer '17]
- Main bottleneck: knowledge of $W$ and $h$ required for implementation

The next EMPC variant fixes the last problem

---

**Multiobjective EMPC**

**Goal:** make the closed loop trajectory converge to $x_s$ while minimizing the economic cost

Lyapunov-based EMPC with $\rho = 0$ solves this problem

The main problem of Lyapunov-based EMPC is the required knowledge of a stabilizing controller $h$ and a corresponding Lyapunov function $W$

Multiobjective EMPC [Zavala '15] avoids this problem by computing $h$ and $W$ via stabilizing MPC with terminal conditions

In each step, two optimal control problems — one with the economic objective and one with a stabilizing objective — are solved and suitably combined

We start by explaining the stabilizing problem

---

**Multiobjective EMPC:** stabilizing subproblem

$$
\min_{u(\cdot|t)} J^{stab}(x(t), u(\cdot|t)) = \sum_{k=0}^{N-1} \ell^{stab}(x(k|t), u(k|t))
$$

subject to

$$
\begin{align*}
x(k+1|t) &= f(x(k|t), u(k|t)), & k &= 0, \ldots, N-1 \\
x(0|t) &= x(t) \\
(x(k|t), u(k|t))^T &\in X \times U, & k &= 0, \ldots, N-1 \\
x(N|t) &= x_s
\end{align*}
$$

with $\ell^{stab}(x_s, u_s) = 0$, $\ell^{stab}(x, u) > 0$ otherwise

$(x(N|t) = x_s$ could be replaced by regional constraint + terminal cost)
Lyapunov function property

Define \( V^{\text{stab}}(x(t)) = \inf_{u(\cdot|t)} J^{\text{stab}}(x(t), u(\cdot|t)) \)

Then, under standard assumptions on the stabilizing MPC scheme, there is \( \alpha_4 \in \mathcal{K}_\infty \) such that for each admissible control sequence \( \hat{u} \) the inequality

\[
V^{\text{stab}}(f(x(t), \hat{u}(0))) \leq J^{\text{stab}}(x(t), \hat{u}) - \alpha_4(|x(t) - x_s|)
\]

holds

Thus, for any \( \sigma \in (0, 1) \) there is an admissible control \( \hat{u} \) with

\[
J^{\text{stab}}(f(x(t), \hat{u}(0)), \hat{u}) \leq J^{\text{stab}}(x(t), \hat{u}) - (1 - \sigma)\alpha_4(|x(t) - x_s|)
\]

\( \leadsto J^{\text{stab}} \) can serve as a Lyapunov function constraint in the economic subproblem of the EMPC scheme

Multiobjective EMPC: economic subproblem

\[
\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t))
\]

subject to

\[
x(k+1|t) = f(x(k|t), u(k|t)), \quad k = 0, \ldots, N - 1
\]

\[
x(0|t) = x(t)
\]

\[
(x(k|t), u(k|t))^T \in \mathcal{X} \times \mathcal{U}, \quad k = 0, \ldots, N
\]

\[
J^{\text{stab}}(x(t), u(\cdot|t)) \leq (1 - \sigma) V^{\text{stab}}(x(t)) + \sigma J^{\text{stab}}(x(t-1), u^*(\cdot|t-1)), \quad t \geq 1
\]

for \( \sigma \in [0, 1) \)

\( \leadsto J^{\text{stab}}(x(t+1), u^*(\cdot|t+1)) \leq J^{\text{stab}}(x(t), u^*(\cdot|t)) - (1 - \sigma)\alpha_4(|x(t) - x_s|) \)

\( \leadsto \sigma \) determines the speed of convergence

Multiobjective EMPC: Example

We illustrate the role of \( \sigma \) by the chemical reactor without dissipativity

\[
\begin{align*}
\dot{x}_1 &= 1 - r_1(x_1, x_3) - x_1 \\
\dot{x}_2 &= r_2(x_1, x_3) - x_2 \\
\dot{x}_3 &= u - x_3
\end{align*}
\]

with

\[
\begin{align*}
r_1(x_1, x_3) &= 10^4 x_1^2 e^{-\frac{1}{x_3}} + 400 x_1 e^{-\frac{0.55}{x_3}}, \\
r_2(x_1, x_3) &= 10^4 x_1^2 e^{-\frac{1}{x_3}}
\end{align*}
\]

