Nonlinear Model Predictive Control

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Contents

Part A: Stabilizing Model Predictive Control

Contents

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(1) Introduction

What is Model Predictive Control (MPC)?

Setup

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \;\; x_{\mathbf{u}}(0) = x_0$$

or, briefly

$$x^+ = f(x, u)$$

with $x \in X$, $u \in U$

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Prototype Problem

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$$x_\mu(n+1) = f(x_\mu(n), \mu(x_\mu(n))), \;\; x_\mu(0) = x_0$$

Additionally, we impose state constraints $x_\mu(n) \in \mathbb{X}$
and control constraints $\mu(x(n)) \in \mathbb{U}$

for all $n \in \mathbb{N}$ and given sets $\mathbb{X} \subseteq X$, $\mathbb{U} \subseteq U$

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Idea of MPC: use an optimal control problem which minimizes the distance to x_* in order to synthesize a feedback law μ

The idea of MPC

For defining the MPC scheme, we choose a stage cost $\ell(x, u)$ penalizing the distance from x_* and the control effort, e.g., $\ell(x, u) = \|x - x_*\|^2 + \lambda \|u\|^2$ for $\lambda \geq 0$

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- minimize the summed stage cost along trajectories generated from our model over a prediction horizon N
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Notation in what follows:

- general feedback laws will be denoted by μ
- the MPC feedback law will be denoted by μ_N

The basic MPC scheme

Formal description of the basic MPC scheme:

At each time instant n solve for the current state $x_{\mu_N}(n)$

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_{\mu_N}(n),\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k),\mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_{\mu_N}(n)$$

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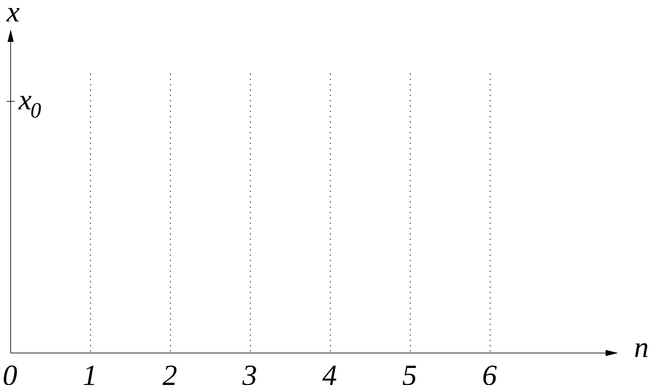

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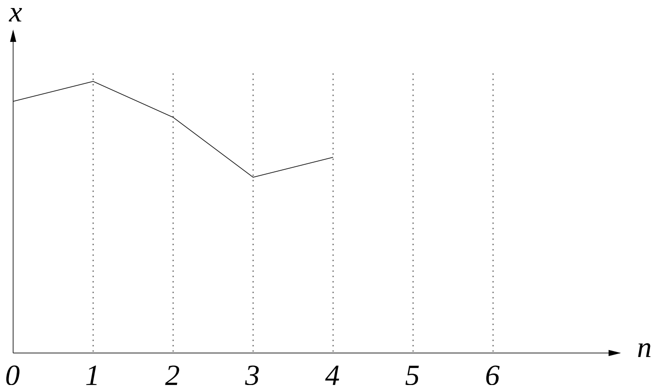

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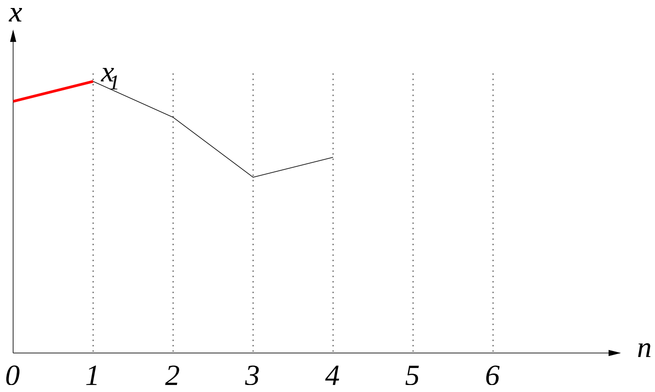

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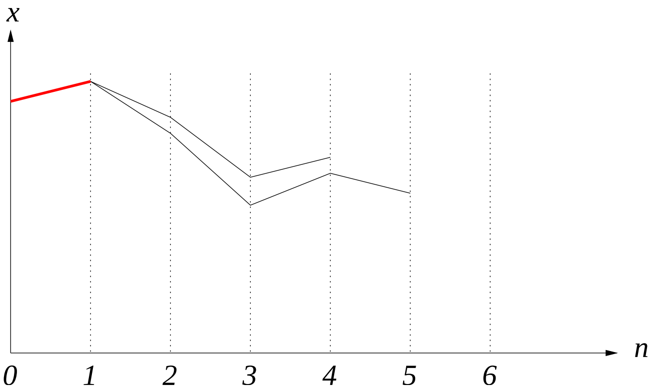

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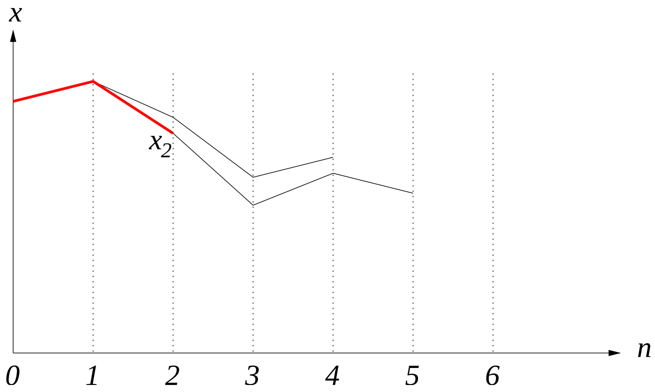

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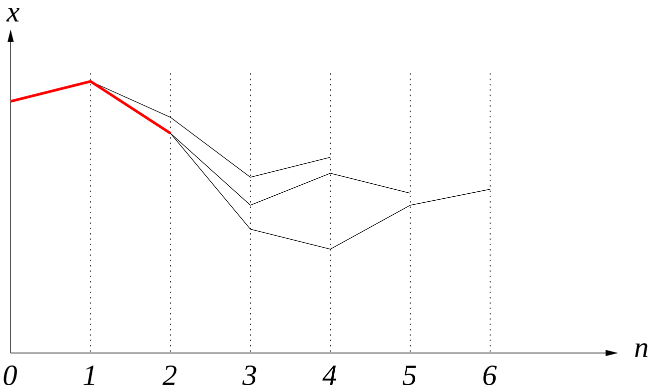

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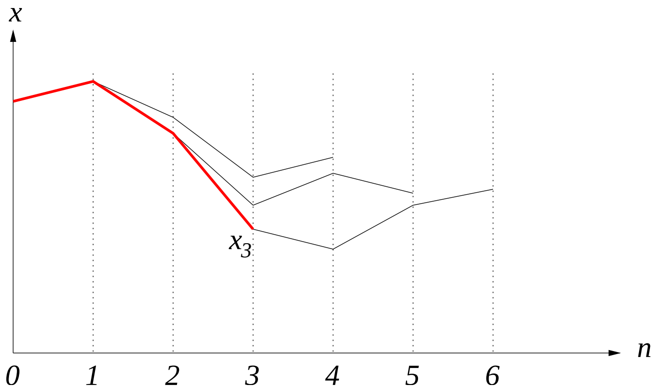

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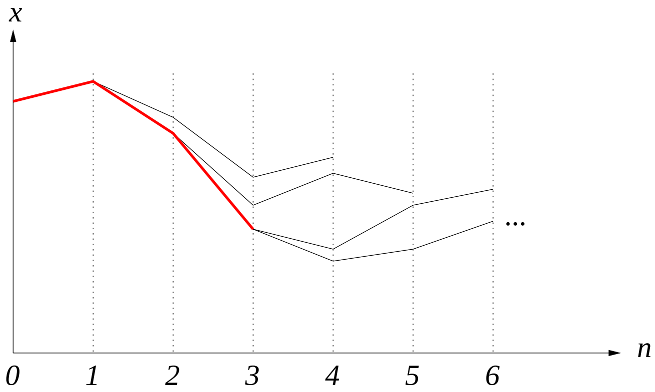

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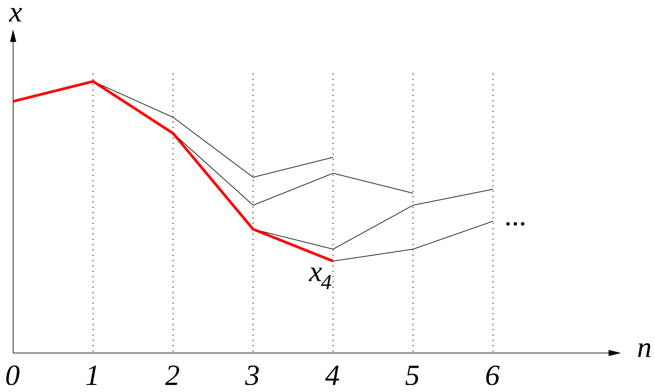

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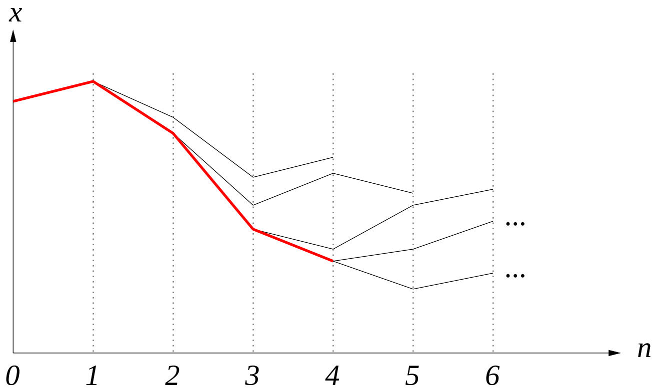

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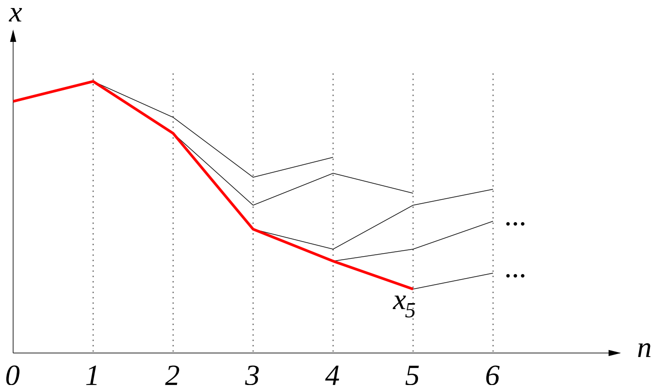

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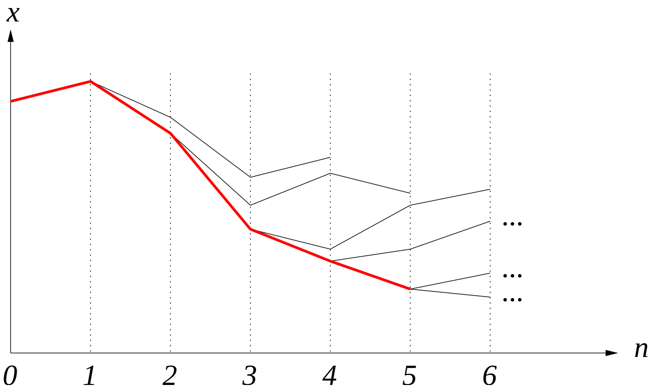

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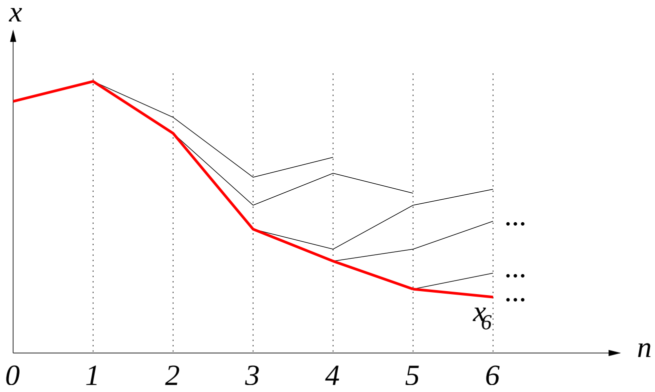

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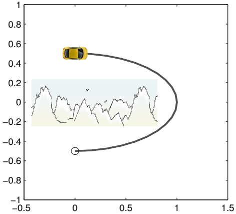
and, of course, the development of good algorithms (not topic of this course)

An example



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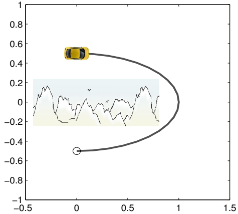

$$x_1^+ = \sin(\varphi + u)$$
$$x_2^+ = \cos(\varphi + u)/2$$

with $\varphi = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$

$\mathbb{X} = \{x \in \mathbb{R}^2 \, : \, \|(x_1, 2x_2)^T\| = 1\}$, $\mathbb{U} = [0, u_{\max}]$

$x_* = (0, -1/2)^T$, $x_0 = (0, 1/2)^T$

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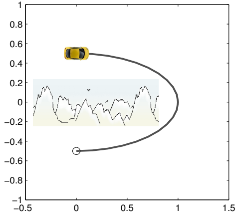

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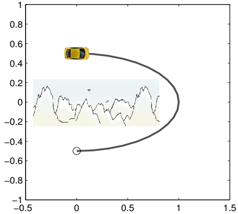

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MPC with $\ell(x,u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$ yields asymptotic stability for $N = 11$ but not for $N \leq 10$

Summary of Section (1)

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- MPC computes the feedback law by iteratively solving finite horizon optimal control problems using the current state $x_0 = x_{\mu_N}(n)$ as initial value
- the feedback value $\mu_N(x_0)$ is the first element of the resulting optimal control sequence
- the example shows that MPC does not always yield an asymptotically stabilizing feedback law

(2a) Background material:

Lyapunov functions

Purpose of this section

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In the subsequent sections, this will be used in order to establish asymptotic stability of the MPC closed loop

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Note: we do not require g to be continuous

Comparison functions

For $\mathbb{R}_0^+ = [0, \infty)$ we use the following classes of comparison functions

$$\mathcal{K} := \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{l} \alpha \text{ is continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array} \right\}$$

$$\mathcal{K}_\infty := \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \alpha \in \mathcal{K} \text{ and } \alpha \text{ is unbounded} \right\}$$

$$\mathcal{KL} := \left\{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{l} \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \in \mathbb{R}_0^+ \\ \text{and } \beta(r, \cdot) \text{ is strictly de-} \\ \text{creasing to } 0 \text{ for all } r \in \mathbb{R}_0^+ \end{array} \right\}$$

Asymptotic stability revisited

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We say that x_* is asymptotically stable for $x^+ = g(x)$ on a forward invariant set Y if there exists $\beta \in \mathcal{KL}$ such that

$$\|x(n) - x_*\| \leq \beta(\|x(0) - x_*\|, n)$$

holds for all $x \in Y$ and $n \in \mathbb{N}$

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How can we check whether this property holds?

Lyapunov function

Let $Y \subseteq X$ be a forward invariant set and $x_* \in X$. A function $V : Y \to \mathbb{R}_0^+$ is called a Lyapunov function for $x^+ = g(x)$ if the following two conditions hold for all $x \in Y$:

(i) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$$

(ii) There exists $\alpha_V \in \mathcal{K}$ such that

$$V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$$

Stability theorem

Theorem: If the system $x^+ = g(x)$ admits a Lyapunov function V on a forward invariant set Y, then x_* is an asymptotically stable equilibrium on Y

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The bounds $\alpha_1(\|x - x_*\|) \le V(x) \le \alpha_2(\|x - x_*\|)$ imply that asymptotic stability holds with $\beta(r,t) = \alpha_1^{-1}(\tilde{\beta}(\alpha_2(r), t))$

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Lyapunov functions — discussion

While the convergence $x(n) \to x_*$ is typically non-monotone for an asymptotically stable system, the convergence $V(x(n)) \to 0$ is strictly monotone

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But it is in general difficult to find a candidate for a Lyapunov function

For MPC, we will use the optimal value functions which we introduce in the next section

(2b) Background material:

Dynamic Programming

Purpose of this section

We define the optimal value functions V_N for the optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \;\; J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \;\; x_{\mathbf{u}}(0) = x_0$$

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We present the dynamic programming principle, which establishes a relation for these functions and will eventually enable us to derive conditions under which V_N is a Lyapunov function

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An admissible control sequence $\mathbf{u}^\star$ is called optimal, if

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Note: an optimal $\mathbf{u}^\star$ does not need to exist in general. In the sequel we assume that $\mathbf{u}^\star$ exists if x_0 is feasible

Dynamic Programming Principle

Theorem: (Dynamic Programming Principle) For any feasible $x_0 \in \mathbb{X}$ the optimal value function satisfies

$$V_N(x_0) = \inf_{u \in \mathbb{U}} \{\ell(x_0, u) + V_{N-1}(f(x_0, u))\}$$

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Moreover, if $\mathbf{u}^\star$ is an optimal control, then

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holds.

Idea of Proof: Follows by taking infima in the identity

$$\begin{aligned} J_N(x_0, \mathbf{u}) &= \ell(x_\mathbf{u}(0), \mathbf{u}(0)) + \sum_{k=1}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)) \\ &= \ell(x_0, \mathbf{u}(0)) + J_{N-1}(f(x_0, \mathbf{u}(0)), \mathbf{u}(\cdot + 1)) \end{aligned}$$

Corollaries

Corollary: Let $x^\star$ be an optimal trajectory of length N with optimal control $u^\star$ and $x^\star(0) = x$.

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(ii) The MPC feedback μ_N satisfies

$$\mu_N(x) = \operatorname*{argmin}_{u \in \mathbb{U}} \{\ell(x,u) + V_{N-1}(f(x,u))\}$$

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and

$$u^\star(k) = \mu_{N-k}(x^\star(k)), \;\; k = 0, \ldots, N-1$$

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In order to see why this can work, in the next section we briefly look at infinite horizon optimal control problems

Moreover, for simple systems the principle can be used for computing V_N and μ_N — we will see an example in the excercises

(2c) Background material:

Relaxed Dynamic Programming

Infinite horizon optimal control

Just like the finite horizon problem we can define the infinite horizon optimal control problem

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

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If we could compute an optimal feedback μ_∞ for this problem (which is — in contrast to computing μ_N — in general a very difficult problem), we would have solved the stabilization problem

Infinite horizon dynamic programming principle

Recall the corollary from the finite horizon dynamic programming principle

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Relaxing dynamic programming

Unfortunately, an equation of the type

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$$\rightsquigarrow \quad V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

"relaxed dynamic programming inequality" [Rantzer et al. '06ff]

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What can we conclude from this inequality?

Relaxed dynamic programming

We define the infinite horizon performance of the MPC closed loop system $x^+ = f(x, \mu_N(x))$ as

$$J_\infty^{cl}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \;\; x_{\mu_N}(0) = x_0$$

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Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a forward invariant set for the MPC closed loop and assume that

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

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holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

Then for all $x \in Y$ the infinite horizon performance satisfies

$$J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$$

Relaxed dynamic programming

Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the inequalities

$$V_N(x) \le \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x,u) \ge \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N.

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hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N.

Proof: The assumed inequalities immediately imply that $V = V_N$ is a Lyapunov function for $x^+ = g(x) = f(x, \mu_N(x))$ with

$$\alpha_1(r) = \alpha_3(r), \quad \alpha_V(r) = \alpha\, \alpha_3(r)$$

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Relaxed dynamic programming

Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the inequalities

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x,u) \geq \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N.

Proof: The assumed inequalities immediately imply that $V = V_N$ is a Lyapunov function for $x^+ = g(x) = f(x, \mu_N(x))$ with

$$\alpha_1(r) = \alpha_3(r), \quad \alpha_V(r) = \alpha\, \alpha_3(r)$$

$\Rightarrow$ asymptotic stability

Relaxed dynamic programming

For proving the performance estimate $J^{cl}_\infty(x_0, \mu_N) \le V_N(x_0)/\alpha$, the relaxed dynamic programming inequality implies

$$\begin{aligned} &\alpha \sum_{n=0}^{K-1} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \\ \le\ & \sum_{n=0}^{K-1} \Big(V_N(x_{\mu_N}(n)) - V_N(x_{\mu_N}(n+1)) \Big) \\ =\ & V_N(x_{\mu_N}(0)) - V_N(x_{\mu_N}(K)) \ \le\ V_N(x_{\mu_N}(0)) \end{aligned}$$

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\end{aligned}
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Since all summands are ≥ 0, this implies that the limit for $K \to \infty$ exists and we get

$$\alpha J_\infty^{cl}(x_0, \mu_N) = \alpha \sum_{n=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \le V_N(x_{\mu_N}(0))$$

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$\Rightarrow$ assertion

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- The performance of the MPC controller can be measured by looking at the infinite horizon value along the MPC closed loop trajectories
- Relaxed dynamic programming gives us conditions under which both asymptotic stability and performance results can be derived

Application of background results

The main task will be to verify the assumptions of the relaxed dynamic programming theorem, i.e.,

$$V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

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To this end, we present two different approaches:

- modify the optimal control problem in the MPC loop by adding terminal constraints and costs
- derive assumptions on f and ℓ under which MPC works without terminal constraints and costs

(3) Stability with stabilizing constraints

V_N as a Lyapunov Function

Problem: Prove that the MPC feedback law μ_N is stabilizing

V_N as a Lyapunov Function

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$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

of the relaxed dynamic programming theorem for the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \;\; x_{\mathbf{u}}(0) = x_0$$

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(where “small” means that the error can be compensated replacing $\ell(x, \mu_N(x))$ by $\alpha\ell(x, \mu_N(x))$ with $\alpha \in (0, 1)$)

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$\rightsquigarrow$ additional stabilizing constraints were proposed

(3a) Equilibrium terminal constraint

Equilibrium terminal constraint

Optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

Equilibrium terminal constraint

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Assumption: $f(x_*, 0) = x_*$ and $\ell(x_*, 0) = 0$

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Idea: add equilibrium terminal constraint

$$x_{\mathbf{u}}(N) = x_*$$

[Keerthi/Gilbert '88, . . .]

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[Keerthi/Gilbert '88, ...]

$\rightsquigarrow$ we now solve

$$\underset{\mathbf{u}\in\mathbb{U}^N_{x_*}(x_0)}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \quad x_\mathbf{u}(0) = x_0$$

with $\mathbb{U}^N_{x_*}(x_0) := \{\mathbf{u} \in \mathbb{U}^N$ admissible and $x_\mathbf{u}(N) = x_*\}$

Prolongation of control sequences

Let $\tilde{\mathbf{u}} \in \mathbb{U}^{N-1}_{x_*}(x_0)$

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Moreover, since

$$\ell(x_{\mathbf{u}_N}(N-1), \mathbf{u}_N(N-1)) = \ell(x_*, 0) = 0,$$

the prolongation has zero stage cost

Reversal of $V_{N-1} \leq V_N$

Now, let $\tilde{\mathbf{u}}^\star \in \mathbb{U}^{N-1}_{x_*}(x_0)$ be the optimal control for J_{N-1}, i.e.,

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Note: $V_{N-1} \leq V_N$ does no longer hold now

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$$= \sum_{n=0}^{N-2} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + \underbrace{\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1))}_{=0}$$

$$= \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) = J_N(x_0, \mathbf{u}) \geq V_N(x_0)$$

$\rightsquigarrow$ The inequality $V_{N-1} \leq V_N$ is reversed to $V_{N-1} \geq V_N$

Note: $V_{N-1} \leq V_N$ does no longer hold now

But: the dynamic programming principle remains valid

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Relaxed dynamic programming inequality

From the reversed inequality

$$V_{N-1}(x) \geq V_N(x)$$

and the dynamic programming principle

$$V_N(x) \geq \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

we immediately get

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$\rightsquigarrow$ stability follows if we can ensure the additional inequalities

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

Feasible sets

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$\rightsquigarrow$ the feasible set $\mathbb{X}_N$ is the “natural” operating region of MPC with equilbrium terminal constraints

Stability theorem

Theorem: Consider the MPC scheme with equilibrium terminal constraint $x_{\mathbf{u}}(N) = x_*$ where x_* satisfies $f(x_*, 0) = x_*$ and $\ell(x_*, 0) = 0$.

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Note: The constraint $x_{\mathbf{u}}(N) = x_*$ does not imply $x_{\mu_N}(N) = x_*$

Stability theorem — sketch of proof

Sketch of proof: All assertions follow from the relaxed dynamic programming theorem if we prove forward invariance of $\mathbb{X}_N$ for the MPC closed loop system $x^+ = f(x, \mu_N(x))$

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Equilibrium terminal constraint — Discussion

The additional condition

$$x(N) = x_*$$

ensures asymptotic stability in a rigorously provable way

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- if we want a large feasible set $\mathbb{X}_N$ we typically need a large optimization horizon N
  (see the car-and-mountains example)
- system needs to be controllable to x_* in finite time
- not very often used in industrial practice

(3b) Regional terminal constraint and terminal cost

Regional constraint and terminal cost

Optimal control problem

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \; J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \;\; x_{\mathbf{u}}(0) = x_0$$

Regional constraint and terminal cost

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Idea: add local Lyapunov function $F : \mathbb{X}_0 \to \mathbb{R}_0^+$ as terminal cost

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N))$$

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F is defined on a region $\mathbb{X}_0$ around x_* which is imposed as terminal constraint $x(N) \in \mathbb{X}_0$

[Chen & Allgöwer '98, Jadbabaie et al. '98 . . .]

Regional constraint and terminal cost

We thus change the optimal control problem to

$$\underset{\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)}{\text{minimize}} \;\; J_N(x_0,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k),\mathbf{u}(k)) \; + \; F(x_{\mathbf{u}}(N))$$

with

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Which properties do we need for F and $\mathbb{X}_0$ in order to make this work?