\( x_1 = \) concentration of source material \( R \)

\( x_2 = \) concentration of desired product \( P_1 \)

\( x_3 = \) dimensionless temperature of the mixture in the reactor

\( u \) \( \triangleq \) heat flux through the cooling jacket

Constraints: \( x_i \geq 0, i = 1, 2, 3 \) and \( u \in [0.049, 0.449] \)

Objective: maximize \( P_1 \), i.e. the integral over \( L(x, u) = -x_2 \)

Multiobjective EMPC: Example

As seen before: the optimal trajectories are not constant

\( \leadsto \) no optimal equilibrium \( \leadsto \) not strictly dissipative

Standard EMPC
Multiobjective EMPC: Example

In all simulations \( \ell_{stab}(x, u) = |x - x_s|^2 + |u - u_s|^2 \)

Timm Faulwasser, Lars Grüne, and Matthias Müller, Economic Model Predictive Control, p. IV.21

Multiobjective EMPC: Example

Theorem: Consider the Multiobjective EMPC scheme under the usual stability assumptions for MPC with terminal constraints. Then for all \( x(0) \in X \) the EMPC closed loop solution \( x(t) \) converges to \( x_s \) as \( t \to \infty \)

Idea of proof: The constraints enforce the inequality

\[
J_{stab}(x(t+1), u^*(\cdot | t + 1)) \leq J_{stab}(x(t), u^*(\cdot | t)) - (1 - \sigma)\alpha_4(|x(t) - x_s|)
\]

yielding \( J_{stab}(x(t), u^*(\cdot | t)) \to 0 \) as \( t \to \infty \) and thus \( x(t) \to x_s \)

Note: Asymptotic stability may not hold! This is due to the fact that there is no upper bound on \( J_{stab}(x(0), u(\cdot | 0)) \). Thus, the open loop optimal trajectory may move far away from \( x^* \) for \( x(0) \approx x_s \); in fact even for \( x(0) = x_s \)

In all simulations \( \ell_{stab}(x, u) = |x - x_s|^2 + |u - u_s|^2 \)

Timm Faulwasser, Lars Grüne, and Matthias Müller, Economic Model Predictive Control, p. IV.22
Multiobjective EMPC: Discussion

Discussion of Multiobjective MPC
- Theorem does not require strict dissipativity
- Average performance guaranteed, but no transient performance estimates known
- Under convexity assumptions, the (finite horizon) solution can be interpreted as a Pareto optimum
- Main drawback: two optimization problems need to be solved in each time step

Summary

We compare the EMPC-variants discussed so far with respect to the following characteristics
- Asymptotic stability
- Average performance
- Transient performance

as well as
- Assumptions on the problem
- Ingredients of the algorithm (functions, sets), other than system dynamics $f$ and stage cost $\ell$

### EMPC with | without terminal conditions

<table>
<thead>
<tr>
<th>Characteristics</th>
<th>EMPC with</th>
<th>EMPC without</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymptotic stability</td>
<td>yes</td>
<td>yes (practical)</td>
</tr>
<tr>
<td>Average performance</td>
<td>yes</td>
<td>yes (with error term)</td>
</tr>
<tr>
<td>Transient performance</td>
<td>yes</td>
<td>yes (with $T$-dep. error)</td>
</tr>
<tr>
<td>Assumptions on the problem</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- optimal operation at steady state</td>
<td>strict dissipativity (for average performance)</td>
<td></td>
</tr>
<tr>
<td>- strict dissipativity (for asymptotic stability and transient performance)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ingredients of the algorithm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- optimal steady state</td>
<td>none</td>
<td></td>
</tr>
<tr>
<td>- terminal constraint set and cost</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Remarks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>- potentially small feasible set</td>
<td>recursive feasibility only for suff. large $N$</td>
<td></td>
</tr>
</tbody>
</table>
Lyapunov-based Multiobjective EMPC

- Asymptotic stability: yes | only convergence
- Average performance: yes | yes
- Transient performance: unknown | unknown

Assumptions on the problem
- optimal operation at steady state | optimal operation
- Ingredients of the algorithm
- optimal steady state | optimal steady state
- controller with Lyapunov function | terminal constraint set and cost

Remarks
- requires knowledge of Lyapunov function | requires solution of two Lyapunov function optimal control problems

Remarks and Conclusion
- All considered schemes guarantee (approximate) averaged optimality under mild conditions on the problem structure
- In the absence of an optimal steady state, the advantage of EMPC over stabilizing MPC lies in its ability to find better solutions than the equilibrium (e.g., periodic ones)
- In the presence of an optimal steady state, average optimality is a rather weak optimality concept, which is moreover also satisfied by stabilizing MPC
- In this case, the advantage of EMPC lies in the transient performance. This has been confirmed in many simulations, but rigorously proved only for basic schemes
- So far, rigorous transient performance estimates have only been achieved under strict dissipativity. Is this property really necessary...?

Literature for Part IV


Fagiano, L. and A. R. Teel, 2013, Generalized terminal state constraint for model predictive control, Automatica 49(9), 2622–2631


V. Advanced topics and open problems

In this section we discuss a selection of schemes which go beyond the previous setting. Particularly, we consider discounted optimal control problems and problems which do not exhibit an optimal equilibrium.

Outline:
- Discounted optimal control problems
- Optimal control problems with periodic optimal solutions
- Time-varying optimal control problems
- Uncertain Systems (Matthias)

Discounted optimal control problems

Discounted optimal control problems are of the form
\[
\min_{u \in \mathcal{U}} \sum_{k=0}^{N-1} \beta^k \ell(x(k), u(k))
\]
with \(N \in \mathbb{N}\) or \(N = \infty\), with discount factor \(\beta \in (0, 1)\).

For discounted optimal control, the averaged optimality does not make sense, because for bounded \(\ell\)
\[
\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \beta^k \ell(x(k), u(k)) = 0
\]
\(\Rightarrow\) transient optimality is of interest

Transient performance theorem

Consider discounted EMPC without terminal conditions

Theorem [Grüne/Semmler/Stieler ’15] If the discounted optimal control problem has the turnpike property and the optimal value function is continuous at \(x_s\) uniformly in \(\beta\), then there is \(\delta \in \mathcal{L}\) with
\[
J_{\infty}^{\delta}(x_0, \mu_N) \leq V_{\infty}(x_0) + \frac{\delta(N)}{1 - \beta}
\]

Note: The \(\beta\)-dependence of the error term is the counterpart of the \(T\)-dependence in the non-discounted case.

It is unknown whether this result also holds (or even improves) with suitable terminal conditions.
Relation to dissipativity

Dissipativity concepts have been developed for discounted problems as well [Grüne/Kellett/Weller '16, Grüne/Müller CDC '17]

The discounted strict dissipativity inequality reads

\[ \beta \lambda (f(x,u)) \leq \lambda (x) + \ell (x,u) - \ell (x_s, u_s) - \alpha (\| x - x_s \|) \]

But: In general, discounted strict dissipativity only implies the turnpike property for \( \beta \approx 1 \) [Gaitsgory/Grüne/Höger/Kellett/Weller '17]

Discounted problems: example

We consider a classical economic growth model [Brock/Mirman '72]

\[ x(t + 1) = u(t), \quad \ell(x,u) = - \ln(Ax^\alpha - u) \]

Trajectories for \( A = 5, \alpha = 0.34, x_0 = 5, \beta = 0.95 \)

Problems with time varying optimal operation

Our final two schemes concern problems without optimal operation at steady states

Instead, the system is optimally operated at periodic or more general time varying solutions

Here we distinguish two cases:

- Periodic optimal solutions generated by time invariant dynamics \( f \) and cost \( \ell \)
- Time varying (possibly periodic) solutions generated by time varying dynamics \( f \) and/or cost \( \ell \)

We start with the first situation
Periodic optimal trajectories

We first consider a simple example showing that periodic trajectories may be optimal even if $f$ and $\ell$ are time invariant.