Regional constraint and terminal cost

Assumptions on $F : \mathbb{X}_0 \to \mathbb{R}_0^+$ and $\mathbb{X}_0$

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There exists a controller $\kappa : \mathbb{X}_0 \to \mathbb{U}$ with the following properties:

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There exists a controller $\kappa : \mathbb{X}_0 \to \mathbb{U}$ with the following properties:

(i) $\mathbb{X}_0$ is forward invariant for $x^+ = f(x, \kappa(x))$:
for each $x \in \mathbb{X}_0$ we have $f(x, \kappa(x)) \in \mathbb{X}_0$

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(ii) F is a Lyapunov function for $x^+ = f(x, \kappa(x))$ on $\mathbb{X}_0$ which is compatible with the stage cost ℓ in the following sense:
for each $x \in \mathbb{X}_0$ the inequality

$$F(f(x, \kappa(x))) \le F(x) - \ell(x, \kappa(x))$$

holds

Prolongation of control sequences

Let $\tilde{\mathbf{u}} \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$

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$\rightsquigarrow$ every $\tilde{\mathbf{u}} \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ can be prolonged to an $\mathbf{u} \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$

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By (ii) the stage cost of the prolongation is bounded by

$$\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1)) \leq F(x_{\mathbf{u}}(N-1)) - F(x_{\mathbf{u}}(N))$$

Reversal of $V_{N-1} \leq V_N$

Let $\tilde{\mathbf{u}}^\star \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ be the optimal control for J_{N-1}, i.e.,

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Reversal of $V_{N-1} \leq V_N$

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$\rightsquigarrow$ again we get $V_{N-1} \geq V_N$

Feasible sets

Define the feasible set

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Stability theorem

Theorem: Consider the MPC scheme with regional terminal constraint $x_{\mathbf{u}}(N) \in \mathbb{X}_0$ and Lyapunov function terminal cost F compatible with ℓ.

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Proof: Almost identical to the equilibrium constrained case

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Compared to the equilibrium constraint, the regional constraint

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In Section (5) we will see how stability can be proved without stabilizing terminal constraints

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- equilibrium constraints demand more properties of the system than regional constraints but do not require a Lyapunov function terminal cost
- in both cases, the operating region is restricted to the feasible set $\mathbb{X}_N$

(4) Inverse optimality and suboptimality

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- Inverse Optimality: show that μ_N is optimal for an altered running cost $\tilde{\ell} \neq \ell$
- Suboptimality: derive upper bounds for $J_\infty^{cl}(x_0, \mu_N)$

Inverse optimality

Theorem: [Poubelle/Bitmead/Gevers '88, Magni/Sepulchre '97]

For both types of terminal constraints, μ_N is optimal for

$$\operatorname*{minimize}_{\mathbf{u} \text{ admissible}} \; \widetilde{J}_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \tilde{\ell}(x_{\mathbf{u}}(n), \mathbf{u}(n)), \;\; x_{\mathbf{u}}(0) = x_0$$

with $\tilde{\ell}(x,u) := \ell(x,u) + V_{N-1}(f(x,u)) - V_N(f(x,u))$

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Idea of proof: By the dynamic programming principle

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  is unknown and difficult to compute
- knowing that μ_N is optimal for $\widetilde{J}_\infty(x_0, u)$ doesn't give us a simple way to estimate $J_\infty^{cl}(x_0, \mu_N)$

Suboptimality

Recall: For both stabilizing terminal constraints the relaxed dynamic programming theorem yields the estimate

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But: How large is V_N ?

Without terminal constraints, the inequality $V_N \leq V_\infty$ is immediate

However, the terminal constraints also reverse this inequality, i.e., we have $V_N \geq V_\infty$ and the gap is very difficult to estimate

Suboptimality — example

We consider two examples with $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \mathbb{R}$ for $N = 2$

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General estimates for fixed N appear difficult to obtain. But we can give an asymptotic result for $N \to \infty$

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Theorem: For both types of terminal constraints the assumptions of the stability theorems ensure

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and thus

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and thus

$$J_\infty^{cl}(x, \mu_N) \to V_\infty(x)$$

as $N \to \infty$ uniformly on compact subsets of the feasible sets, i.e., the MPC performance converges to the optimal one

Idea of proof: uses that any approximately optimal trajectory for J_∞ converges to x_* and can thus be modified to meet the constraints with only moderately changing its value

Summary of Section (4)

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- $V_N \gg V_\infty$ is possible under terminal constraints
- $V_N \to V_\infty$ holds for $N \to \infty$

(5) Stability and suboptimality without stabilizing constraints

MPC without stabilizing terminal constraints

We return to the basic MPC formulation

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0 = x_{\mu_N}(n)$$

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In order to motivate why we want to avoid terminal constraints and costs, we consider an example of P double integrators in the plane

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A motivating example for avoiding terminal constraints

Example: [Jahn '10] Consider P 4-dimensional systems

$$\dot{x}_i = f(x_i, u_i) := (x_{i2}, u_{i1}, x_{i4}, u_{i2})^T, \quad i = 1, \ldots, P$$

Interpretation: $(x_{i1}, x_{i3})^T$ = position, $(x_{i2}, x_{i4})^T$ = velocity

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Stage cost: $\ell(x,u) = \sum_{i=1}^{P} \|(x_{i1}, x_{i3})^T - x_d\| + \|(x_{i2}, x_{i4})^T\|/50$

with $x_d = (0,0)^T$ until $t = 20s$ and $x_d = (3,0)^T$ afterwards

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The simulation shows MPC for $P = 128$ ($\rightsquigarrow$ system dimension 512) with sampling time $T = 0.02s$ and horizon $N = 6$

Stabilizing NMPC without terminal constraint

(Some) stability and performance results known in the literature:

[Alamir/Bornard '95]

[Shamma/Xiong '97, Primbs/Nevistić '00]

[Jadbabaie/Hauser '05]

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Here we explain the last approach

Bounds on the optimal value function

Recall the definition of the optimal value function

$$V_N(x) := \inf_{\mathbf{u}\text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k,x), \mathbf{u}(k))$$

Boundedness assumption: there exists $\gamma > 0$ with

$$V_N(x) \leq \gamma \ell^\star(x) \quad \text{for all } x \in \mathbb{X}, N \in \mathbb{N}$$

where $\ell^\star(x) := \min_{u \in \mathbb{U}} \ell(x,u)$

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(sufficient conditions for and relaxations of this bound will be discussed later)

Stability and performance index

We choose ℓ, such that

$$\alpha_3(\|x - x_*\|) \leq \ell^\star(x) \leq \alpha_4(\|x - x_*\|)$$

holds for $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ (again, $\ell(x,u) = \|x - x_*\|^2 + \lambda\|u\|^2$ works)

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$$V_N(f(x, \mu_N(x))) \leq V_N(x) - \alpha_N \ell(x, \mu_N(x))$$

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We can compute α_N from the bound $V_N(x) \le \gamma \ell^\star(x)$

Computing α_N

We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$ $(*)$

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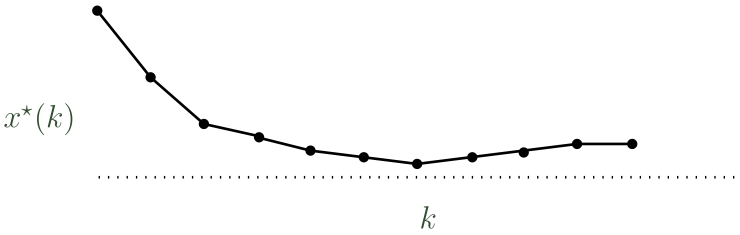

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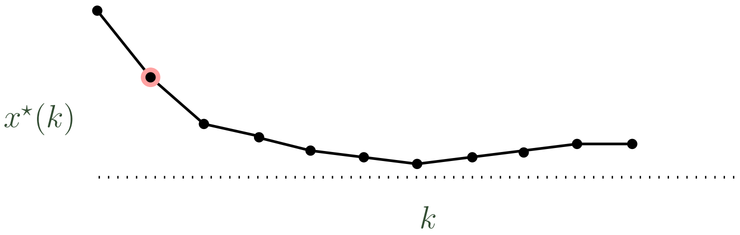

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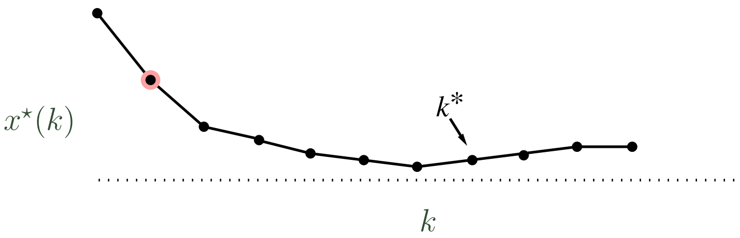

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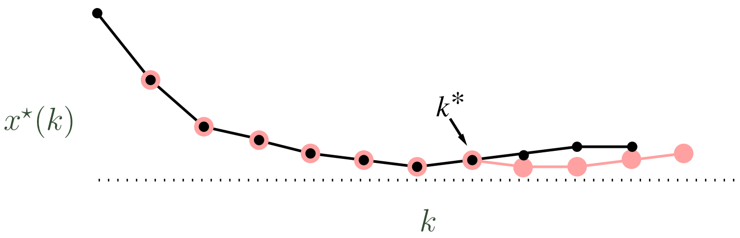

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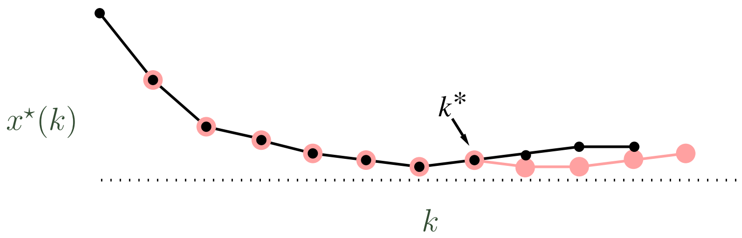

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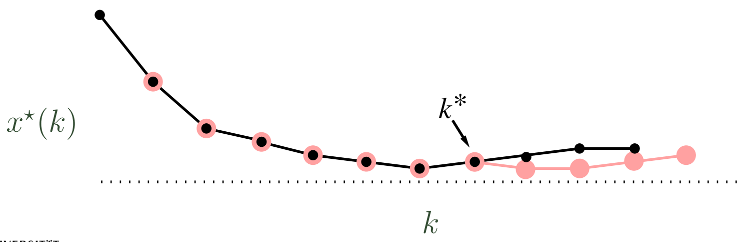

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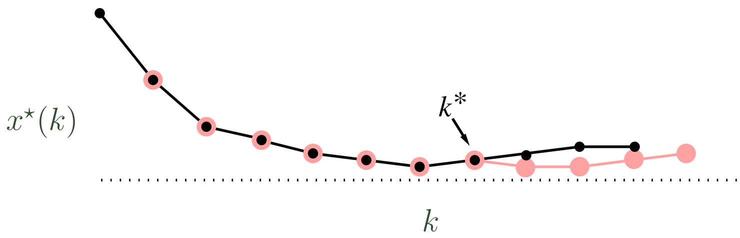

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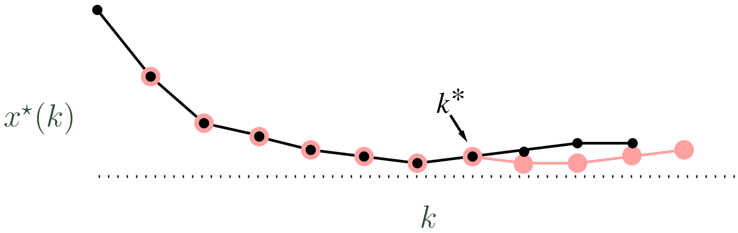

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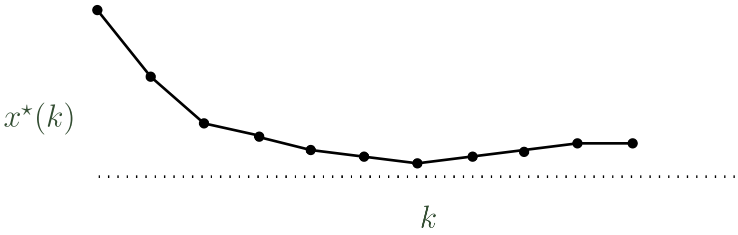

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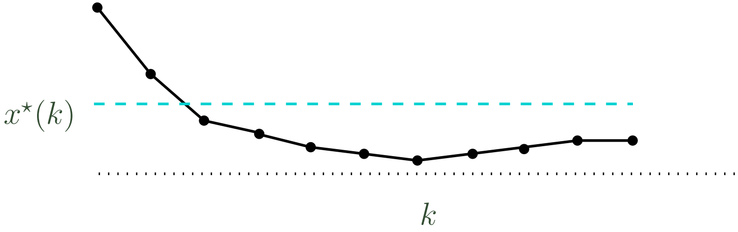

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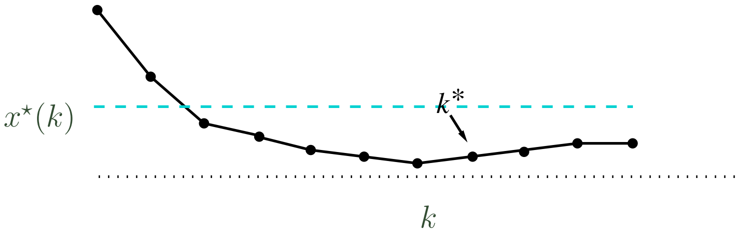

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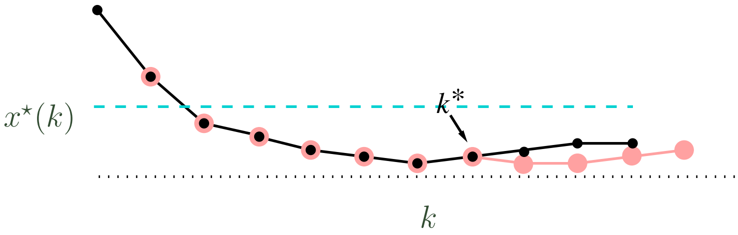

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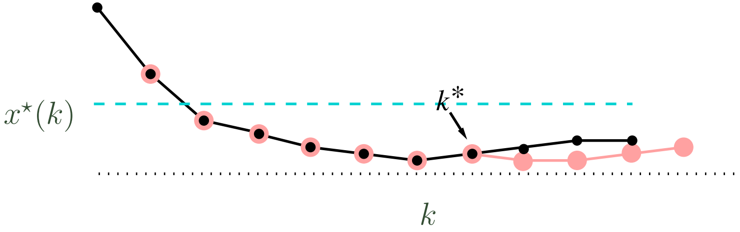

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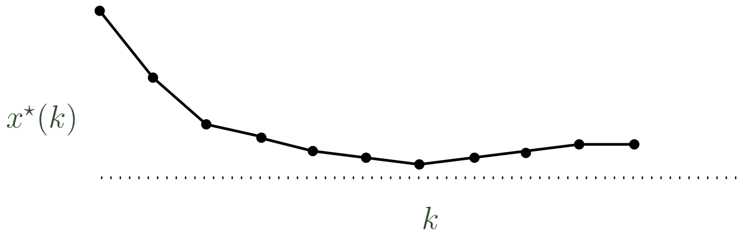

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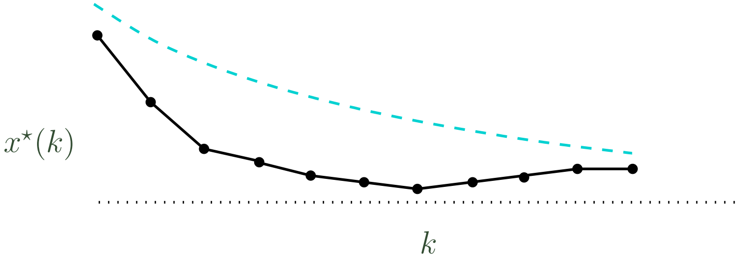
Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

$$V_N(x) \leq \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \leq \gamma \left(\frac{\gamma-1}{\gamma}\right)^k \ell^\star(x)$$



Decay of the optimal trajectory

We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

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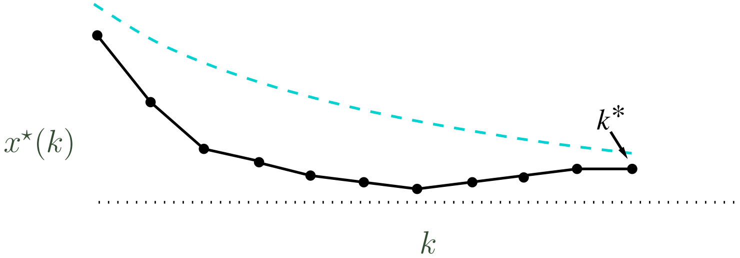

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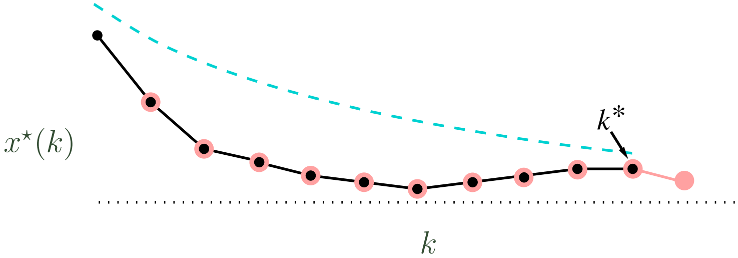

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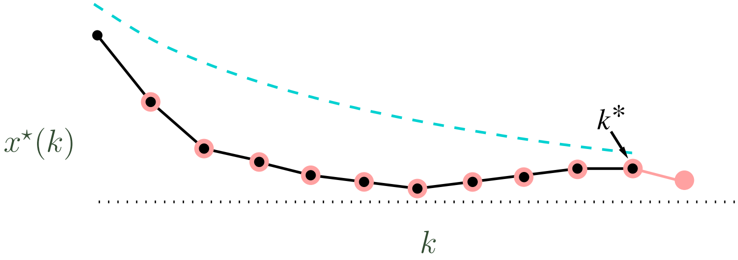

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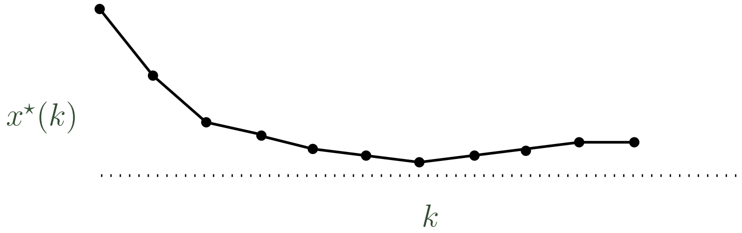

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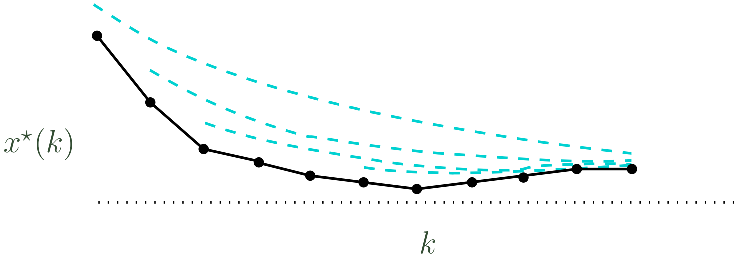
Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

$V_N(x) \leq \gamma \ell^\star(x)$ $\Rightarrow$ formulate all constraints



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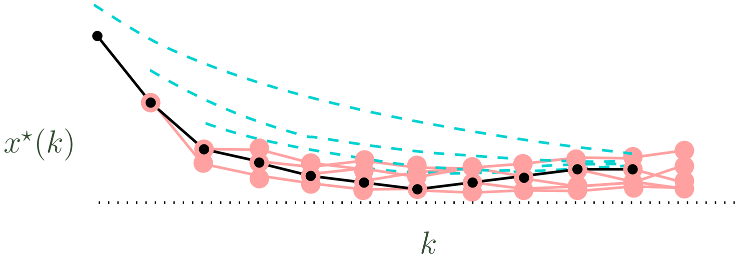
$V_N(x) \leq \gamma\ell^\star(x)$ $\Rightarrow$ formulate all constraints and trajectories



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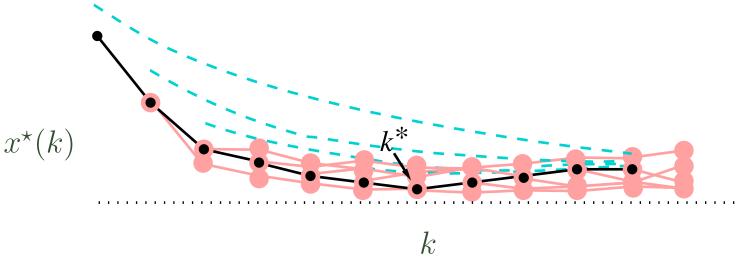

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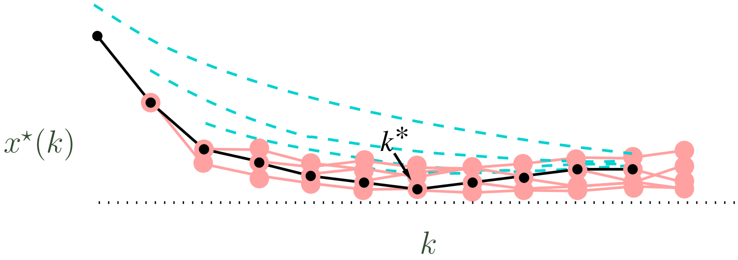

Optimization approach to compute α_N

We explain the optimization approach (Variant 3) in more detail. We want α_N such that

$$V_N(x^\star(1)) \leq V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$$

holds for all optimal trajectories $x^\star(n), \mathbf{u}^\star(n)$ for V_N

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The bound and the dynamic programming principle imply:

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\vdots \quad & \quad \vdots \quad \quad \vdots
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Optimization approach to compute α_N

$\rightsquigarrow V_N(x^\star(1))$ is bounded by sums over $\ell(x^\star(n), \mathbf{u}^\star(n))$

For sums of these values, in turn, we get bounds from the dynamic programming principle and the bound

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Optimization approach to compute α_N

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Verifying the relaxed Lyapunov inequality

Find α_N, such that for all optimal trajectories $x^\star$, $\mathbf{u}^\star$:

$$V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0)) \qquad (*)$$

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The inequalities from the last slides translate to

$$\sum_{n=k}^{N-1} \lambda_n \leq \gamma \lambda_k, \quad k = 0, \ldots, N-2 \qquad (1)$$

$$\nu \leq \sum_{n=1}^{j} \lambda_n + \gamma \lambda_{j+1}, \quad j = 0, \ldots, N-2 \qquad (2)$$

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Optimization problem

⇒ if α_N is such that the inequality

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holds for all admissible λ_n and ν, then the desired inequality will hold for all optimal trajectories

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This is a linear optimization problem whose solution can be computed explicitly (which is nontrivial) and reads

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}}$$

Stability and performance theorem

Theorem: [Gr./Pannek/Seehafer/Worthmann '10]: Assume $V_N(x) \leq \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$. If

$$\alpha_N > 0$$

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Conversely, if $N < 2 + \frac{\ln(\gamma-1)}{\ln\gamma - \ln(\gamma-1)}$, then there exists a system for which $V_N(x) \leq \gamma \ell^\star(x)$ holds but the NMPC closed loop is not asymptotically stable.

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Horizon dependent γ-values

The theorem remains valid if we replace the bound condition

$$V_N(x) \le \gamma \ell^\star(x)$$

by

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for horizon-dependent bounded values $\gamma_N \in \mathbb{R}$, $N \in \mathbb{N}$

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$$\rightsquigarrow \quad \alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)}$$

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This allows for tighter bounds and a refined analysis

Controllability condition

A refined analysis can be performed if we compute γ_N from a controllability condition

Controllability condition

A refined analysis can be performed if we compute γ_N from a controllability condition, e.g., exponential controllability:

Assume that for each $x_0 \in \mathbb{X}$ there exists an admissible control $\mathbf{u}$ such that

$$\ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \leq C\sigma^k \ell^\star(x_0), \quad k = 0, 1, 2, \ldots$$

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A refined analysis can be performed if we compute γ_N from a controllability condition, e.g., exponential controllability:

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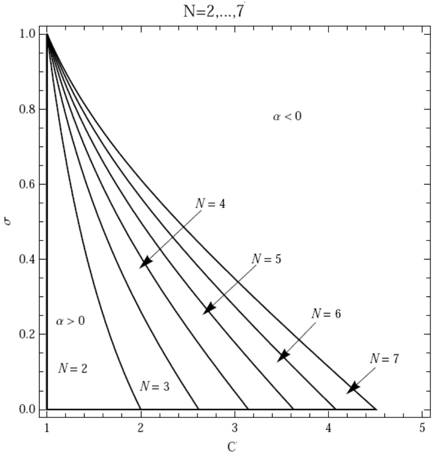
This allows to compute the minimal stabilizing horizon

$$\min\{N \in \mathbb{N} \,|\, \alpha_N > 0\}$$

depending on C and σ

Stability chart for C and σ



(Figure: Harald Voit)

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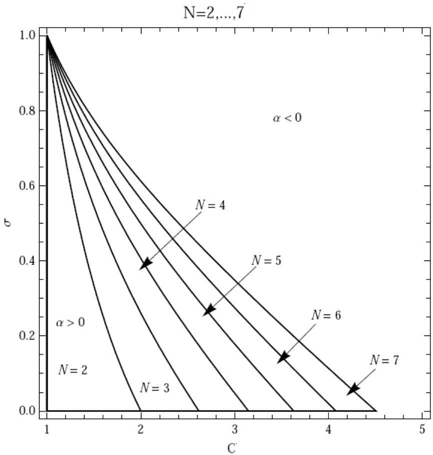

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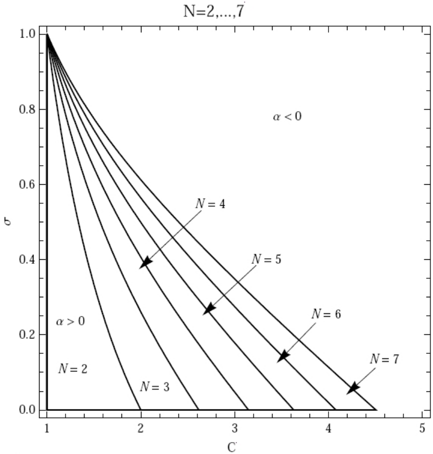

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(we will see in the next section how to use this information)

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- positive definiteness of ℓ can be replaced by a detectability condition [Grimm/Messina/Tuna/Teel '05]

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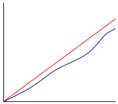

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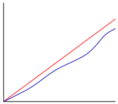

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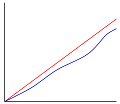

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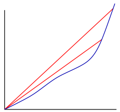

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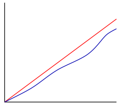

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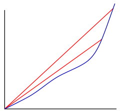

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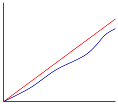

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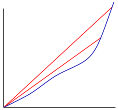

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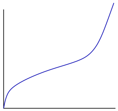

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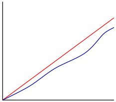

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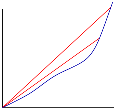

- $\rho \in \mathcal{K}_\infty$ arbitrary
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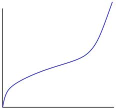

[Grimm/Messina/Tuna/Teel ’05, Gr./Pannek ’11]

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- An optimization approach allows to compute the best possible α_N in the relaxed dynamic programming theorem
- The γ or γ_N can be computed from controllability properties, e.g., exponential controllability
- The overshoot bound $C > 0$ plays a crucial role or obtaining small stabilizing horizons

(6) Examples for the design of MPC schemes

Design of “good” MPC running costs ℓ

We want small overshoot C in the estimate

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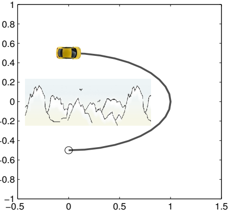
$$\ell(x_{\mathbf{u}}(n), \mathbf{u}(n)) \leq C\sigma^n \ell^\star(x_0)$$

The trajectories $x_{\mathbf{u}}(n)$ are given, but we can use the running cost ℓ as design parameter

The car-and-mountains example reloaded



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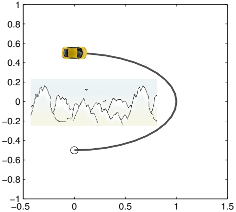

MPC with $\ell(x,u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$

$\rightsquigarrow$ asymptotic stability for $N = 11$ but not for $N \leq 10$

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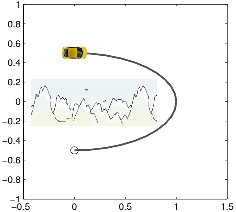

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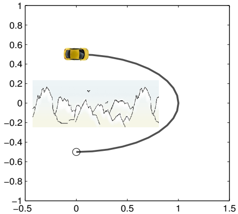

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Remedy: put larger weight on x_2:

$\ell(x,u) = (x_1 - x_{*,1})^2 + 5(x_2 - x_{*,2})^2 + |u|^2$

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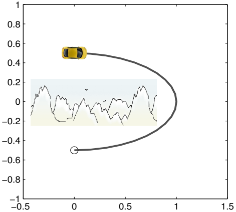

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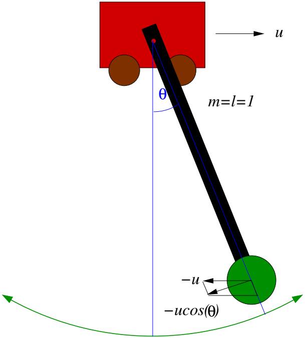
Example: pendulum on a cart



$x_1 = \theta =$ angle
$x_2 =$ angular velocity
$x_3 =$ cart position
$x_4 =$ cart velocity
$u =$ cart acceleration

⇝ control system

$$
\begin{aligned}
\dot{x}_1 &= x_2(t) \\
\dot{x}_2 &= -g\sin(x_1) - kx_2 \\
&\quad -u\cos(x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{aligned}
$$

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Example: Inverted Pendulum

Reducing overshoot for swingup of the pendulum on a cart:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = g\sin(x_1) - kx_2 + u\cos(x_1)$$
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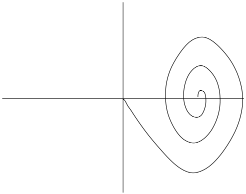

Typical swingup trajectory
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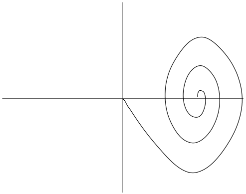
Let $\ell(x) = \sqrt{\ell_1(x_1, x_2) + x_3^2 + x_4^2}$



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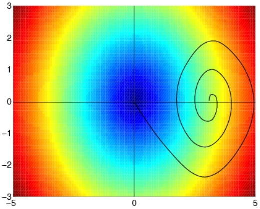

$\ell_1(x_1,x_2) = x_1^2 + x_2^2$

$N = 15$

sampling time $T = 0.15$

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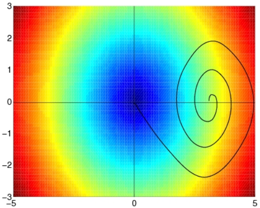

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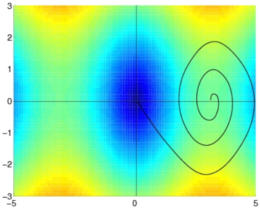

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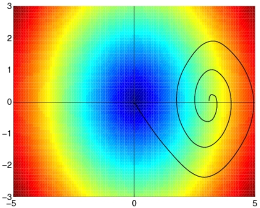

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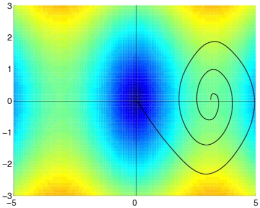
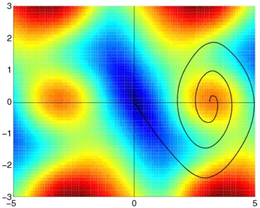


$4(1 - \cos x_1) + x_2^2$

$N = 10$



$(\sin x_1, x_2)P(\sin x_1, x_2)^T$
$+2((1 - \cos x_1)(1 - \cos x_2)^2)^2$

$N = 4$ (swingup only)

sampling time $T = 0.15$

A PDE example

We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0,1]$

solution $y = y(t,x)$

boundary conditions $y(t,0) = y(t,1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

and distributed control $u : \mathbb{R} \times \Omega \to \mathbb{R}$

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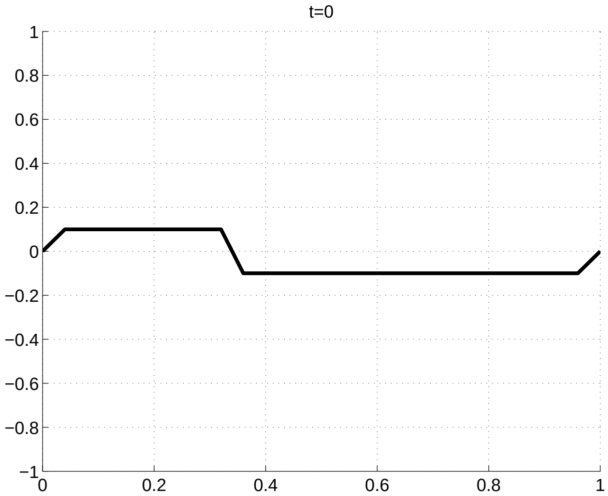
Discrete time system: $y(n) = y(nT, \cdot)$, sampling time $T = 0.025$

The uncontrolled PDE



uncontrolled ($u \equiv 0$)

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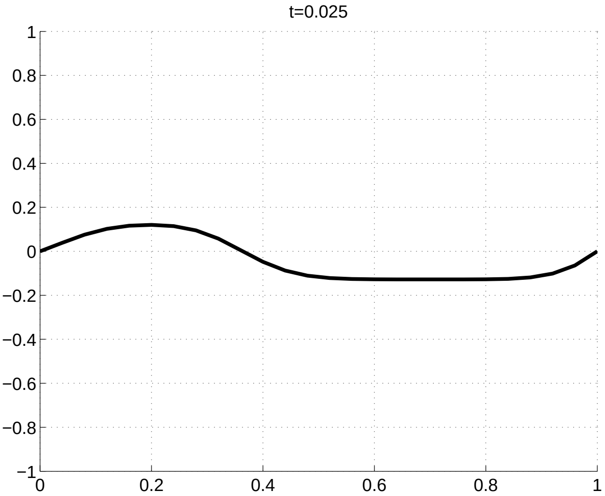

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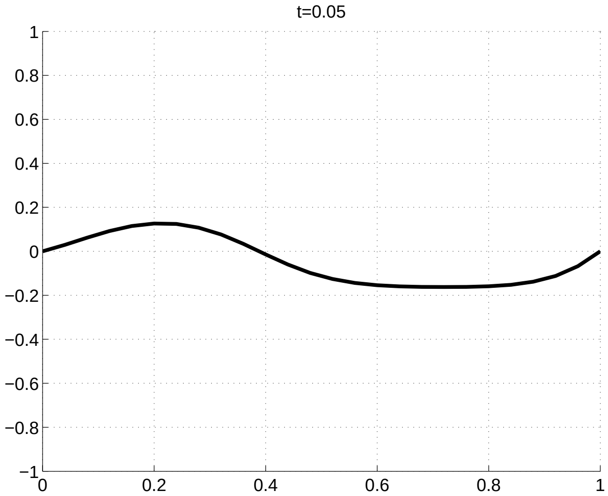

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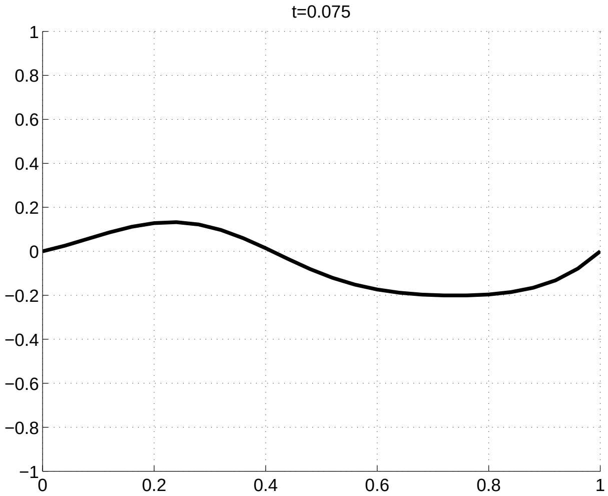

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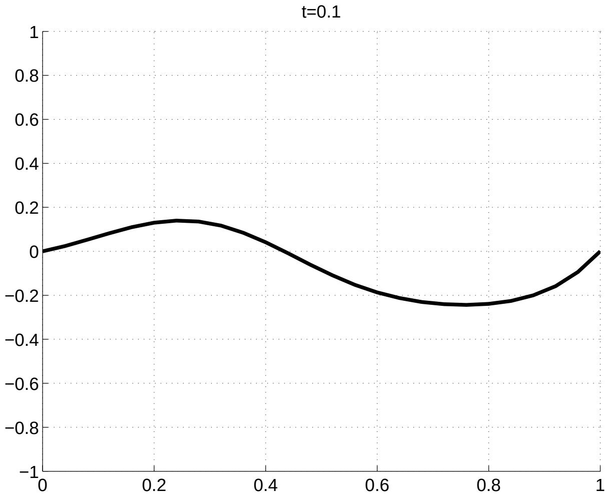

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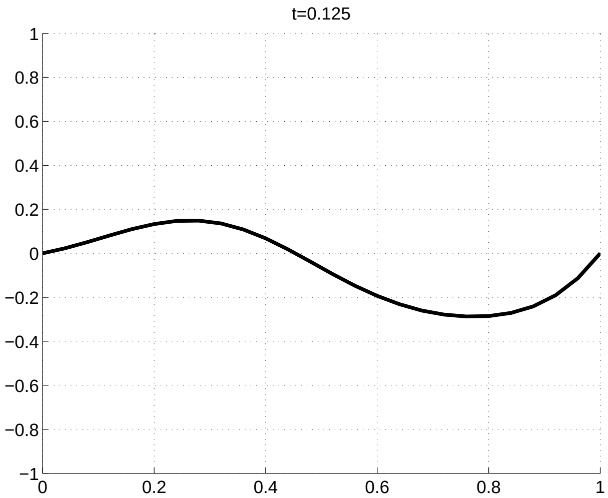
The uncontrolled PDE



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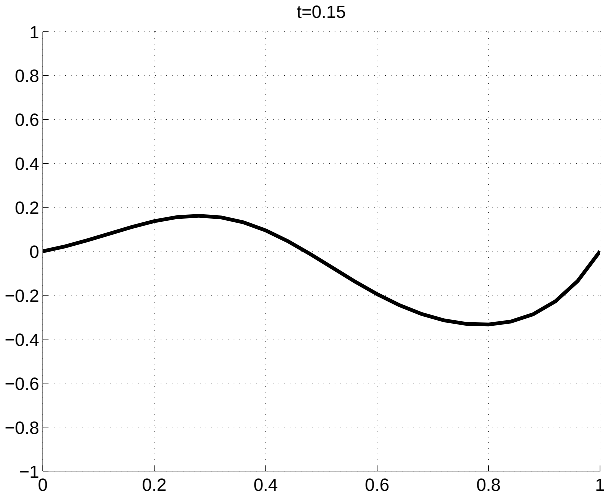

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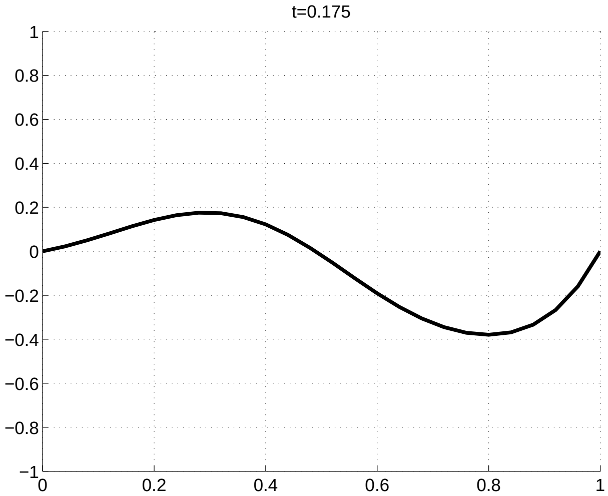

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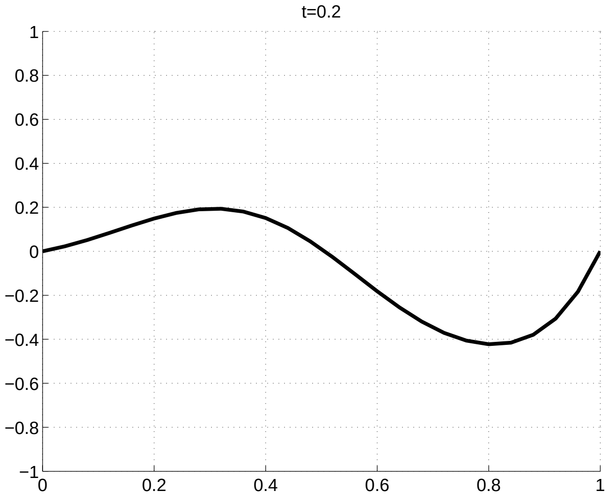

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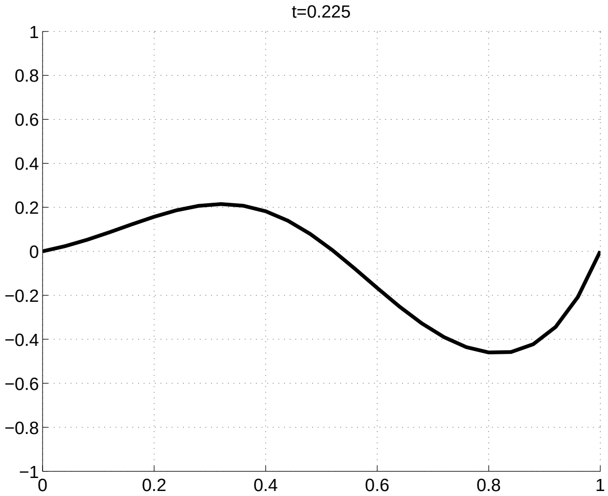

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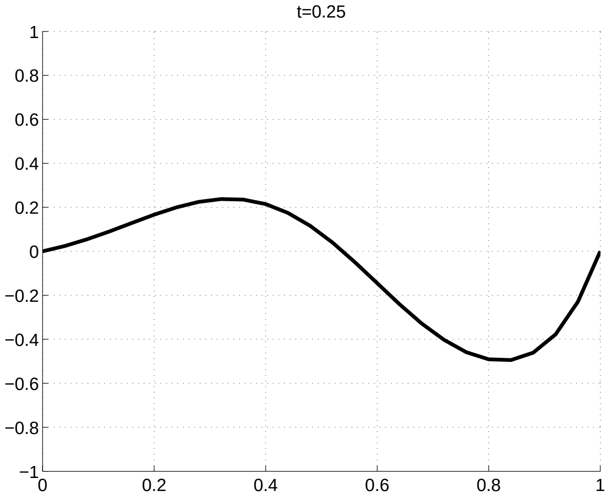

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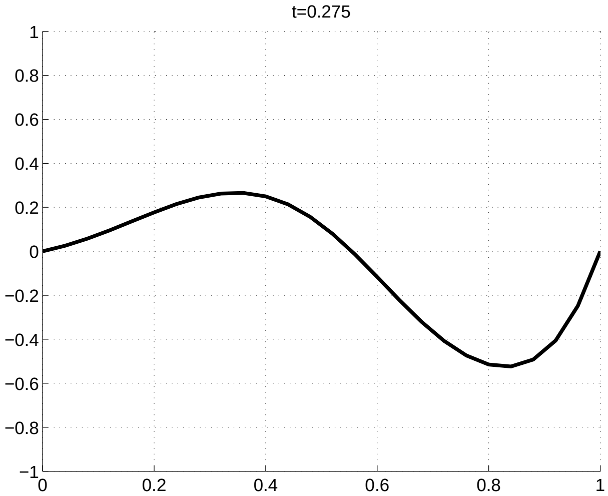

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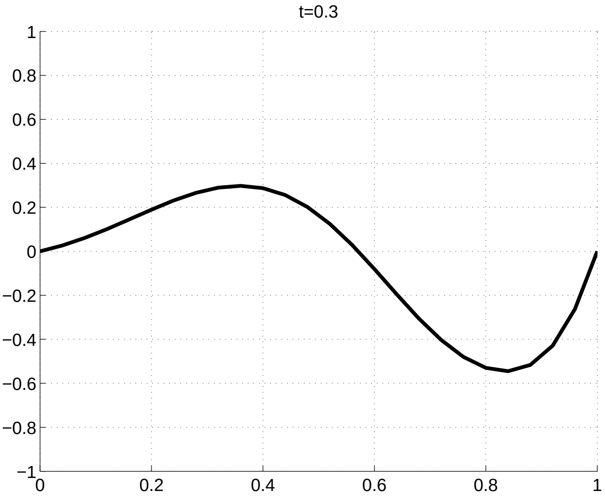

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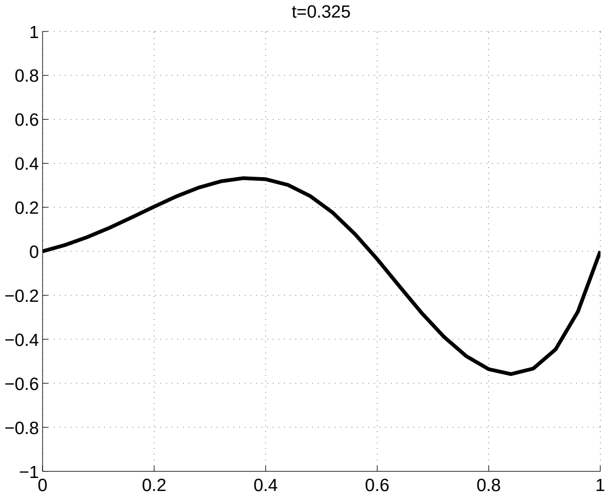

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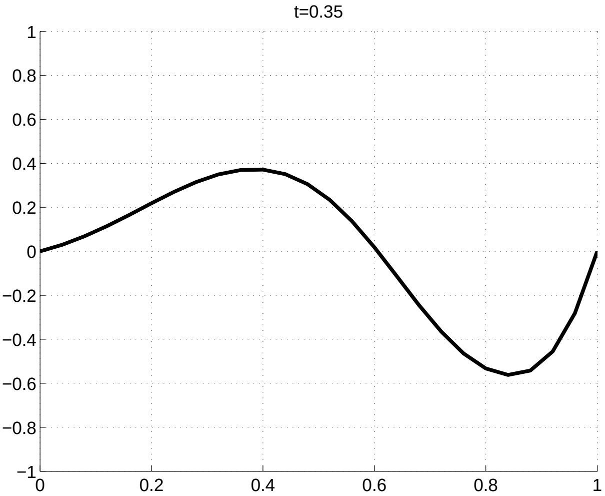

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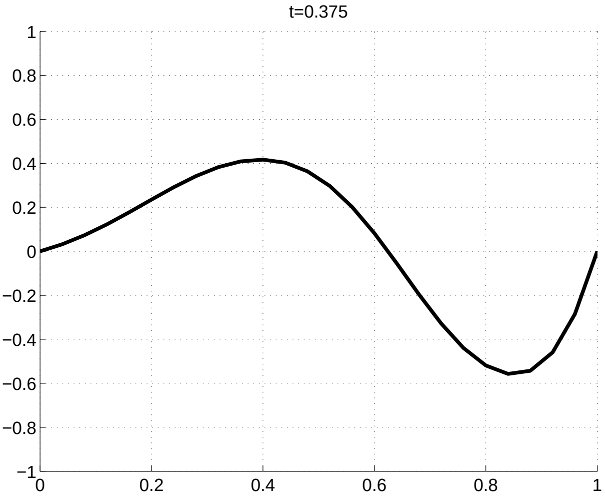

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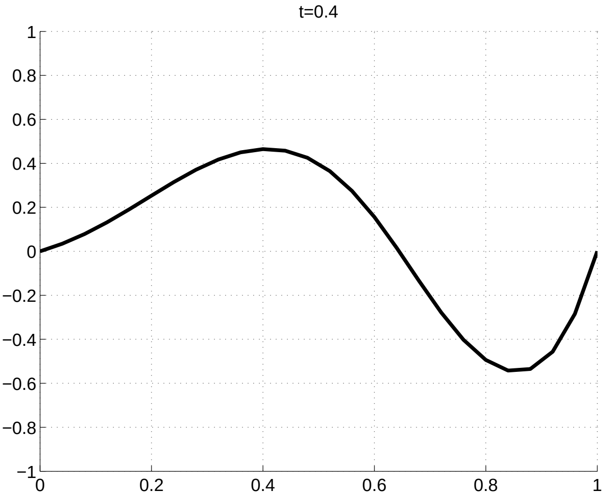

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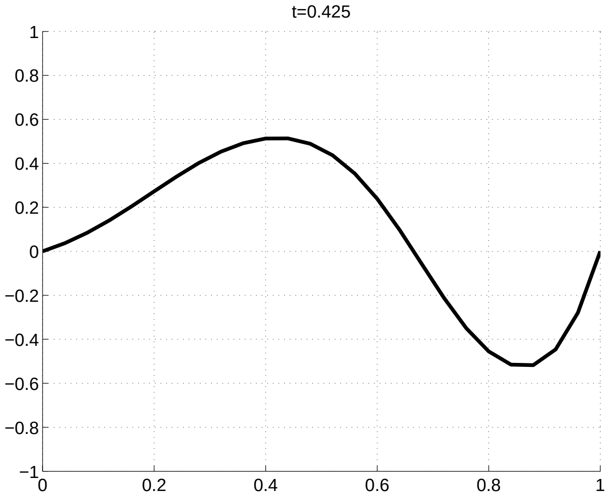

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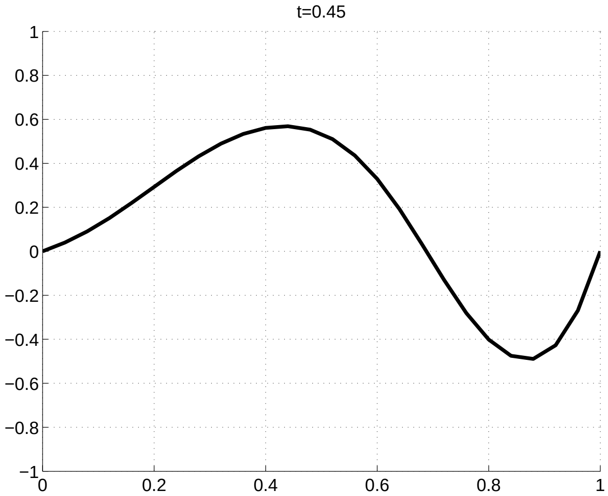

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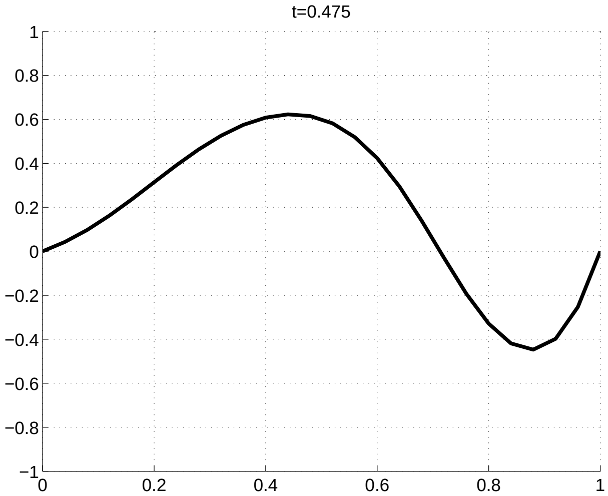

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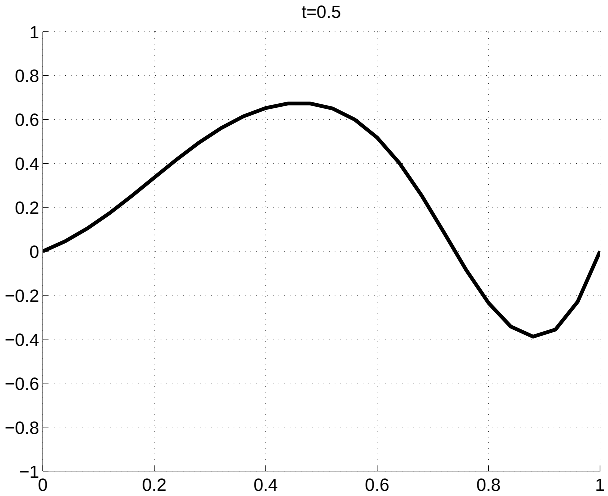