We choose $X = U = \{-1, 0, 1\}$ and dynamics and cost indicated in the following figure:

\[
\begin{align*}
\ell(x,u) &= 1 & u = -1 \\
\ell(x,u) &= 0 & u = 0 \\
\ell(x,u) &= 1 & u = 1 \\
\ell(x,u) &= 1 - \varepsilon & u = 0
\end{align*}
\]

The average cost of the steady state $x = -1$ is 1.
The average cost of the periodic orbit $(0, 1, 0, 1, 0, 1, \ldots)$ is $1 - \varepsilon$.

Will MPC “find” this orbit when starting in $x = -1$?

EMPC and periodic orbits

We start in $x = -1$.
If the horizon $N$ is odd, the trajectory

\[(-1, -1, 0, 1, 0, 1, \ldots, 0, 1)\]

is optimal $\leadsto$ the closed loop system will stay in $-1$ forever.

Conclusion: MPC does not necessarily find optimal periodic orbits, even if $N$ is arbitrarily large.

Remedy: In order to find an optimal $p$-periodic orbit $(\hat{x}_0, \ldots, \hat{x}_{p-1})$, EMPC can be modified in two ways:

- impose periodic terminal constraints, e.g., $x(t|N) = \hat{x}_{tp}$ with $tp = t \mod p$ (regional constraints also possible) [Angeli/Amrit/Rawlings ’09ff, Zanon/Grüne/Diehl ’17]
- use the periodic optimization horizon $N_t = N - tp$ [Müller/Grüne ’16]

Note: The second approach without terminal conditions needs no information about the periodic orbit except its period, but — similar to the steady state case — yields weaker results.

Periodic strict dissipativity

The formal results rely on a periodic variant of strict dissipativity

\[
\lambda_{k+1}(f(x,u)) \leq \lambda_k(x) + \ell(x,u) - \ell(\hat{x}_k, \hat{u}_k) - \sigma(x,u)
\]

for $k = 0, \ldots, p - 1$, where $\lambda_p = \lambda_0$.

or on a strict dissipativity condition for the stacked system

\[
x^p = \begin{bmatrix} x_0 \\ \vdots \\ x_{p-1} \end{bmatrix}, \quad u^p = \begin{bmatrix} u_0 \\ \vdots \\ u_{p-1} \end{bmatrix}, \quad f^p(x^p, u^p) := \begin{bmatrix} f(x_{p-1}, u_0) \\ f(x_{p-2}, u_1) \\ \vdots \end{bmatrix}
\]

(The relation between these two conditions is still waiting to be explored.)
Properties of periodic EMPC scheme

**Theorem:** (a) Under the periodic strict dissipativity condition and suitable technical conditions (continuity), the optimal periodic orbit is asymptotically stable for the EMPC scheme with periodic terminal constraints and averaged optimality holds.

(The precise asymptotic stability property in (a) depends on the form of the function $\sigma$ in the periodic strict dissipativity condition)

(b) Under the stacked strict dissipativity condition and suitable technical conditions (continuity), the closed loop of the EMPC scheme with periodic optimization horizon converges to the optimal periodic orbit and approximate averaged optimality holds.

(in (b), asymptotic stability does not hold in general. This is due to a strange feature of the periodic turnpike property)

EMPC for time varying problems

Consider a problem with time varying dynamics and stage cost

$$x(k+1) = f(k, x(k), u(k)), \quad \ell(k; x, u)$$

Obviously, the extension of the EMPC scheme is straightforward, at least without terminal conditions.

However, carrying over the previous results is nontrivial:

- what is the time varying counterpart of the optimal equilibrium / periodic orbit?
- which kind of approximate infinite horizon optimal performance can be expected?

We start by studying a simple example.

Example problem

Prototype problem: Keep the temperature in a room in a desired range with minimal energy consumption for heating and cooling.