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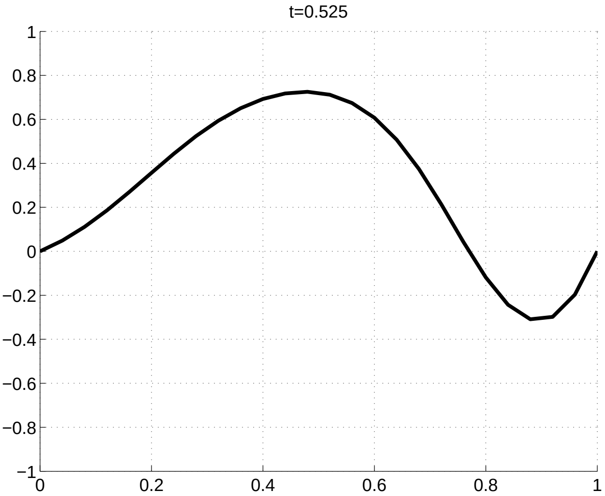

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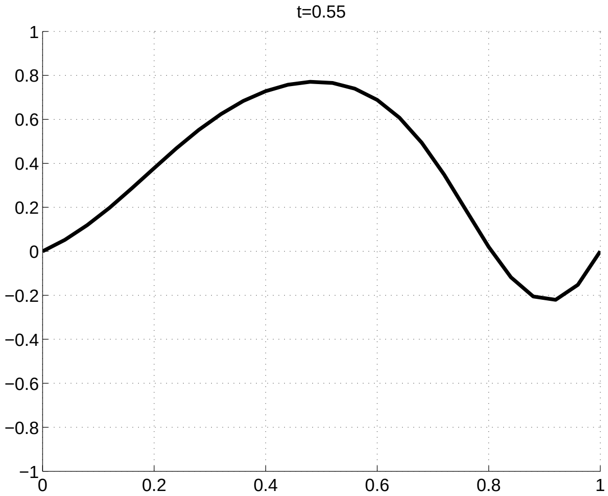

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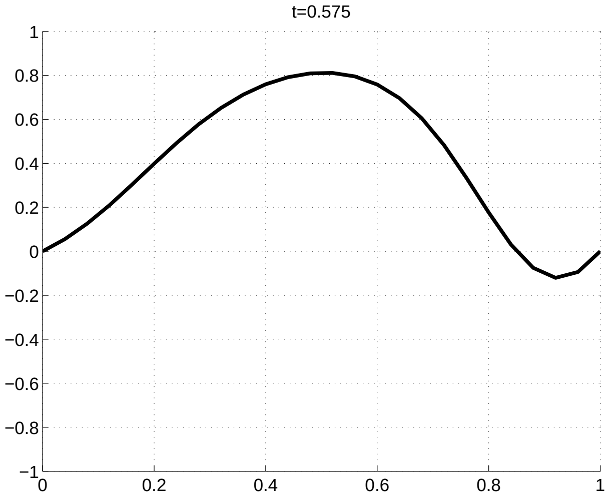

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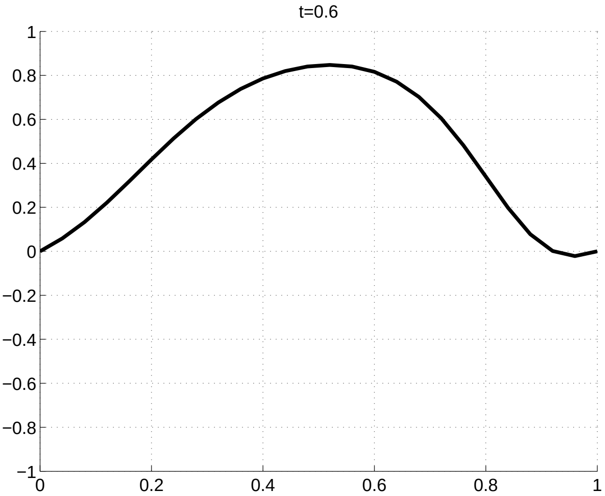

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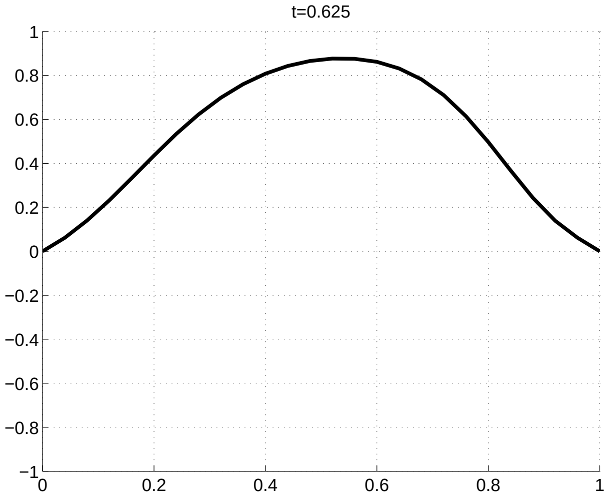

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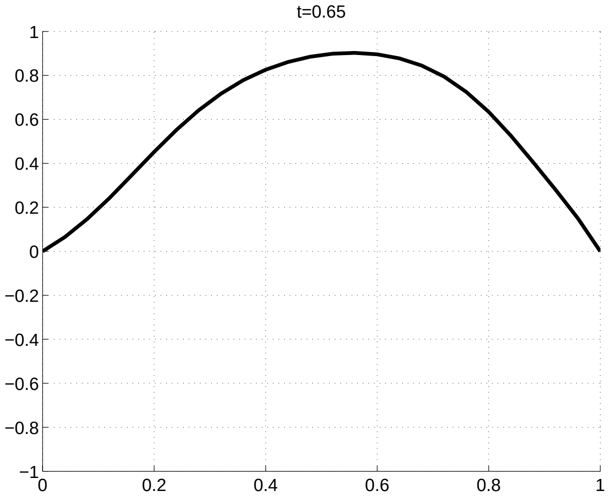

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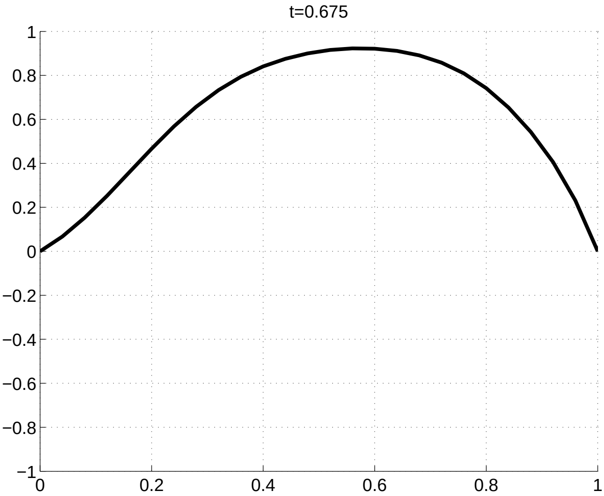

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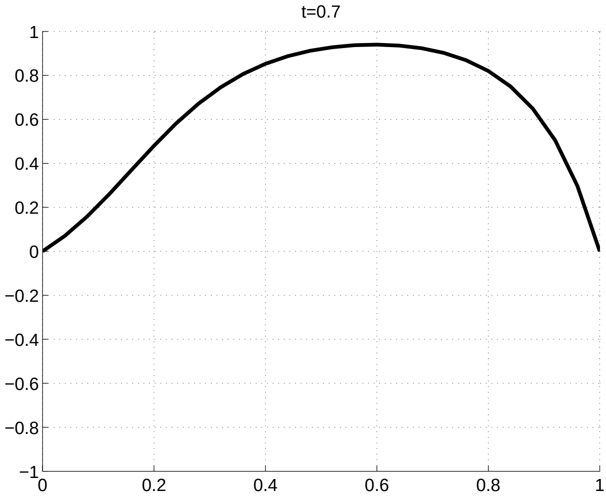

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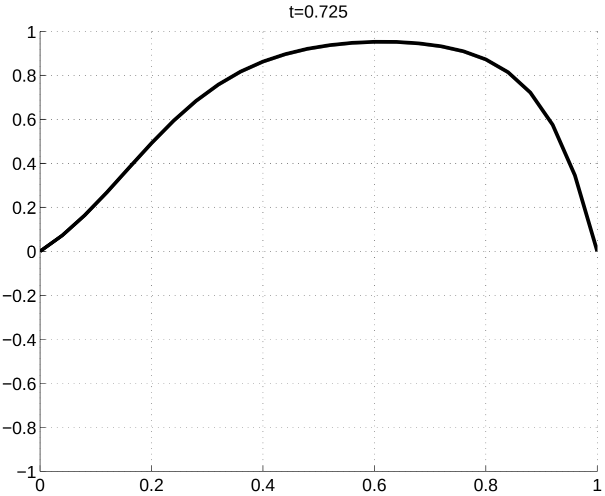

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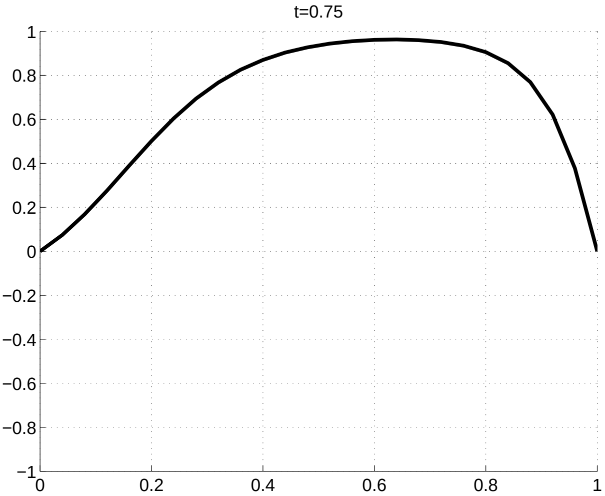

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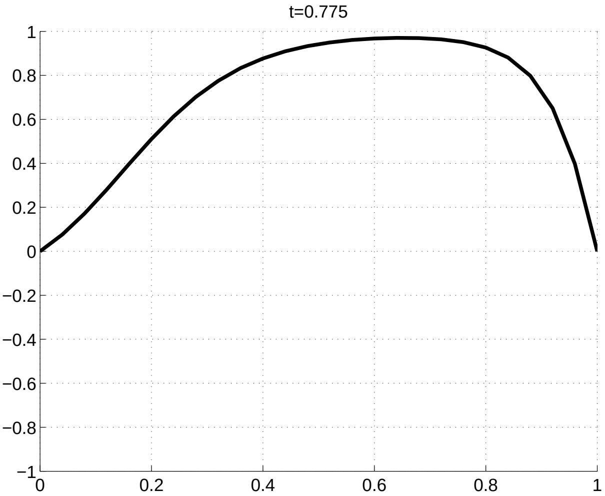

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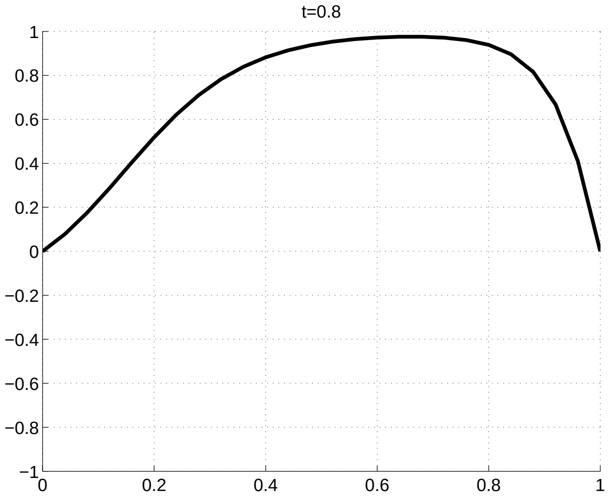

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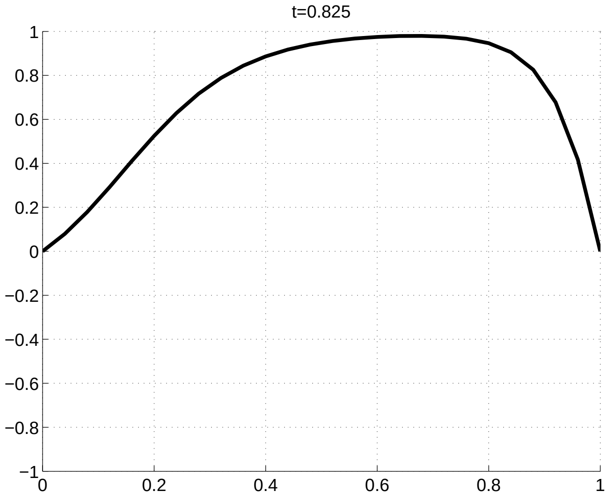

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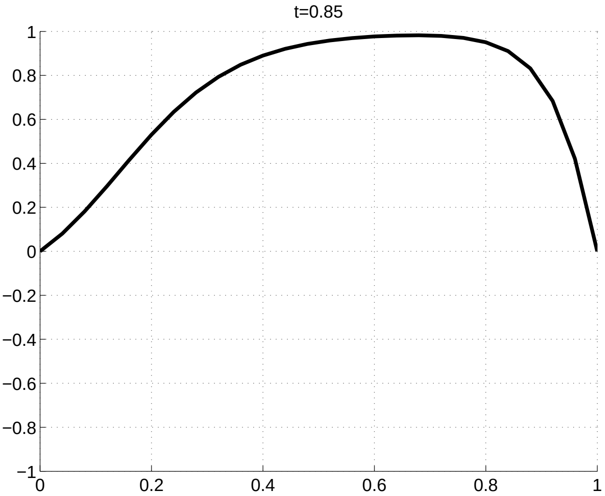

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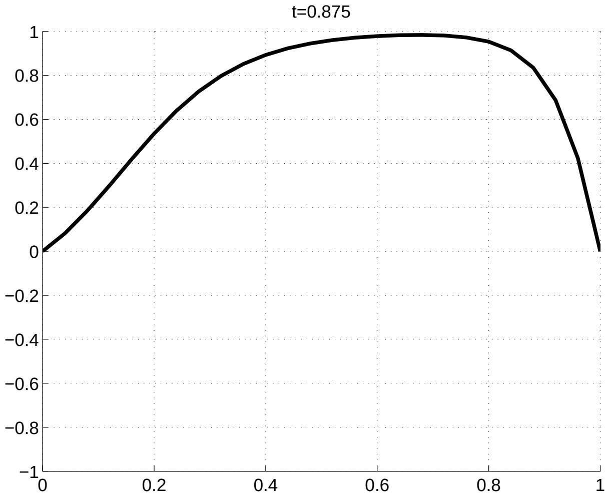

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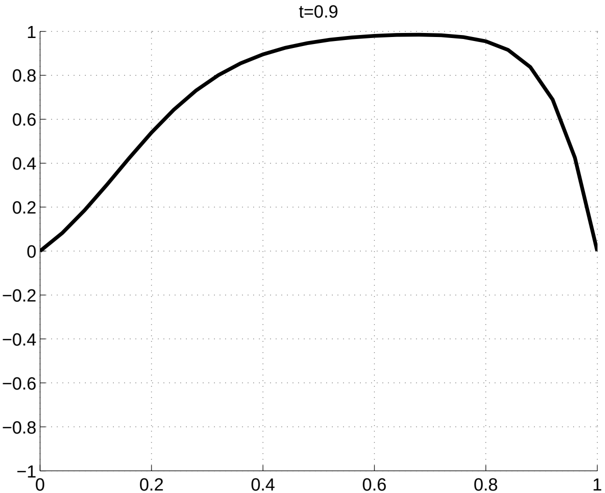

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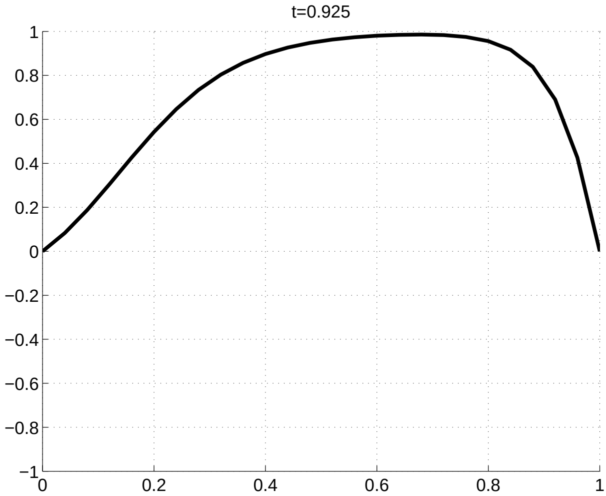

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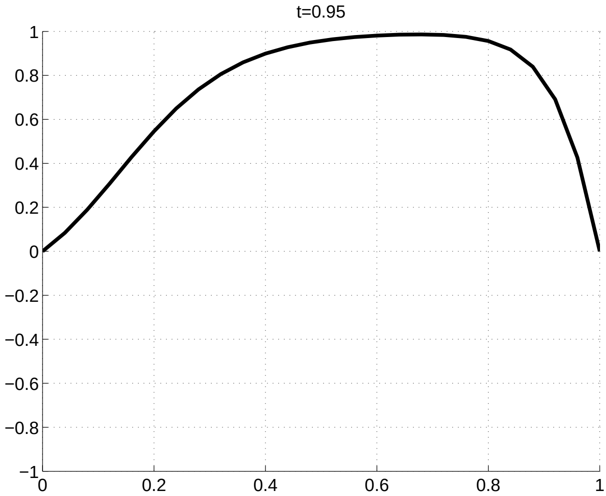

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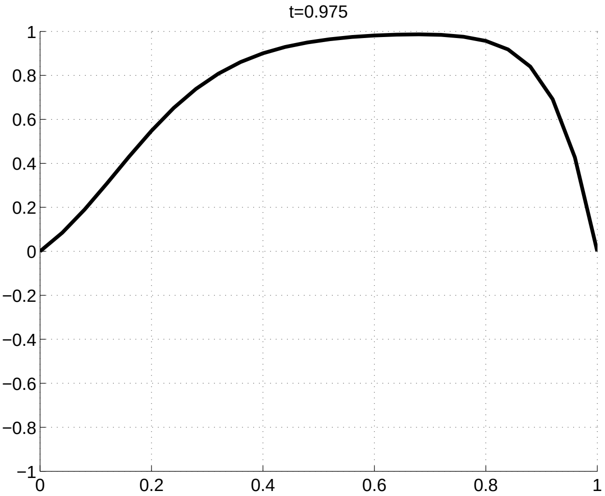

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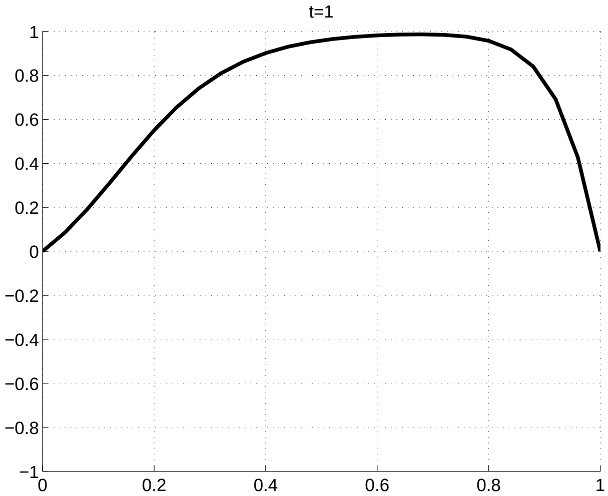

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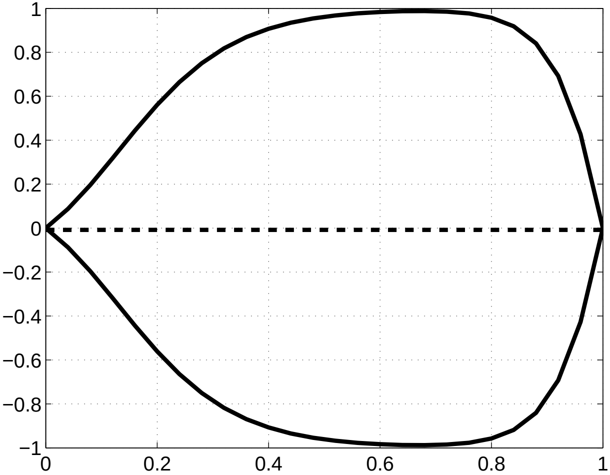

all equilibrium solutions

MPC for the PDE example

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

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Goal: stabilize the sampled data system $y(n)$ at $y \equiv 0$

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Usual approach: quadratic L^2 cost

$$\ell(y(n), u(n)) = \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2$$

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for $\|y_x\|_{L^2} \gg \|y\|_{L^2}$ this can only hold if $C \gg 0$

MPC for the PDE example

Conclusion: because of

$$\|y(n)\|_{L^2}^2 + \lambda\|y_x(n)\|_{L^2}^2 \leq C\sigma^n\|y(0)\|_{L^2}^2$$

the controllability condition may only hold for very large C

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Remedy: use H^1 cost

$$\ell(y(n), u(n)) = \underbrace{\|y(n)\|_{L^2}^2 + \|y_x(n)\|_{L^2}^2}_{=\|y(n)\|_{H^1}^2} + \lambda \|u(n)\|_{L^2}^2.$$

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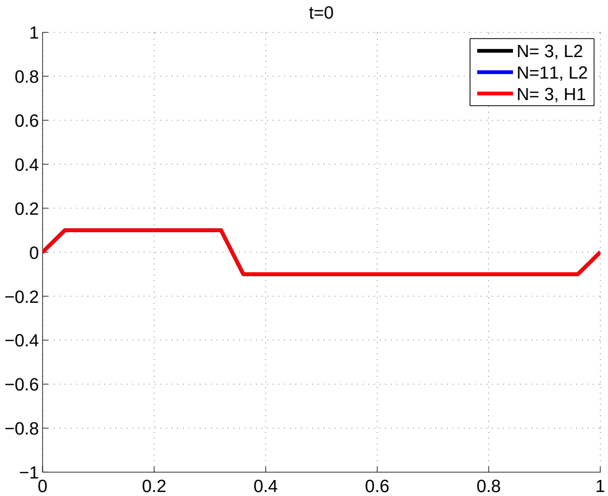
Then an analogous computation yields

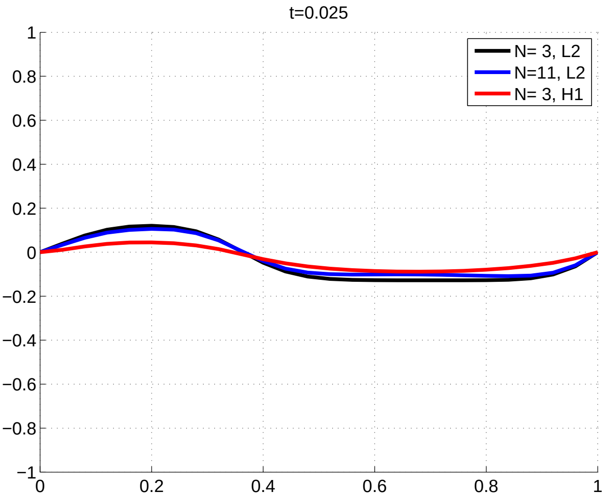
$$\|y(n)\|^2_{L^2} + (1+\lambda)\|y_x(n)\|^2_{L^2} \le C\sigma^n\Big(\|y(0)\|^2_{L^2} + \|y_x(0)\|^2_{L^2}\Big)$$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

MPC with L_2 vs. H_1 cost



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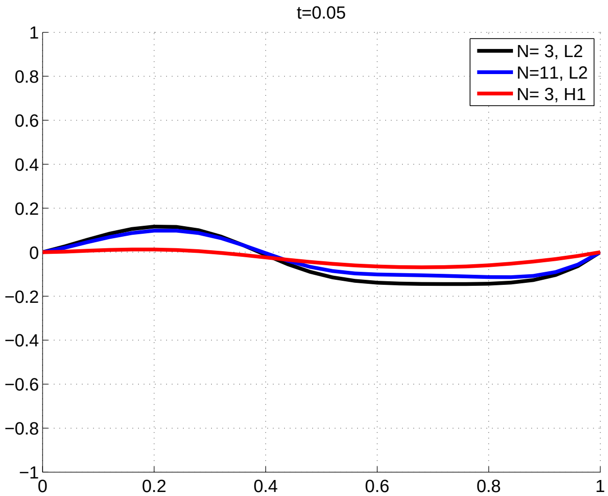

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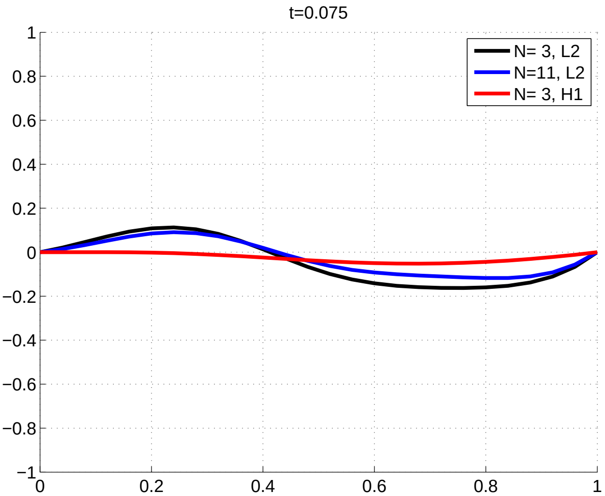

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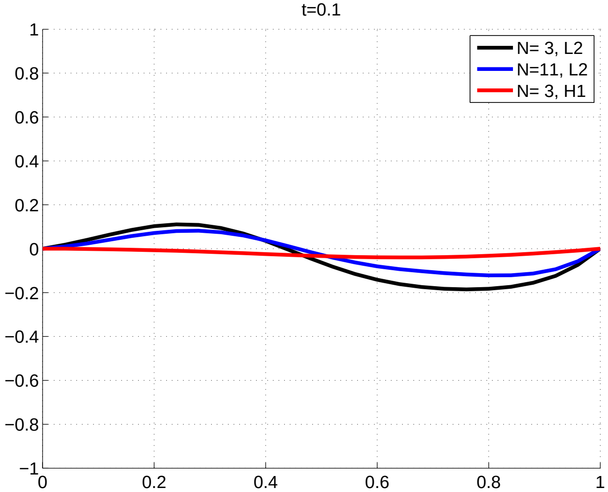

MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

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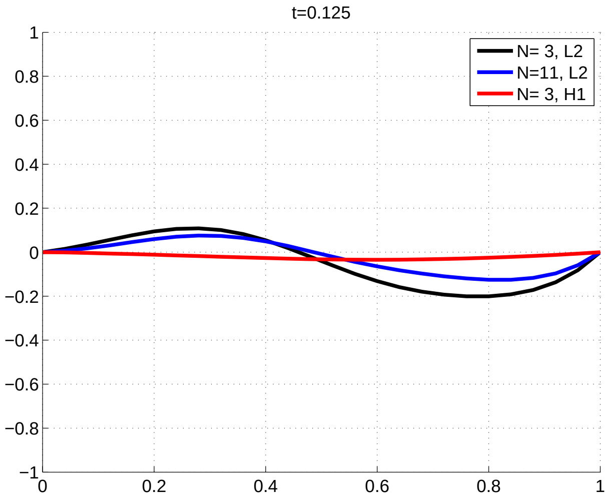

MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

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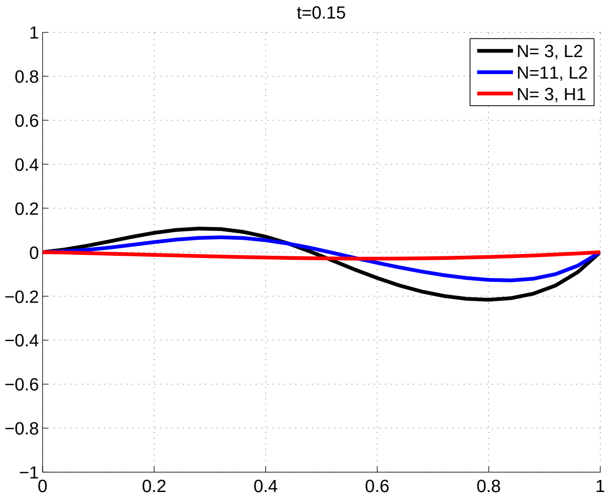

MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

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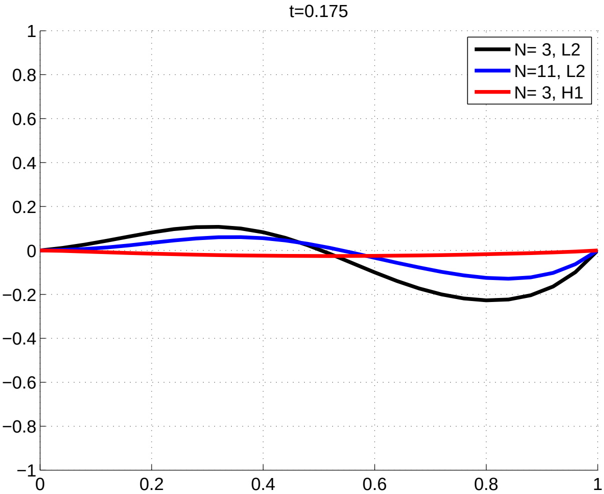

MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

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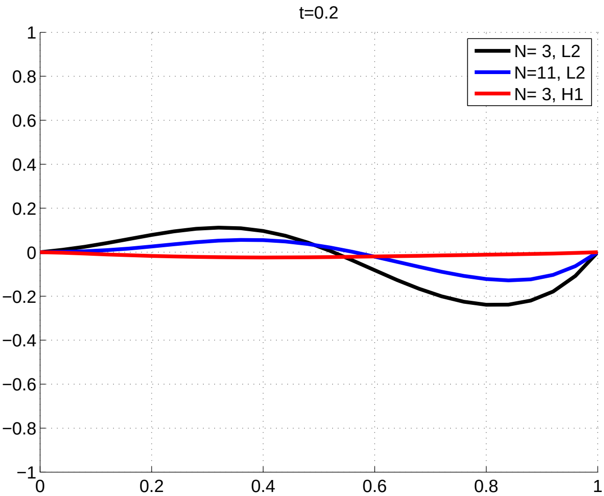

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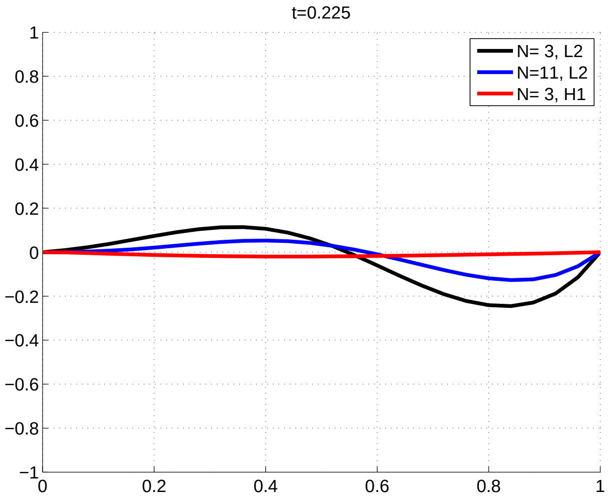

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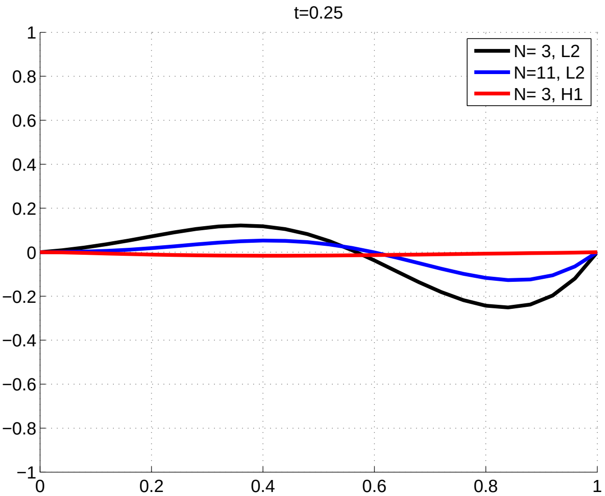

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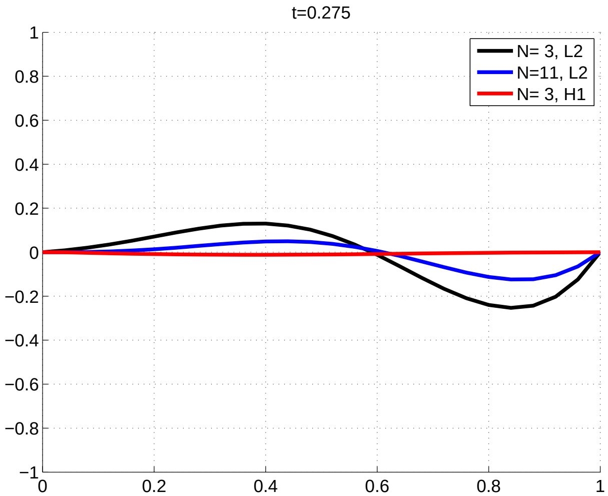
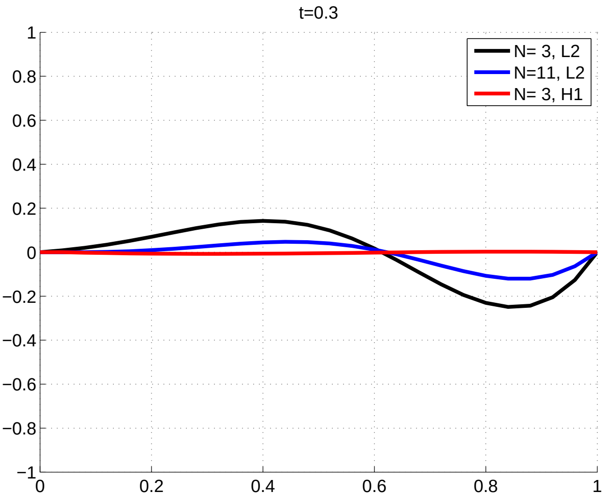


MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

domain $\Omega = [0,1]$

solution $y = y(t,x)$

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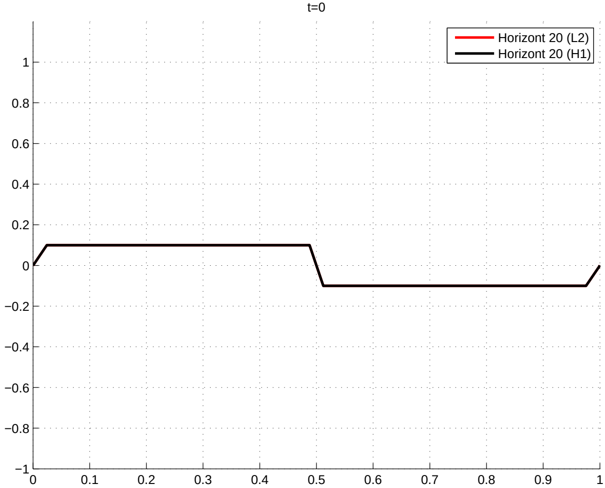
$\rightsquigarrow$ L^2 should perform better that H^1

Boundary control, L_2 vs. H_1, $N = 20$



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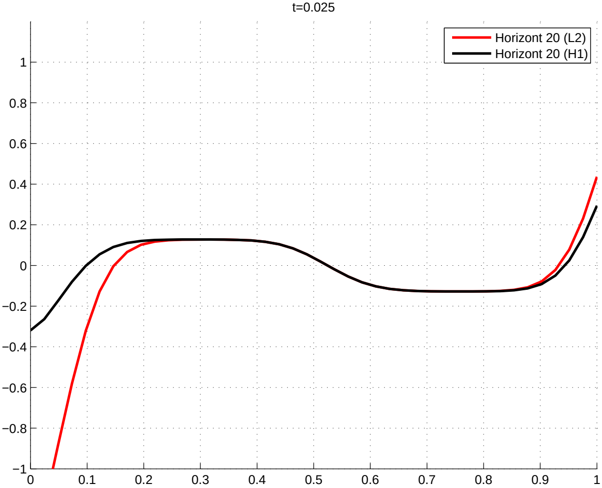

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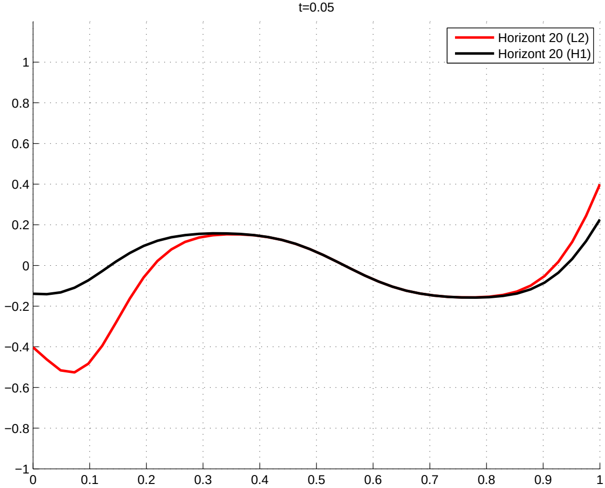

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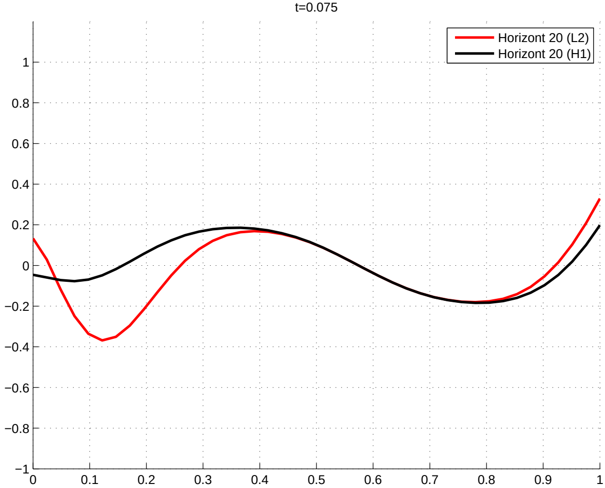

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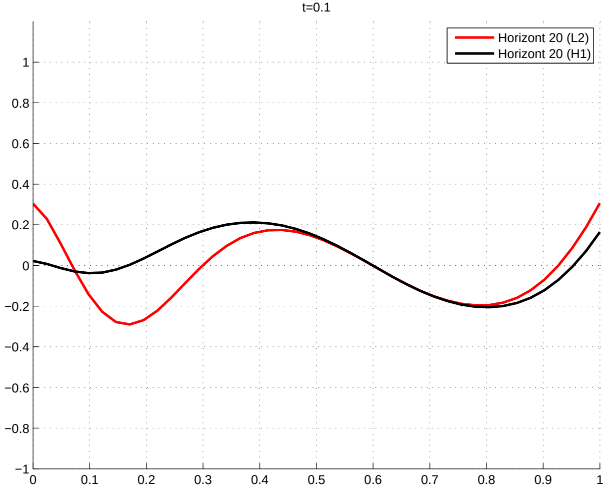

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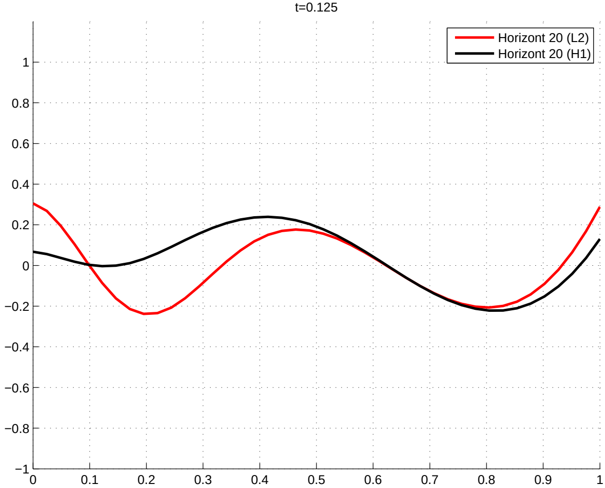

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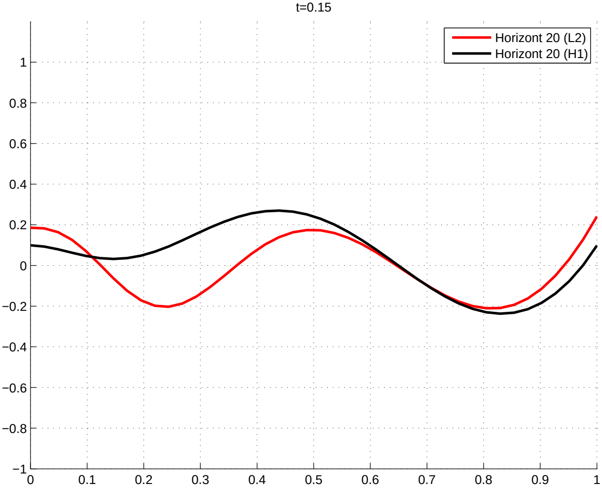

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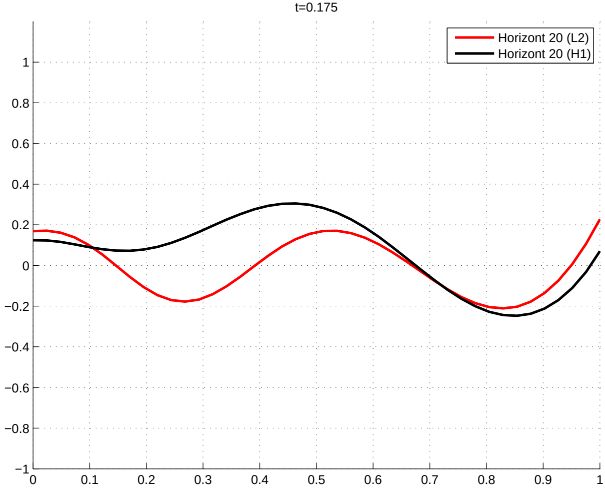

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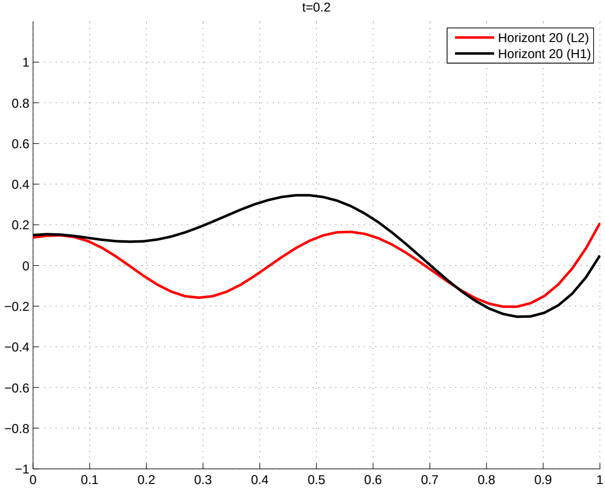

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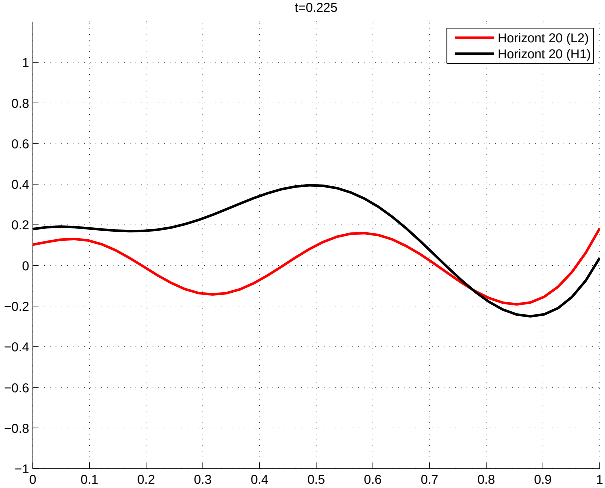

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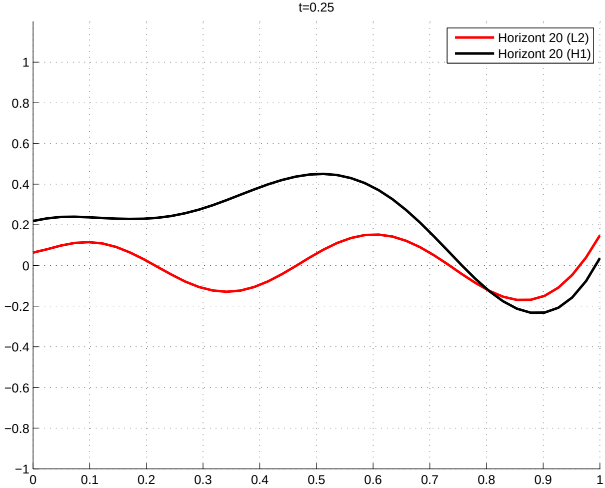

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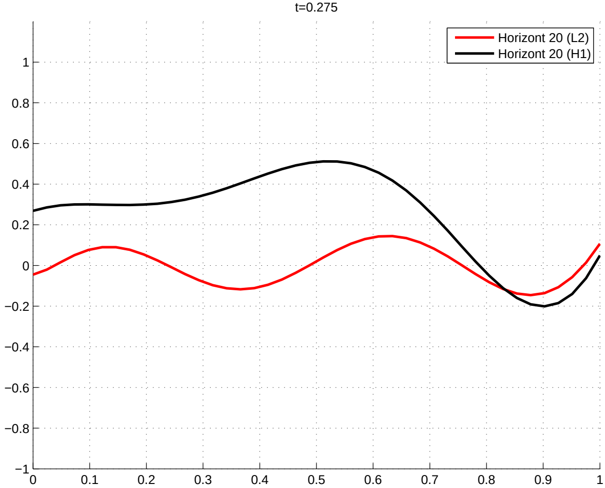

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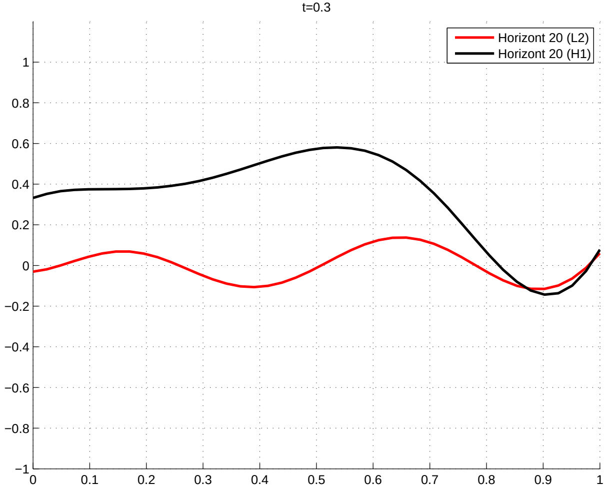

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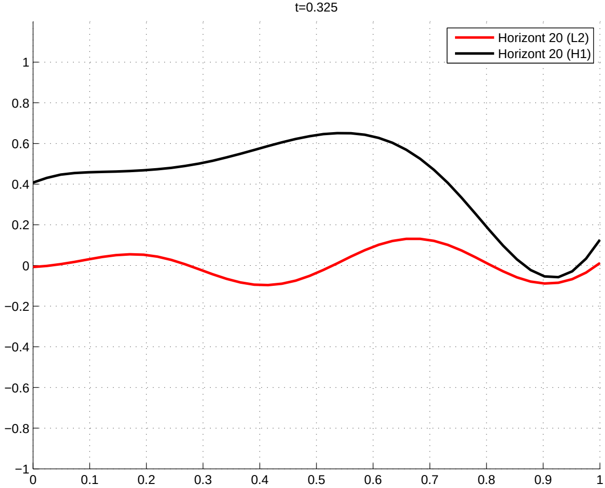

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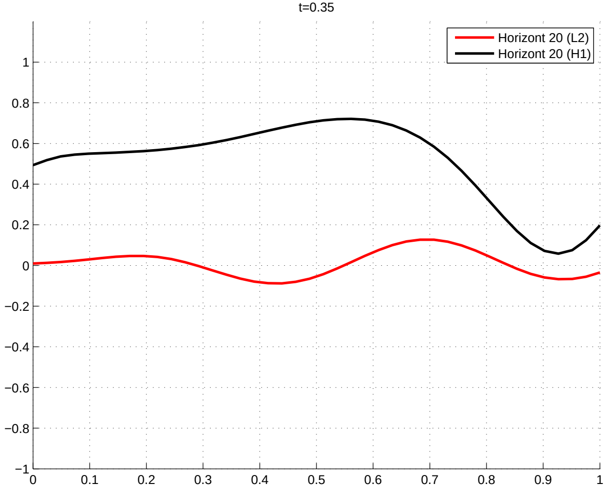

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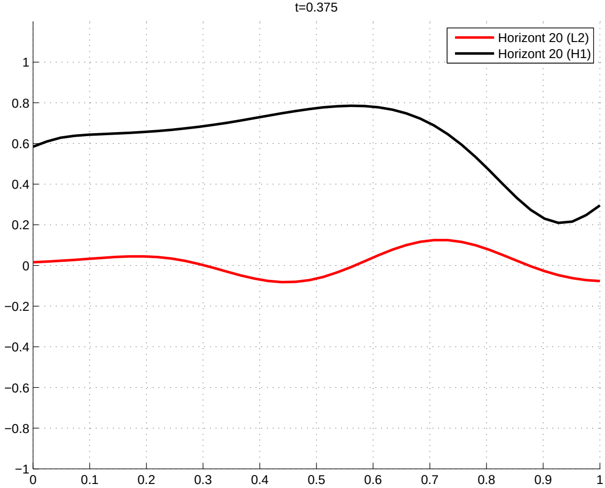

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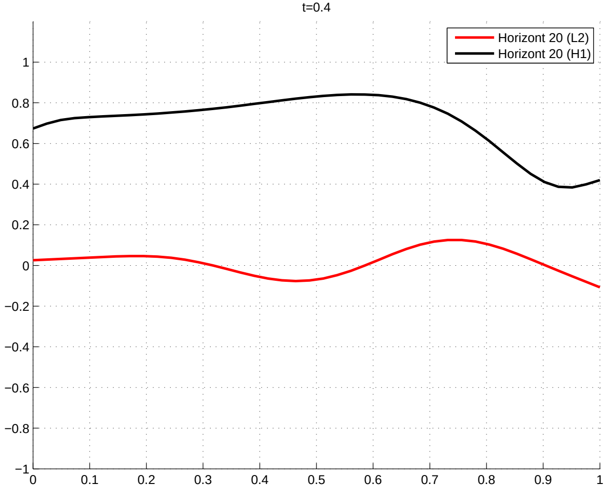

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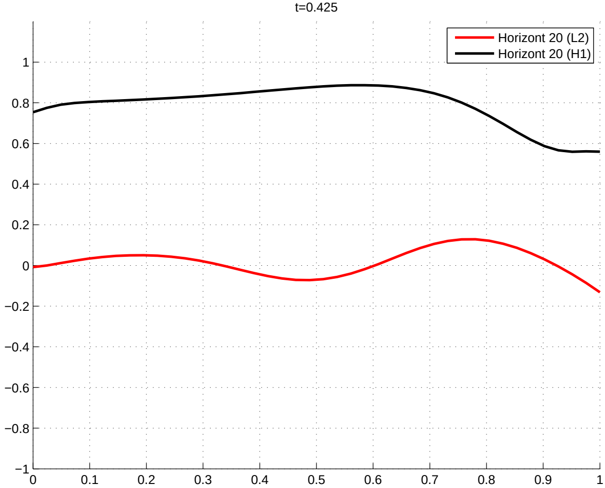