Very simple 1d model:

$$x(n+1) = x(n) + u(n) + w(n)$$

inside temperature heating/cooling outside temperature

with stage cost

$$\ell(x, u) = u^2$$

and time varying $w(n)$ and desired temperature range $\mathcal{X}(n)$

Example: optimal trajectory
Optimality concept
In which infinite horizon sense can we expect that this
trajectory is (near) optimal? Clearly,

\[ \text{“minimize” } J_{\infty}(x, u) = \sum_{n=0}^{\infty} \ell(x_u(n), u(n)) \]
is not meaningful, because the sum will not converge

Remedy: Overtaking Optimality [Gale ’67]
A trajectory \( x^* \) with control \( u^* \) is called overtaking optimal if

\[
\limsup_{K \to \infty} \left( \sum_{n=0}^{K-1} \ell(n, x_u(n), u(n)) - \sum_{n=0}^{K-1} \ell(n, x_u(n), u(n)) \right) \leq 0
\]

holds for all admissible trajectory-control pairs \((x_u, u)\) with
\( x_u(0) = x^*(0) \)
A generalized optimal equilibrium

Obviously, the closed loop trajectories converge to the black limit trajectory. How can we characterize it?

Idea: generalize the definition of optimal operation at a steady state to overtaking optimality:

We say that the system is optimally operated at a trajectory \( \hat{x} \) with control \( \hat{u} \) if

\[
\limsup_{T \to \infty} \left( \sum_{n=0}^{T-1} \ell(n, \hat{x}(n), \hat{u}(n)) - \sum_{n=0}^{T-1} \ell(n, x_u(n), u(n)) \right) \leq 0
\]

holds for all admissible trajectory-control pairs \((x_u, u)\)

Note: this is similar to the definition of overtaking optimality, but now \( x_u(0) \neq \hat{x}(0) \) is allowed

Discussion of Main Result

- the time varying turnpike property can be ensured by a time varying strict dissipativity property
- this strict dissipativity property, in turn, always holds under suitable convexity assumptions (like in the steady state case, but more technical)
- the continuity property can be ensured by a controllability assumption (also in the periodic results before)
- probably the most important feature of the time varying case: in the steady state and in the periodic case, the optimal limit trajectories can be computed beforehand
In the time varying case there is in general no easy way for this
Hence, the fact that EMPC finds this trajectory “automatically” is of utmost importance

Main Result

Theorem: [Grüne/Pirkelmann CDC ’17] Assume that a time varying turnpike property and a continuity property hold. Then there exists an error term \( \delta(N) \to 0 \) as \( N \to \infty \) with

\[
\limsup_{T \to \infty} \left( \sum_{n=0}^{T-1} \ell(n, x_u(N), \mu_N(x_{\mu_N}(n))) - \sum_{n=0}^{T-1} \ell(n, x_u(n), u(n)) - T\delta(N) \right) \leq 0
\]

for all admissible \((x_u, u)\) with \( x_u(0) = x_{\mu_N}(0) \)

In other words: the MPC closed loop trajectory on \( \{0, \ldots, T\} \) is the initial piece of an overtaking optimal trajectory — up to the error \( T\delta(N) \)

Note: The factor “\( T \)” in the error term usually vanishes when looking at the relative error

Literature for Part V


Literature for Part V (cont.)


Economic model predictive control: state of the art and open problems

Timm Faulwasser, Lars Grüne, Matthias A. Müller

Pre-conference workshop CDC 2017

Economic MPC for uncertain systems

- System subject to disturbances/uncertainties: \( x(t+1) = f(x(t), u(t), w(t)) \)
- Motivating example:

\[
\begin{align*}
\text{Cost} \\
0 & \quad 0 \\
\end{align*}
\]

\[
\begin{align*}
x & \quad 0 \\
\end{align*}
\]

Conclusion: just transferring robust stabilizing MPC approaches to economic setting might result in bad performance!
Economic MPC for uncertain systems

Nominal System

\[ z(t+1) = f(z(t), v(t), 0) \]

Error

\[ e(t) = x(t) - z(t) \]

Robust control invariant (RCI) set

\[ e(t) \in \Omega \Rightarrow e(t+1) \in \Omega, \quad \forall w(t) \in W \]

Input parametrization

Use parametrization for the real input

\[ u(t) = \varphi(v(t), x(t), z(t)) \]

(\(v(t)\) input to the nominal system) to determine RCI set \(\Omega\)

Idea:

- Take all possible states within invariant set into account
- Two different approaches:

Idea min-max robust economic MPC

- Consider the worst case within the RCI set
- Use modified stage cost function

\[ \ell_{\text{max}}(z, v) = \max_{\omega \in \Omega} \ell(z + \omega, \varphi(v, z + \omega, z)) \]

Main features:

- All possible real states considered \(x(k|t) \in \lbrace z(k|t) \rbrace \oplus \Omega\)
- Take real input into account \(u(k|t) = \varphi(v(k|t), x(k|t), z(k|t))\)

\[ \Rightarrow \text{Cost of input to stay in RCI set} \]
Economic MPC for uncertain systems

Optimization problem

\[
\min_{z(0), v(t)} \sum_{k=0}^{N-1} \ell^{\text{max}}(z(k|t), v(k|t)) + \nabla_t(z(N|t))
\]

s.t. \( z(k + 1|t) = f(z(k|t), v(k|t), 0) \),
\( x(t) \in \{z(0|t)\} \oplus \Omega \),
\( (z(k|t), v(k|t))^\top \in \mathbb{X} \times \mathbb{U} \), \( k = 0, \ldots, N - 1 \),
\( z(N|t) \in \mathbb{X}_f \)

- Nominal dynamics only
- Free nominal initial state
- Suitably tightened constraint sets \( \mathbb{X}, \mathbb{U}, \mathbb{X}_f \)

Theorem [Bayer, Müller, Allgöwer ‘16]
Under standard assumptions (terminal region/cost, suitable constraint tightening) and given initial feasibility, we have
- recursive feasibility,
- closed-loop constraint satisfaction,
- infinite horizon averaged performance

\[
\limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell(x(t), u(t)) \leq \ell^{\text{max}}(z^m, v^m).
\]

- Optimal steady-state: \( (z^m, v^m) = \arg \min_{z=f(z, v, 0), (z, v) \in \mathbb{Z}} \ell^{\text{max}}(z, v) \)
- Performance result for the real closed-loop system
- Bound usually quite conservative

Idea averaging-based robust economic MPC
- Instead of worst case, consider average over RCI set \( \Omega \)
- Use modified stage cost function

\[
e^{\text{avg}}(z, v) = \int_{\Omega} \ell(z + \omega, \varphi(v, z + \omega, z)) d\omega
\]
Economic MPC for uncertain systems

**Theorem** [Bayer, Müller, Allgöwer '14]

Under standard assumptions (terminal region/cost, suitable constraint tightening) and given initial feasibility, we have

- recursive feasibility,
- closed-loop constraint satisfaction,
- infinite horizon averaged performance

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell^\text{int}(z^*(0|t), v^*(0|t)) \leq \ell^\text{int}(z^*_s, v^*_s).$$

- Optimal steady state: $$(z^*_s, v^*_s) = \arg\min_{z = f(z, v), (z, v) \in \mathbb{Z}} \ell^\text{int}(z, v)$$

- Interpretation: Average performance result for the real closed loop, averaged over all possible disturbances

**Conclusions**

- Just transferring approaches from robust stabilizing MPC is **not enough**
- Different approaches to incorporate disturbances in economic MPC

Min-max approach
- Accounts for worst case
- Typically quite conservative

Averaging approach
- Usually better than min-max
- Poor approx. of real distribution

Stochastic approach
- Real distribution
- More complex
- Results only available for linear case
Conclusions

- Just transferring approaches from robust stabilizing MPC is **not enough**
- Different approaches to incorporate disturbances in economic MPC
- Guaranteed **average performance bounds** for all approaches
- The more information taken into account, the better the performance
- Picture much less complete than in nominal case: transient performance, using no terminal constraints, classification of optimal operating conditions, etc.

Summary and wrap up

- Economic MPC: model predictive control using general performance criterion
- Various different EMPC schemes available with different advantages and disadvantages
- Basic case of optimal steady-state operation by now fairly well understood, closed-loop performance and convergence guarantees available
- Extensions to various settings (periodic optimal behavior, discounted problems, time-varying problems, uncertain systems, . . .), but still many open questions

Further information


Literature for Part VI


Thanks for your attention! Questions?