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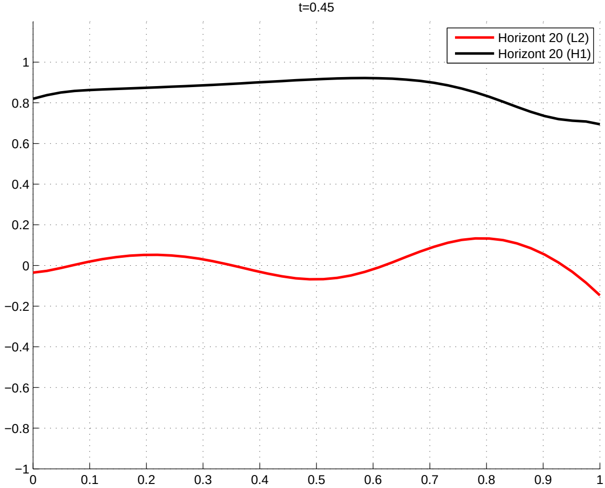

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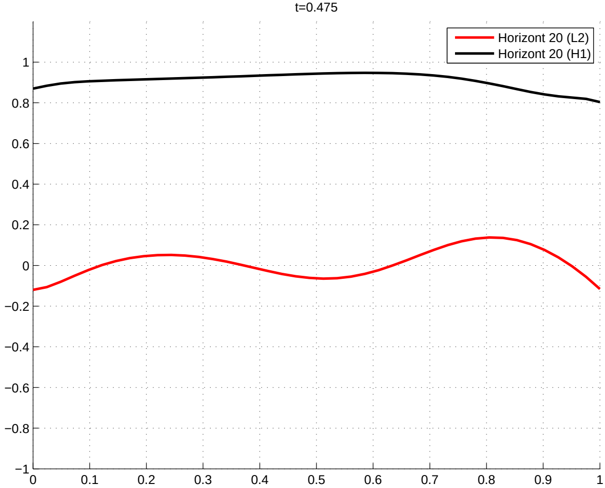

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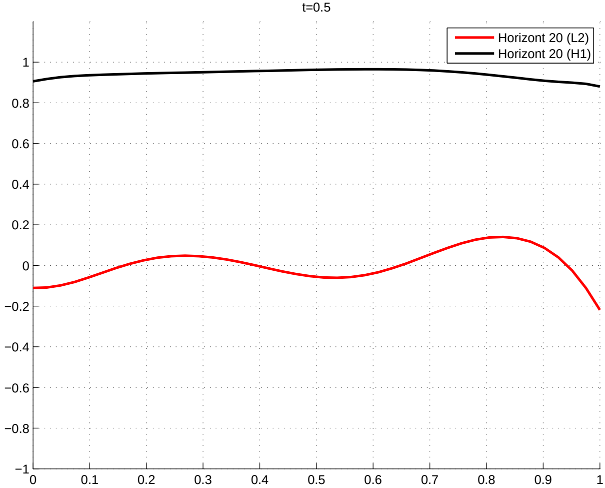

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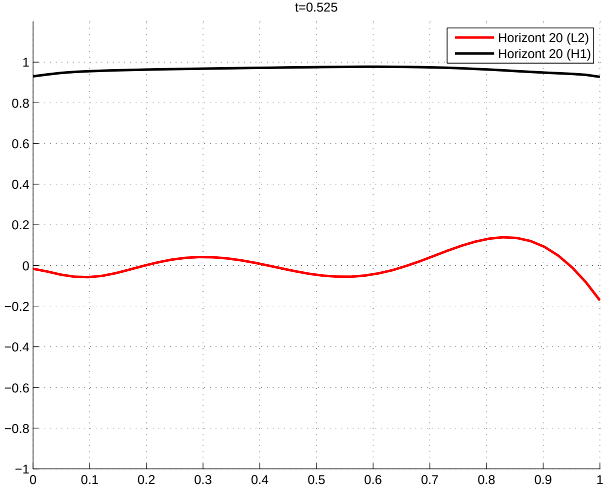

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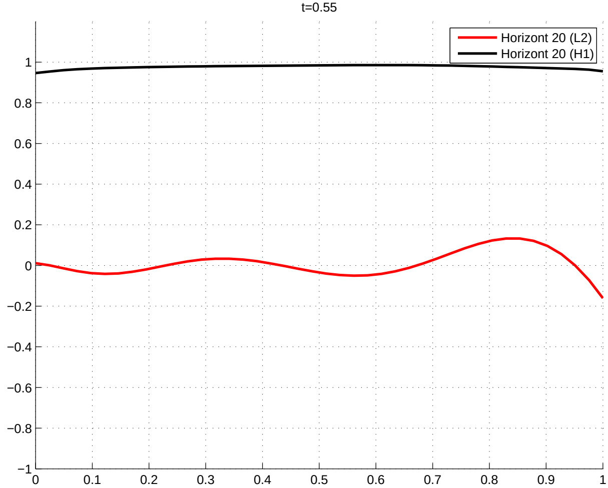

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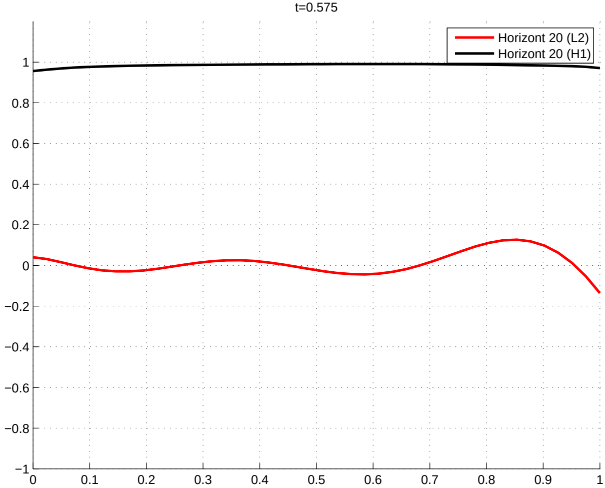

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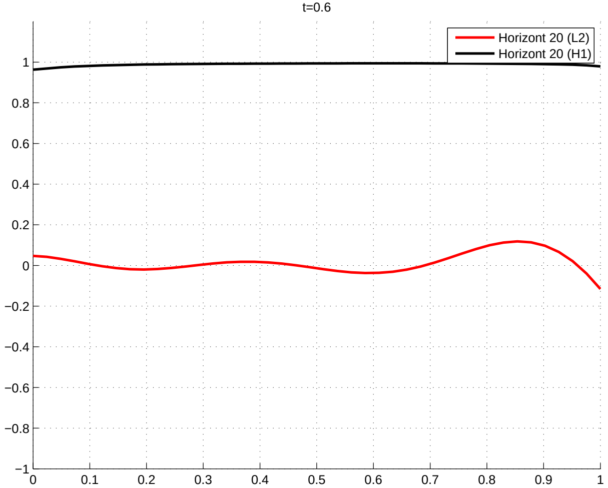

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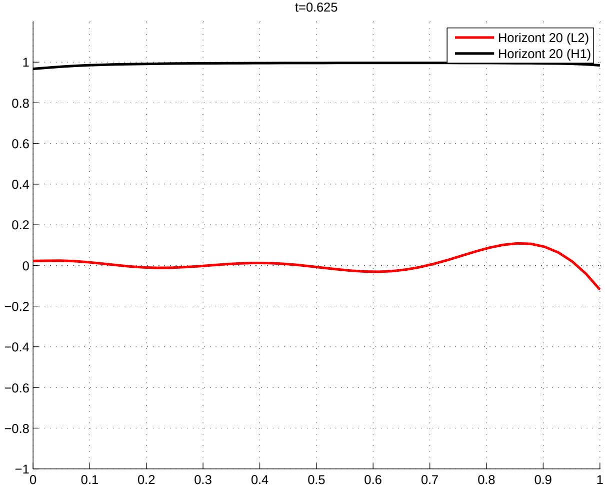

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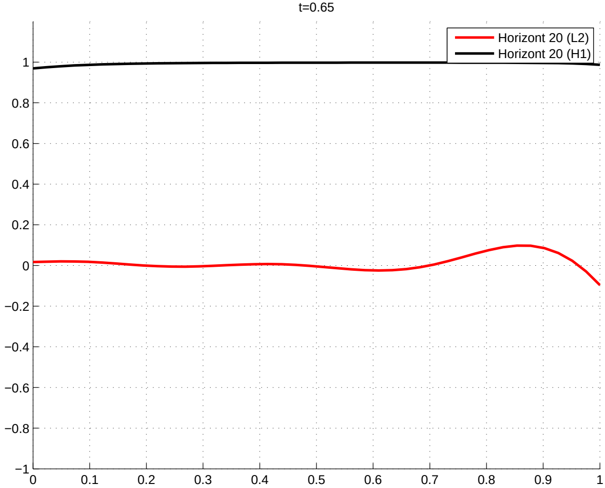

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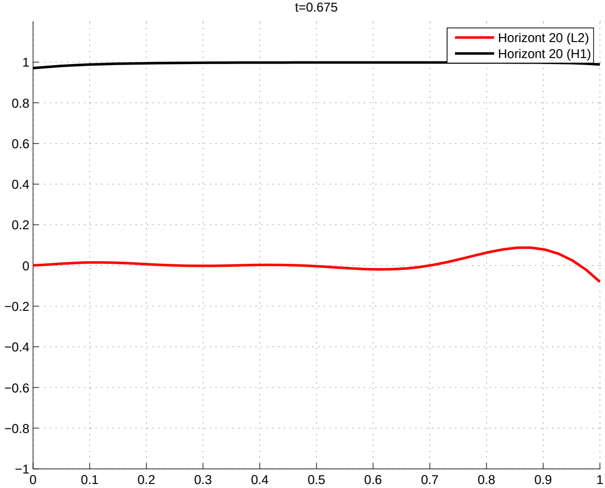

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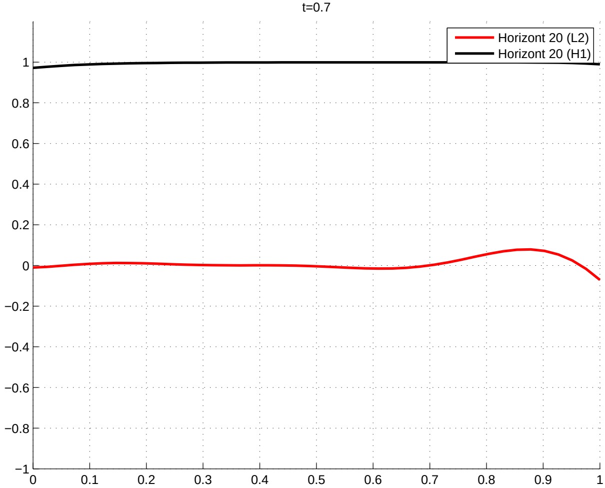

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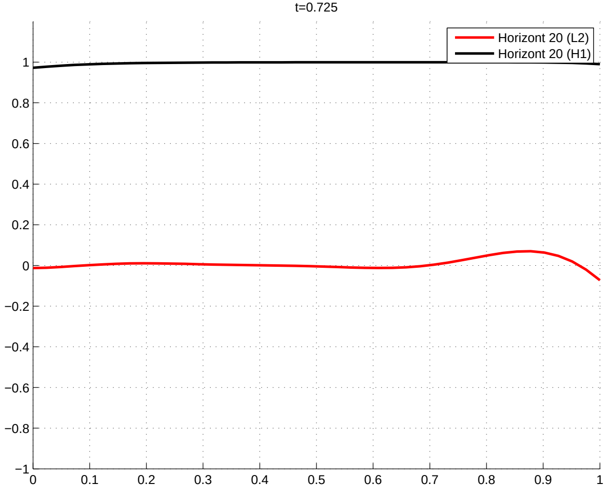

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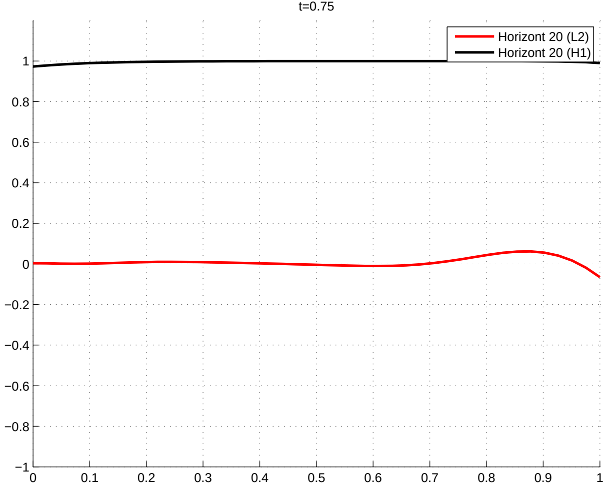

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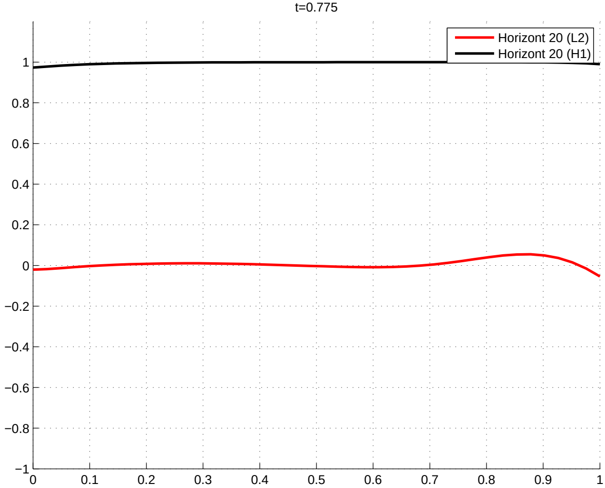

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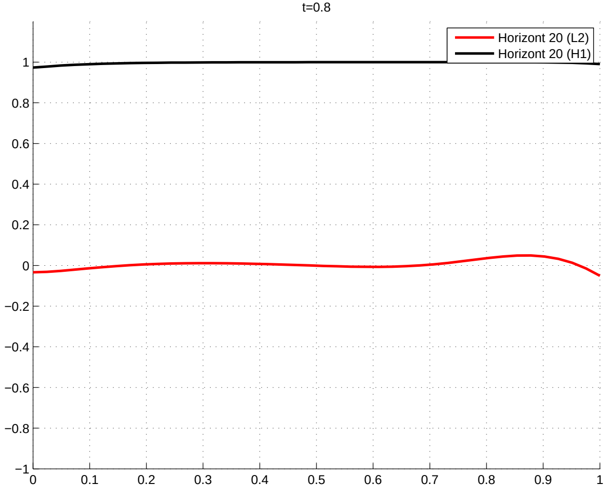

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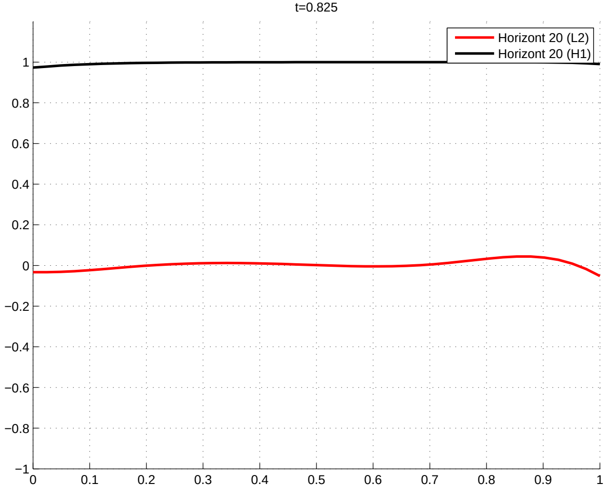

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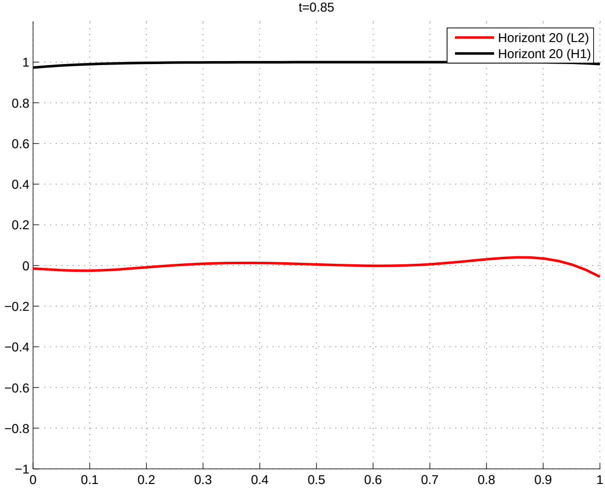

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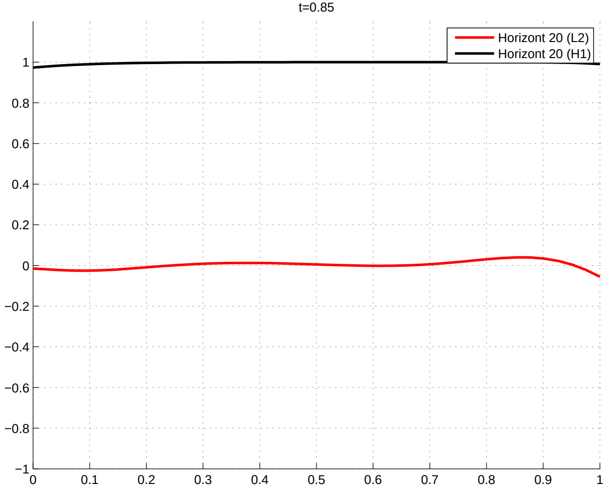

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Can be made rigorous for many PDEs [Altmüller et al. '10ff]

Summary of Section (6)

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- Computing tight estimates for C is in general a difficult if not impossible task
- But structural knowledge of the system behavior can be sufficient for choosing a "good" ℓ

(7) Feasibility

Feasibility

Consider the feasible sets

$$\mathcal{F}_N := \{x \in \mathbb{X} \,|\, \text{there exists an admissible } \mathbf{u} \text{ of length } N\}$$

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What happens if $\mathcal{F}_N \neq \mathbb{X}$ for some $N \in \mathbb{N}$?

The MPC feasibility problem

Even though the open-loop optimal trajectories are forced to satisfy $x^\star(k) \in \mathbb{X}$, the closed loop solutions $x_{\mu_N}(n)$ may violate the state constraints, i.e., $x_{\mu_N}(n) \notin \mathbb{X}$ for some n

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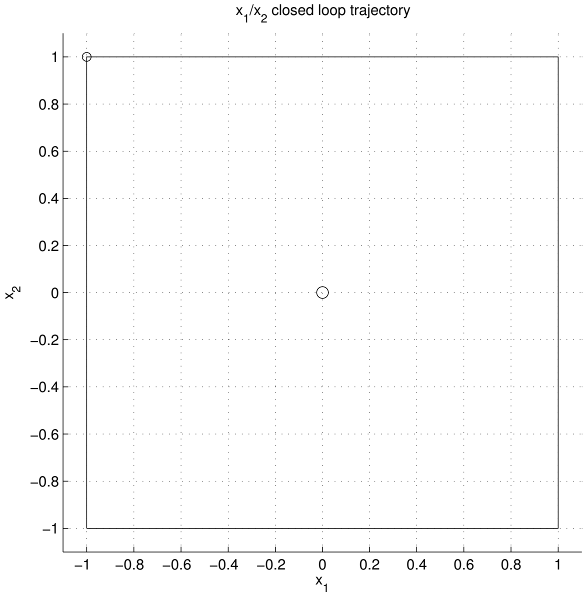
We illustrate this phenomenon by the simple example

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + u/2 \\ x_2 + u \end{pmatrix}$$

with $\mathbb{X} = [-1,1]^2$ and $\mathbb{U} = [-1/4, 1/4]$. For initial value $x_0 = (-1,1)^T$, the system can be controlled to 0 without leaving $\mathbb{X}$

We use MPC with $N = 2$ and $\ell(x,u) = \|x\|^2 + 5u^2$

The MPC feasibility problem: example



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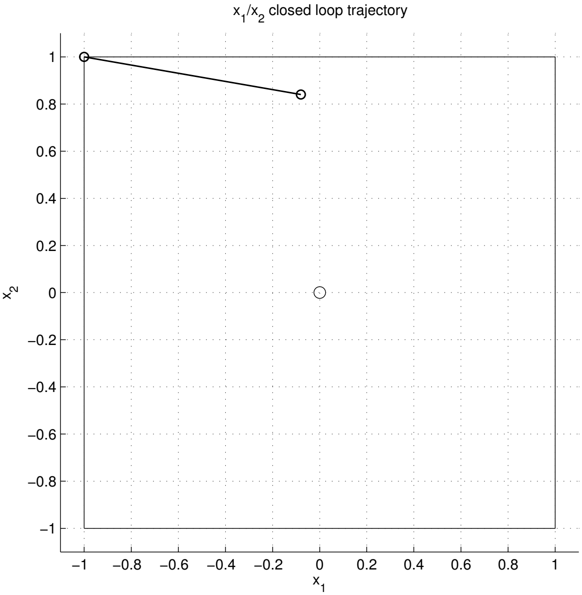

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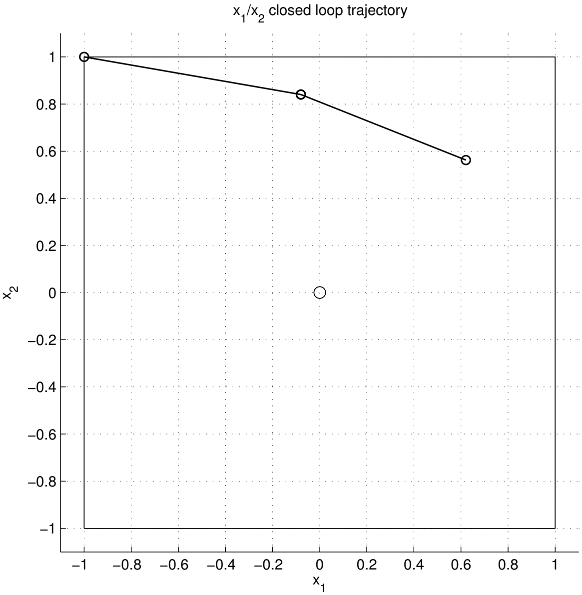

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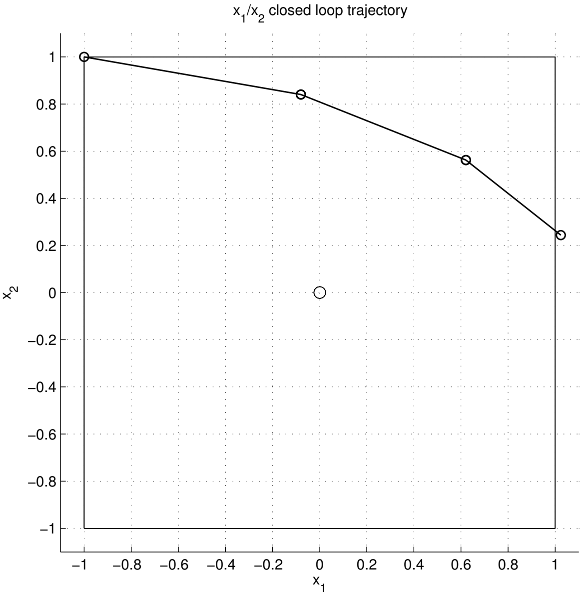

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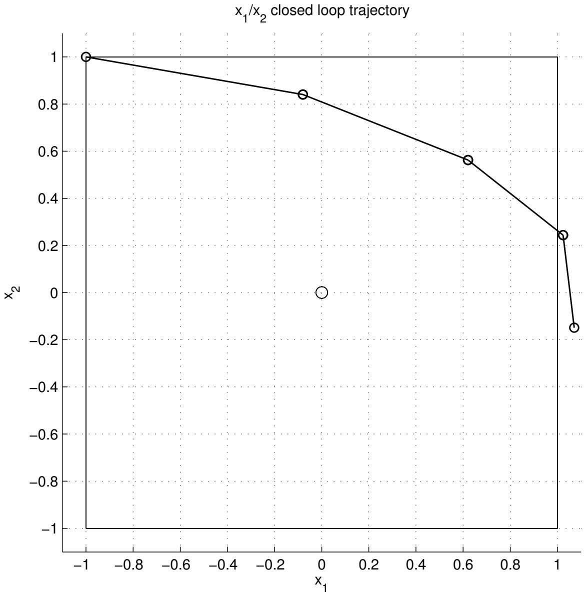

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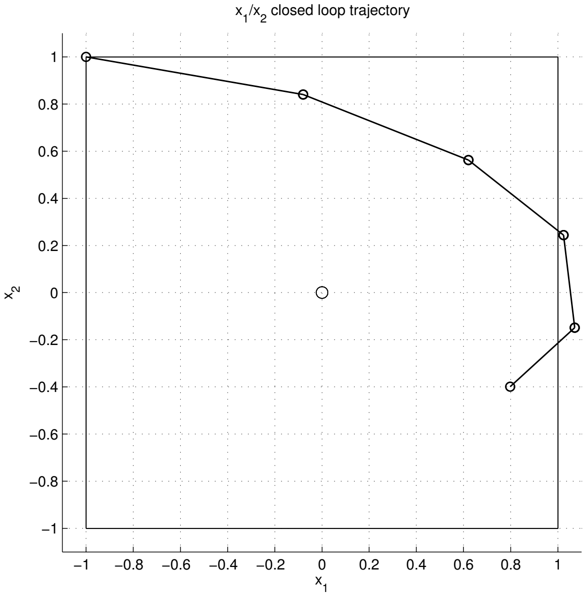

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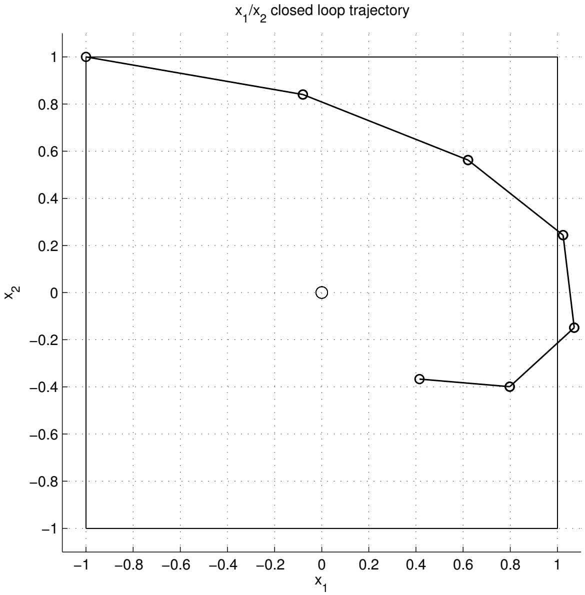

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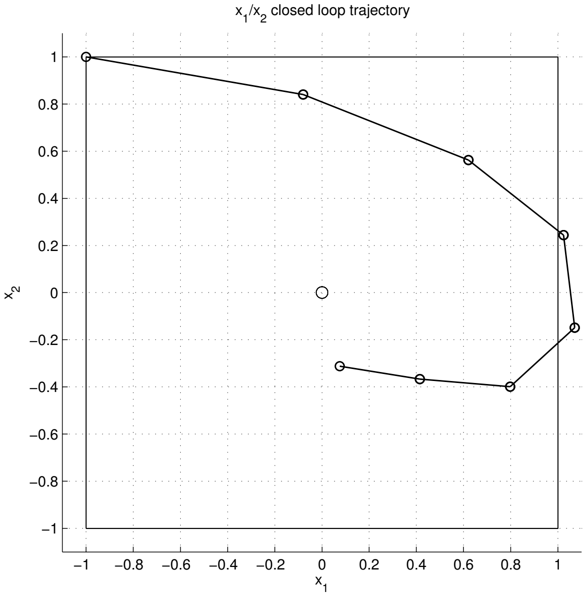

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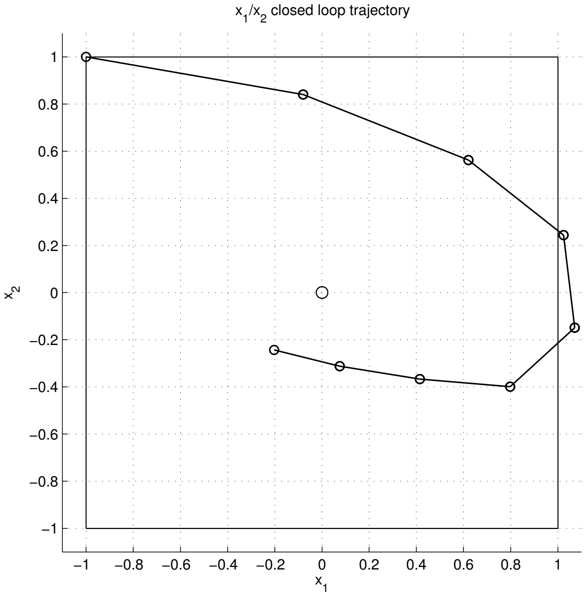

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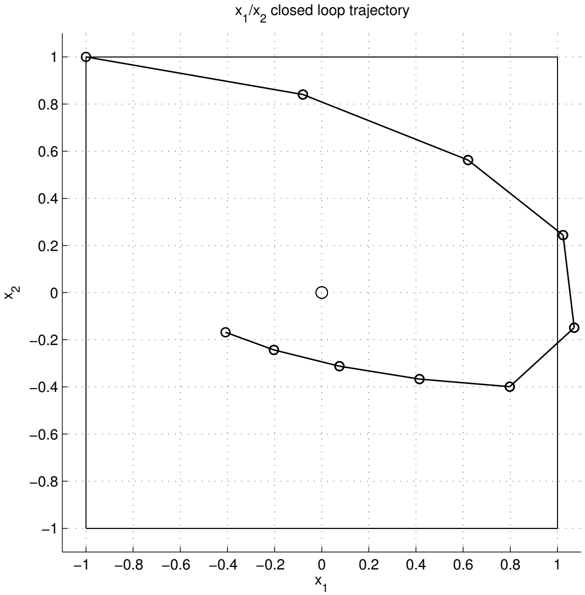

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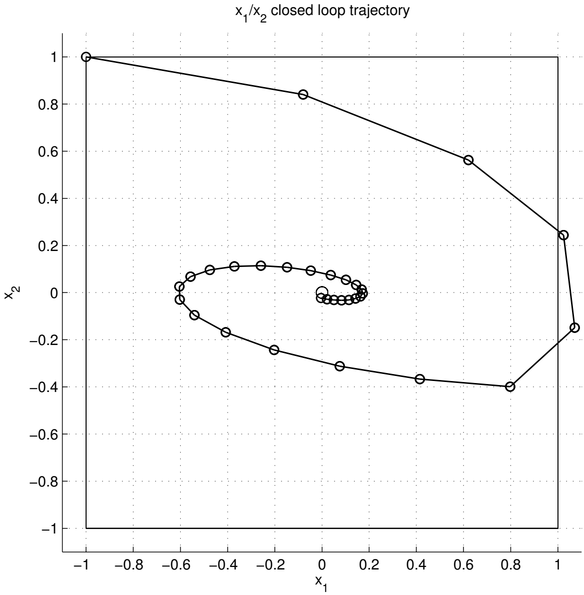

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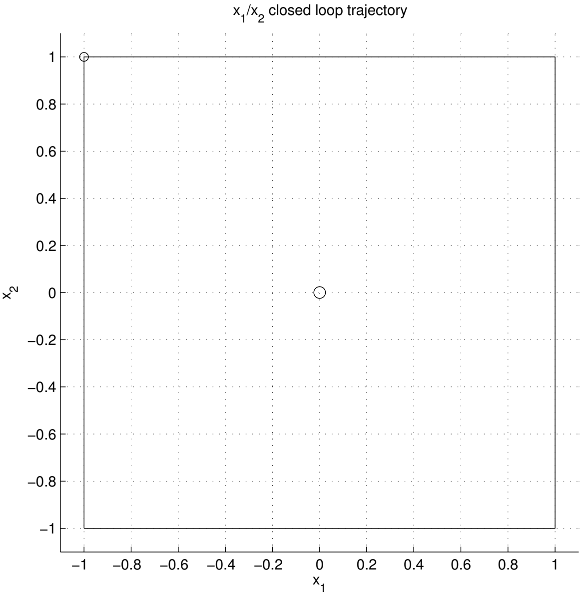

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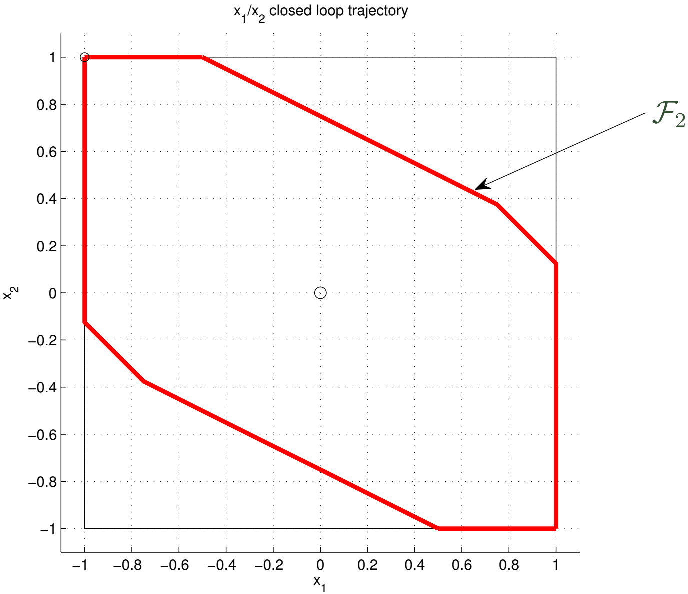

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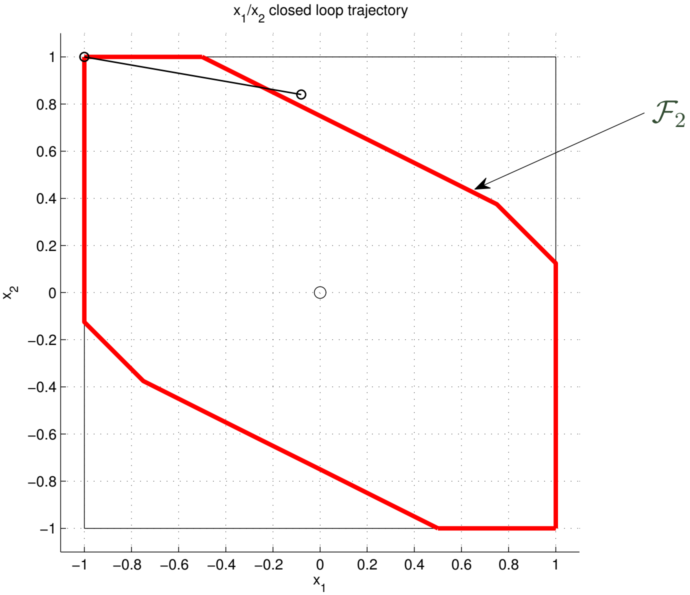

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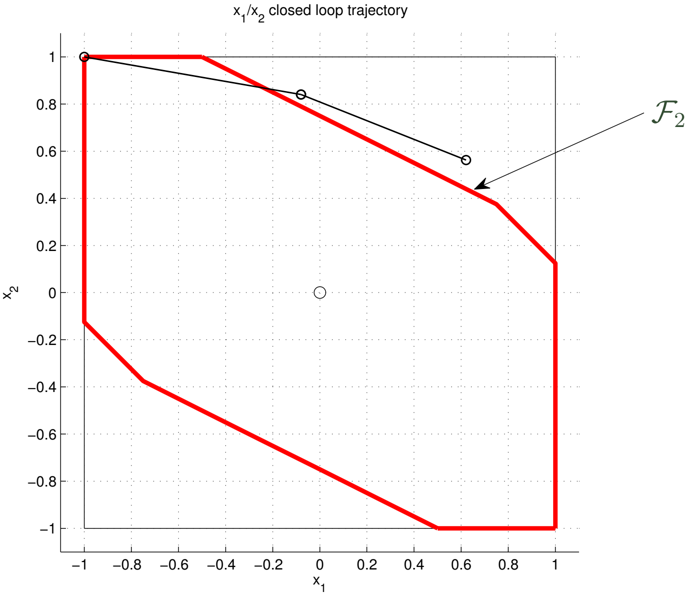

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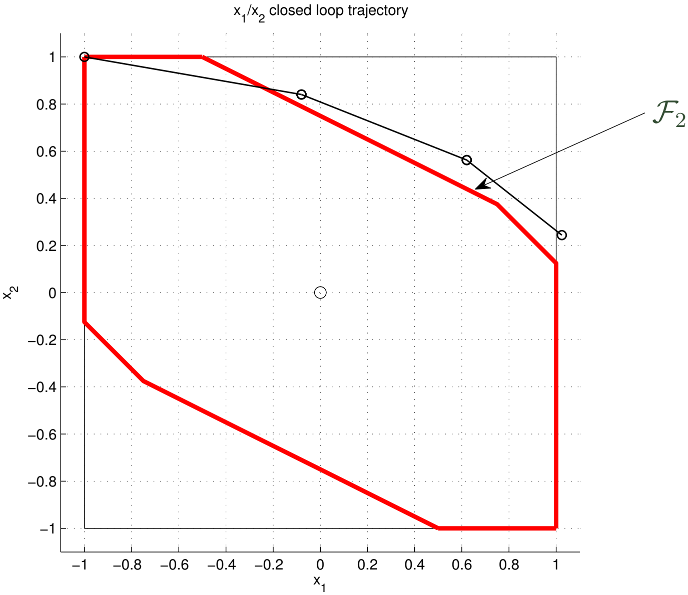

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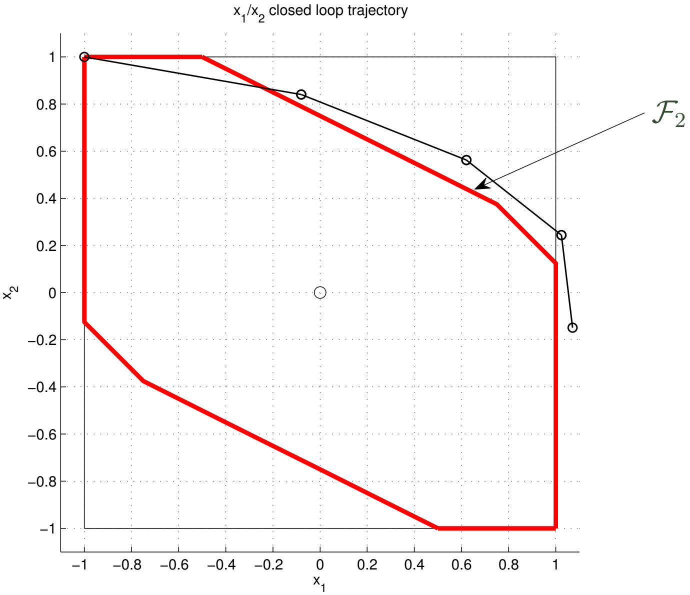

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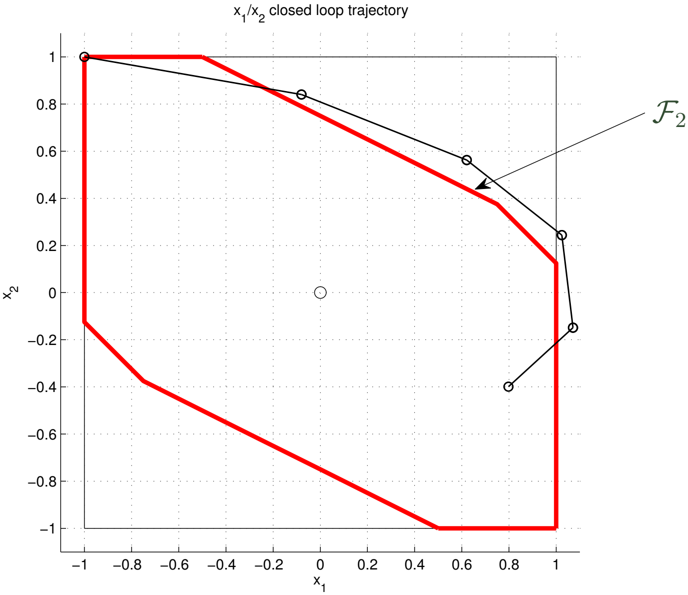

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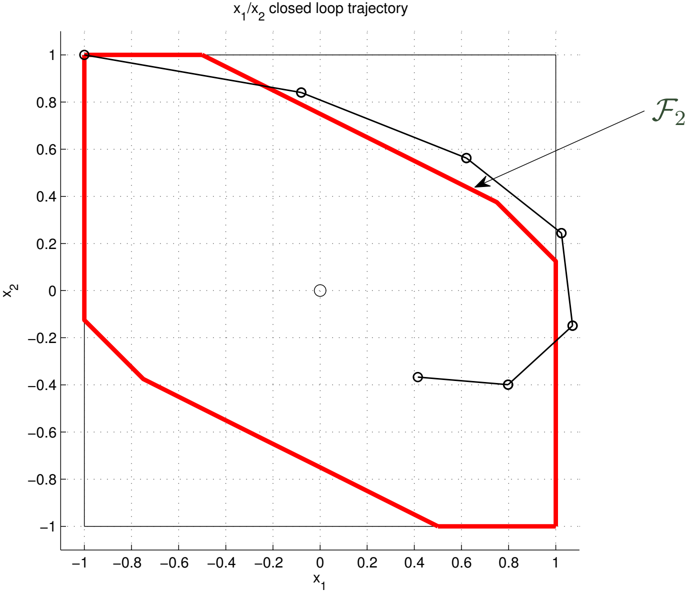

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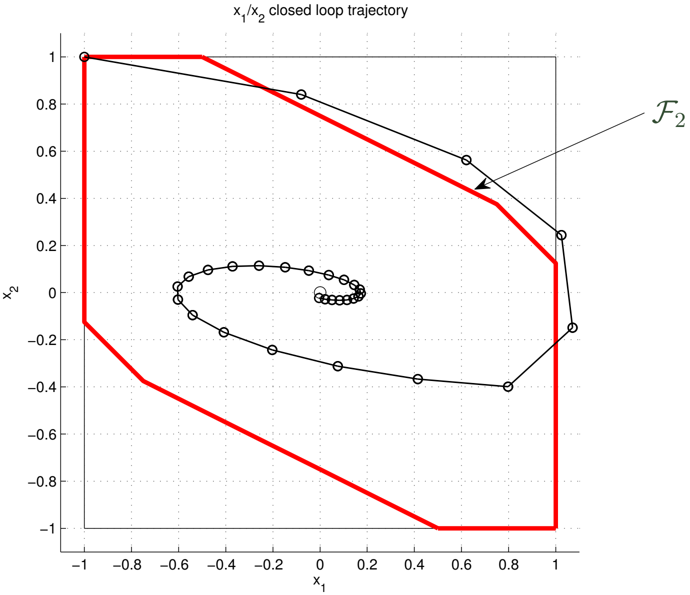

Recursive feasibility

The MPC scheme with horizon N is well defined on a set $A \subseteq \mathcal{F}_N$ if the following recursive feasibility condition holds:

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Can we find recursively feasible sets for NMPC without terminal constraints?

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$\Rightarrow$ recursive feasibility of $\mathcal{F}_N$

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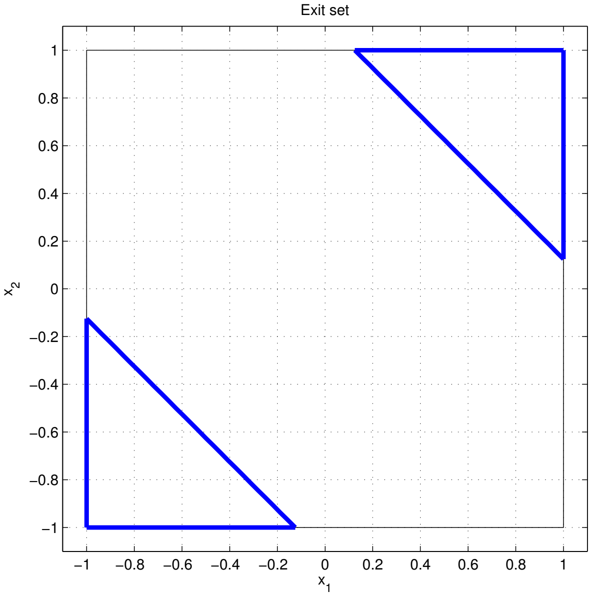
Feasible sets for our example



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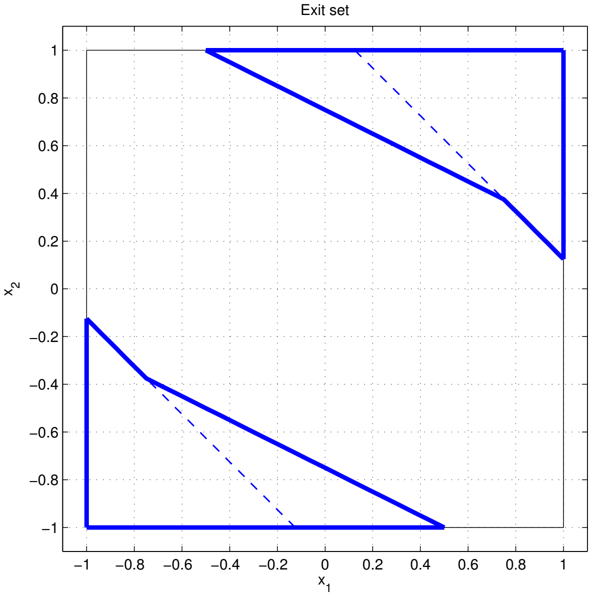

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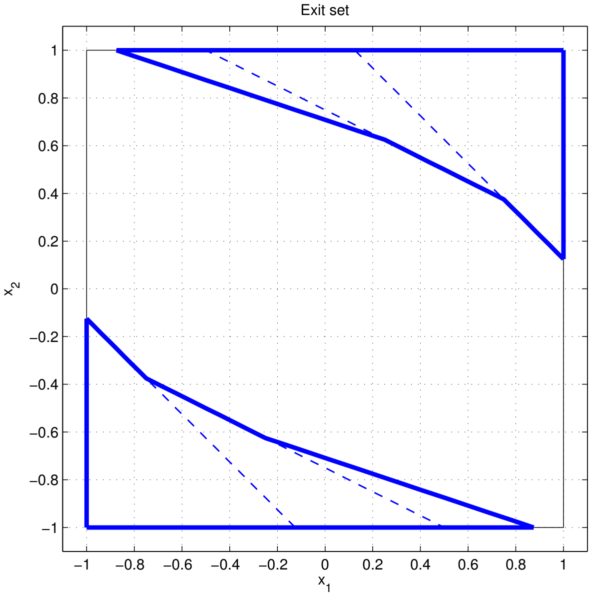

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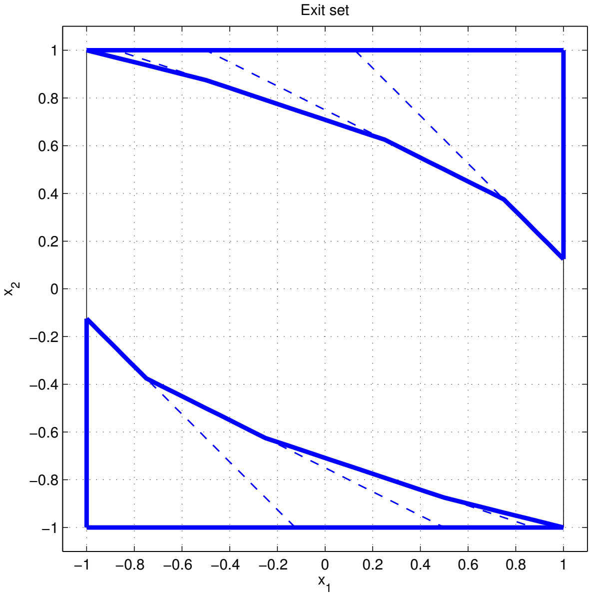

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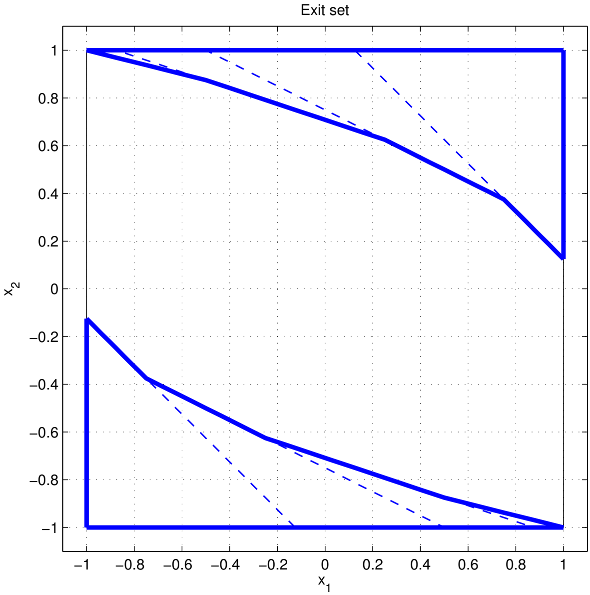

$\mathbb{X} \setminus \mathcal{F}_5$

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If $\mathbb{X}$ is compact, then $A_c = \mathcal{F}_\infty$ for all sufficiently large N

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- ⊖ may need larger N for obtaining stability near x_*

Part B: Economic Model Predictive Control

(8) Economic MPC with terminal constraints

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Idea: Use a stage cost ℓ which does not penalize the distance to some x_* but directly encodes the desired economic criterion

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Mathematical difference of stabilizing and economic MPC

In stabilizing MPC, the stage cost $\ell(x, u)$ penalizes the distance to some equilibrium $(x_*, u_*) \in \mathbb{X} \times \mathbb{U}$. In particular, we required

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We still consider equilibria, but they are now implicitly defined via the optimization criterion. In order to distinguish them from (x_*,u_*) in stabilizing MPC, they are denoted by (x^e,u^e)

Example 1: mimimum energy control

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$$\ell(x,u) = u^2$$

with dynamics

$$x(n+1) = 2x(n) + \mathbf{u}(n)$$

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For this example, it is optimal to control the system to $x^e = 0$ and keep it there with $u^e = 0$ $\quad \rightsquigarrow \quad \ell(x^e, u^e) = 0$

Example 2: a macroeconomic problem

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For this example, the optimal control policy is less obvious

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To this end, recall that $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ is an equilibrium, if

$$f(x^e, u^e) = x^e$$

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Economic MPC with terminal constraints

Theorem: [Angeli/Amrit/Rawlings '09] Consider an economic MPC problem with bounded optimal value function V_N which the optimal control problem

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with terminal constraint $x_{\mathbf{u}}(N) = x^e$ is used to generate the MPC feedback law μ_N. Then the inequality

$$\overline{J}^{cl}_{\infty}(x, \mu_N) \leq \ell(x^e, u^e)$$

holds for the averaged closed loop functional

$$\overline{J}^{cl}_{\infty}(x, \mu_N) := \limsup_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(x_{\mu_N}(k, x), \mu(x_{\mu_N}(k, x)))$$

Sketch of proof

Prolonging an optimal control $\mathbf{u}^\star$ with length N at the end by the control value u^e yields a control $\mathbf{u}$ satisfying

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which in turn yields

$$\ell(x, \mu_N(x)) \leq \ell(x^e, u^e) + V_N(x) - V_N(f(x, \mu_N(x)))$$

Sketch of proof

Prolonging an optimal control $\mathbf{u}^\star$ with length N at the end by the control value u^e yields a control $\mathbf{u}$ satisfying

$$J_{N+1}(x,\mathbf{u}) - V_N(x) \le \ell(x^e,u^e)$$

This implies

$$V_{N+1}(x) - V_N(x) \le \ell(x^e,u^e)$$

which in turn yields

$$\ell(x,\mu_N(x)) \le \ell(x^e,u^e) + V_N(x) - V_N(f(x,\mu_N(x)))$$

Summing and averaging then implies

$$\overline{J}_K^{cl}(x,\mu_N) \le \ell(x^e,u^e) + \frac{1}{K}\Big(V_N(x) - V_N(x_{\mu_N}(K))\Big)$$

which shows the assertion for $K\to\infty$, since V_N is bounded

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Can we give an easily checkable sufficient condition for the existence of such an equilibrium?

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Dissipativity

Given an equilibrium (x^e, u^e), we use the following

Definition: [Willems '72] The optimal control problem is called strictly dissipative if there exists $\lambda : \mathbb{X} \to \mathbb{R}$ and $\alpha \in \mathcal{K}_\infty$ such that

(D) $\quad \ell(x,u) + \lambda(x) - \lambda(f(x,u)) - \ell(x^e,u^e) \geq \alpha(\|x - x^e\|)$

holds for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ and some $\alpha \in \mathcal{K}_\infty$

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strict dissipativity: some amount of energy is dissipated (=lost)

Strict dissipativity

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Strict dissipativity (D) is

- satisfied for affine linear f and linear quadratic ℓ under mild regularity conditions on f, ℓ, $\mathbb{X}$ and $\mathbb{U}$ [Damm/Gr./Stieler/Worthmann '12]

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- more restrictive for nonlinear dynamics, see, e.g., the bilinear example in [Müller/Allgöwer '12]
- sufficient and "close to necessary" for the existence of an infinite horizon averaged optimal equilibrium [Müller/Angeli/Allgöwer '13]

Example 1: mimimum energy control

Example 1:

$$x(n+1) = 2x(n) + \mathbf{u}(n), \qquad \ell(x,u) = u^2$$

with constraints $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

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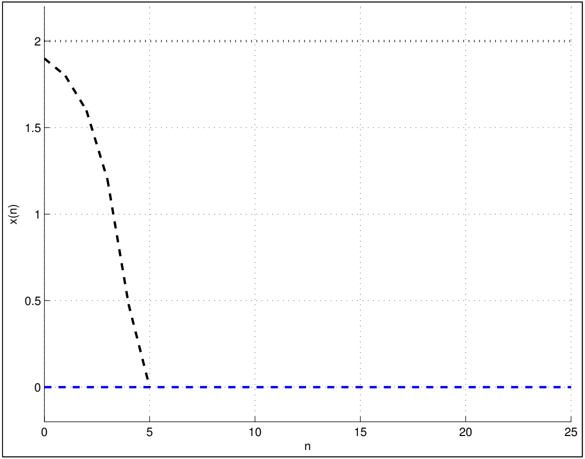
Using the terminal constraint $x_{\mathbf{u}}(N) = 0$, we will see that the closed loop trajectories converge to 0 (and the averaged functional equals 0)

Example 1: trajectories



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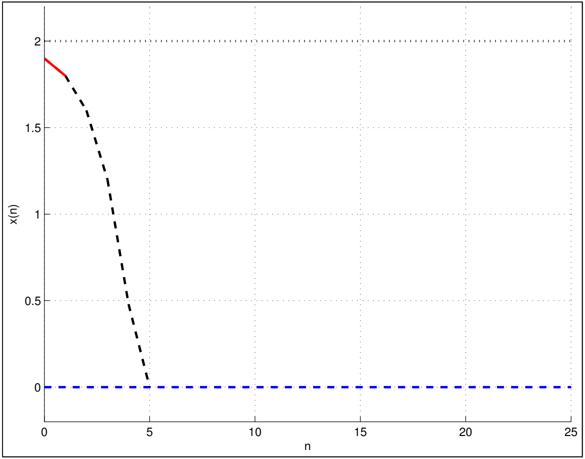

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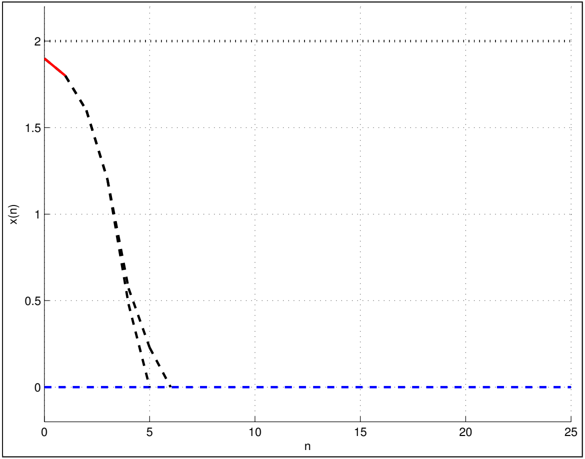

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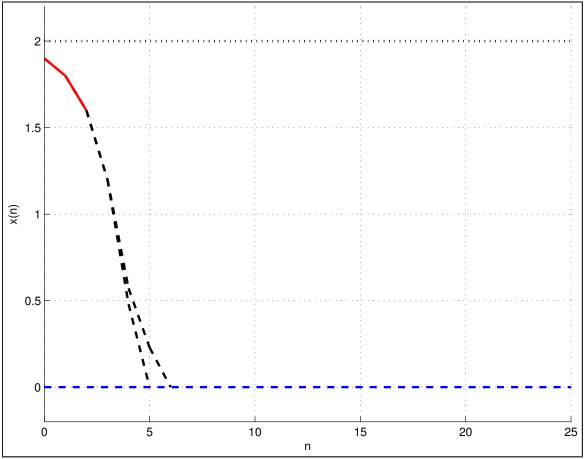

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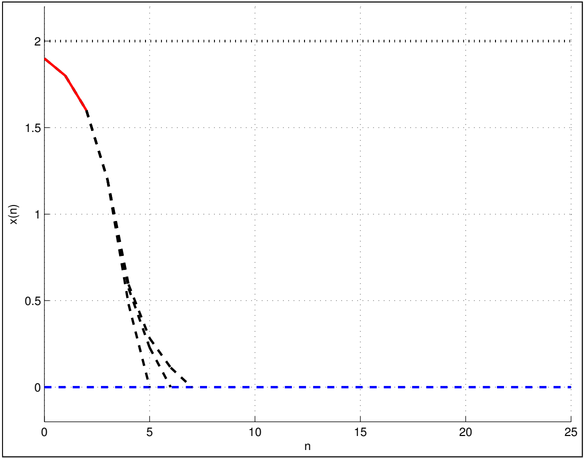

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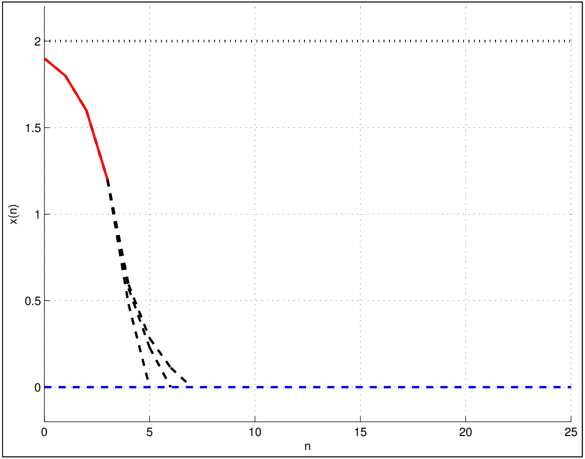

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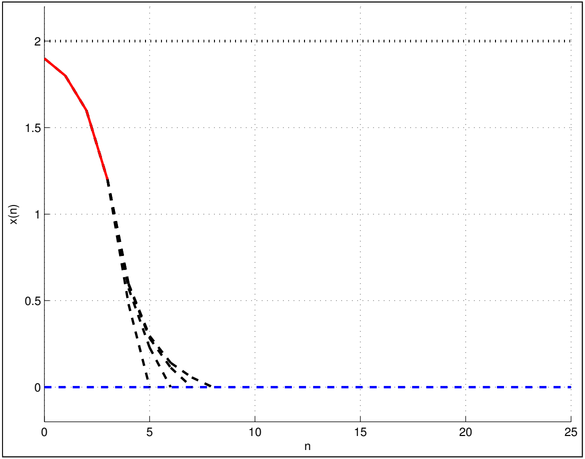

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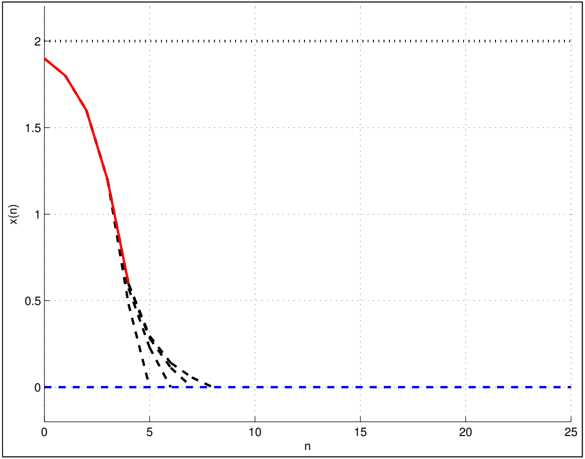

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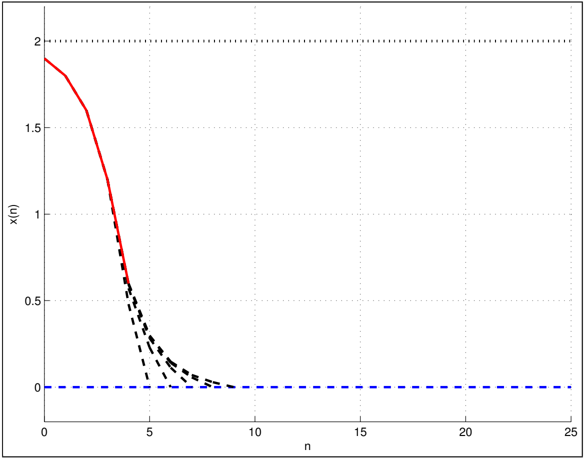

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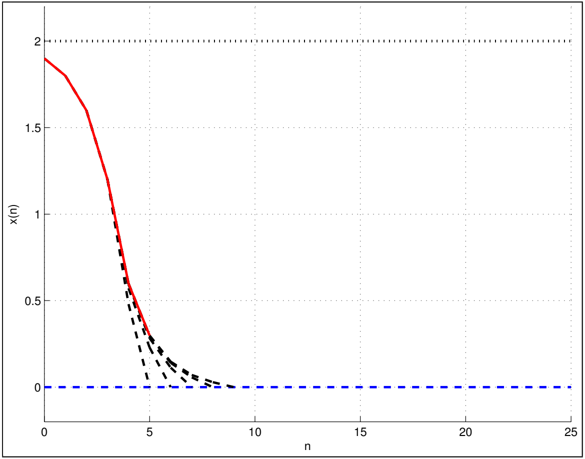

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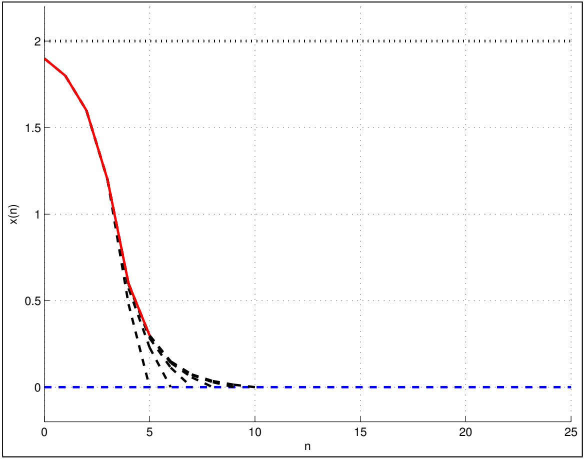

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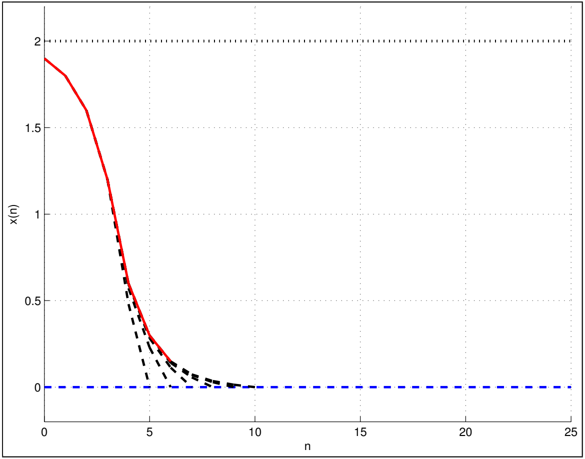

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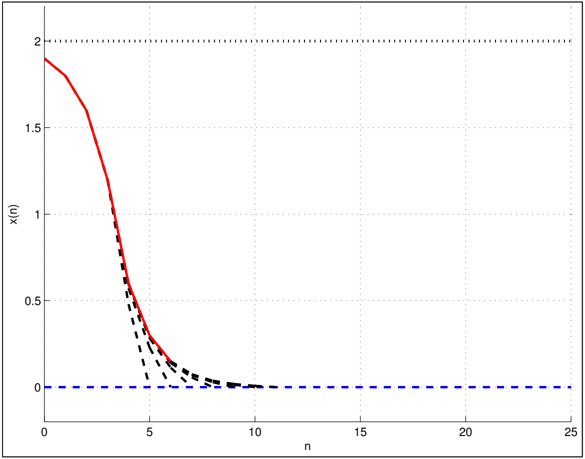

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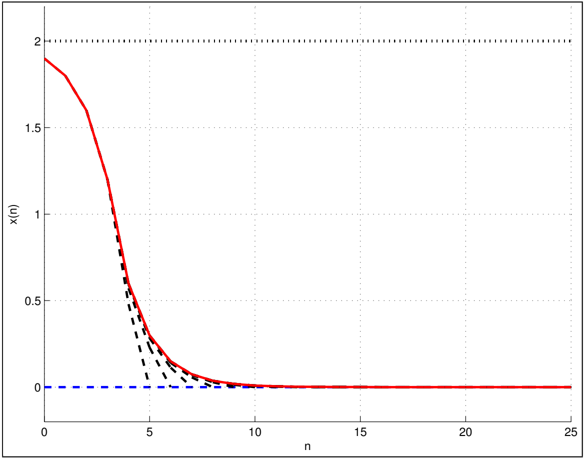

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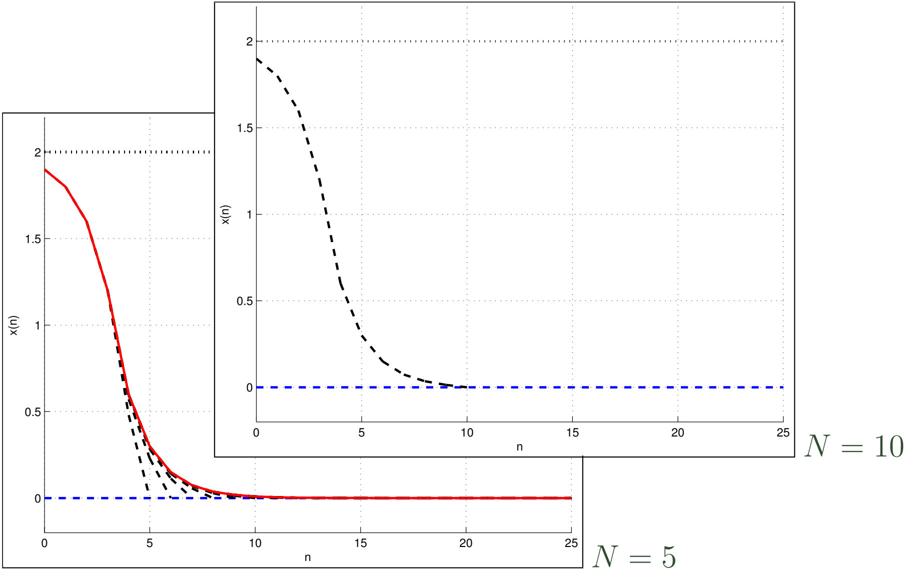

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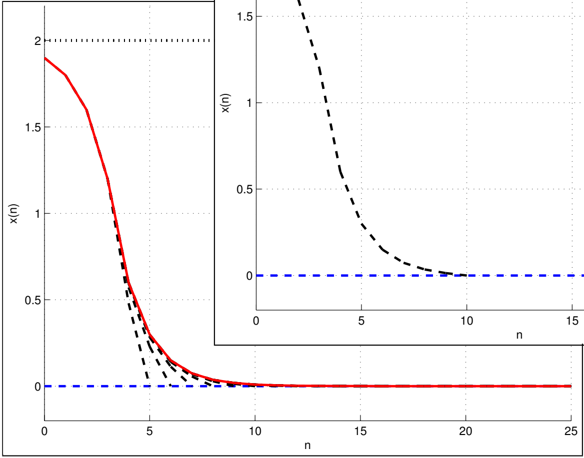

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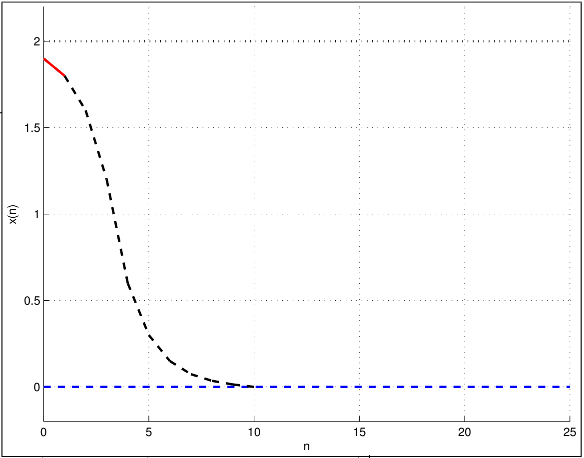

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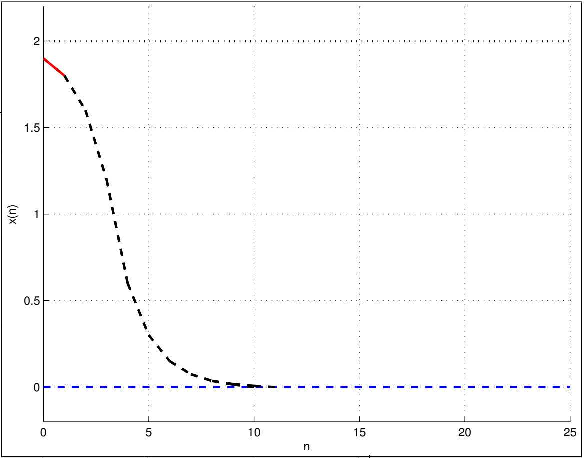

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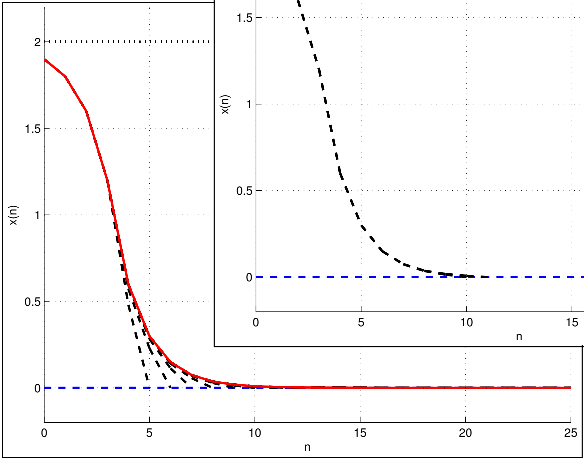

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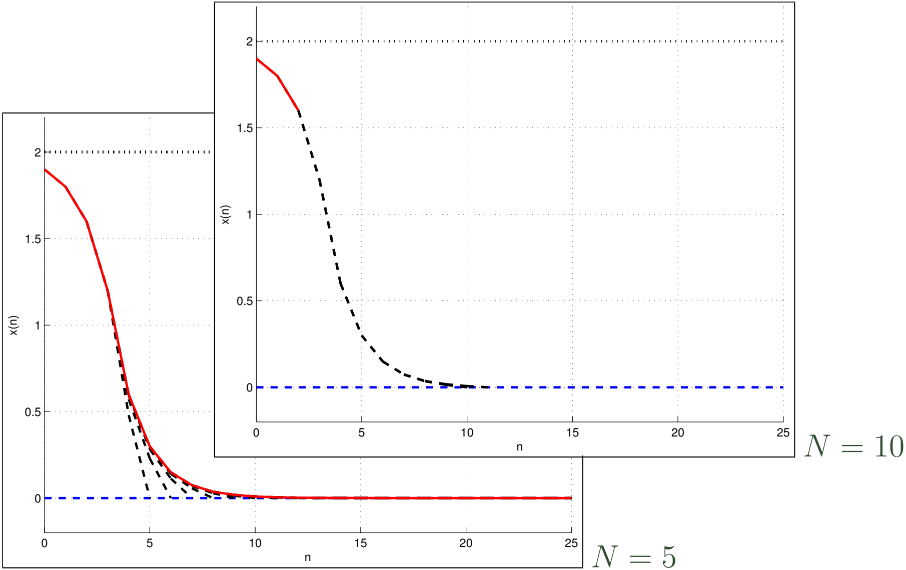

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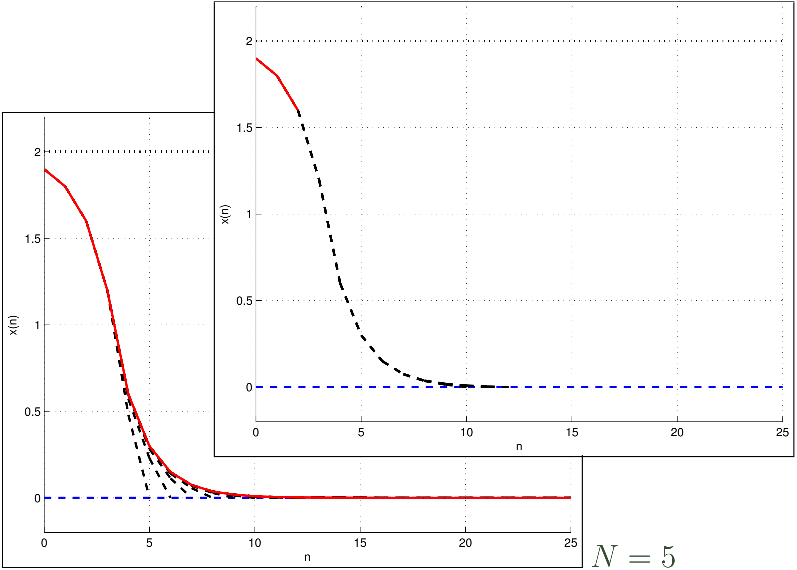

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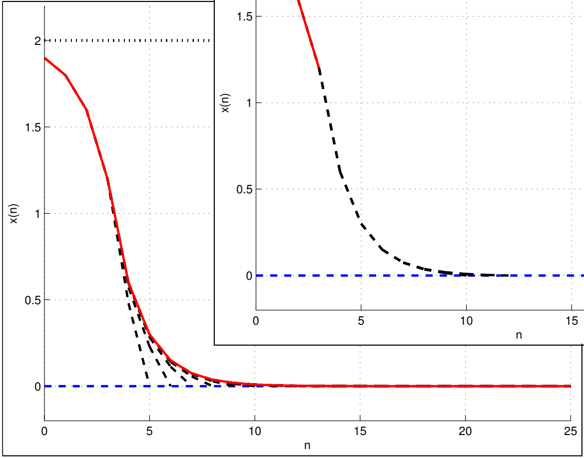

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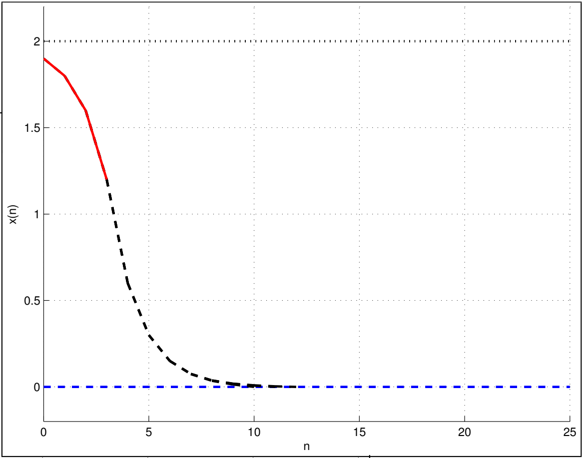

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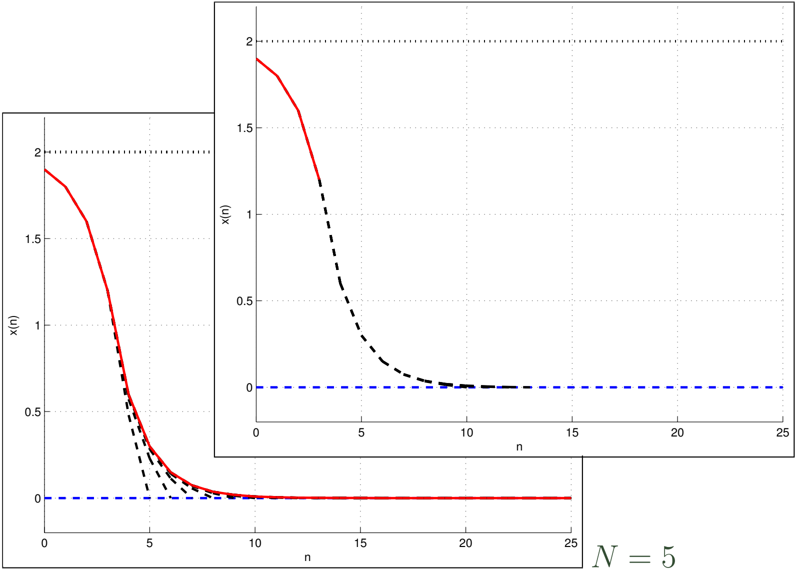

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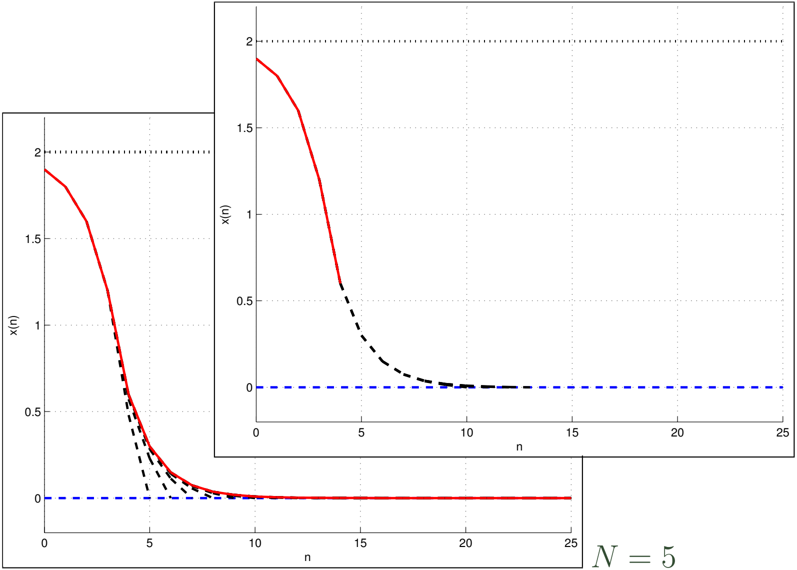

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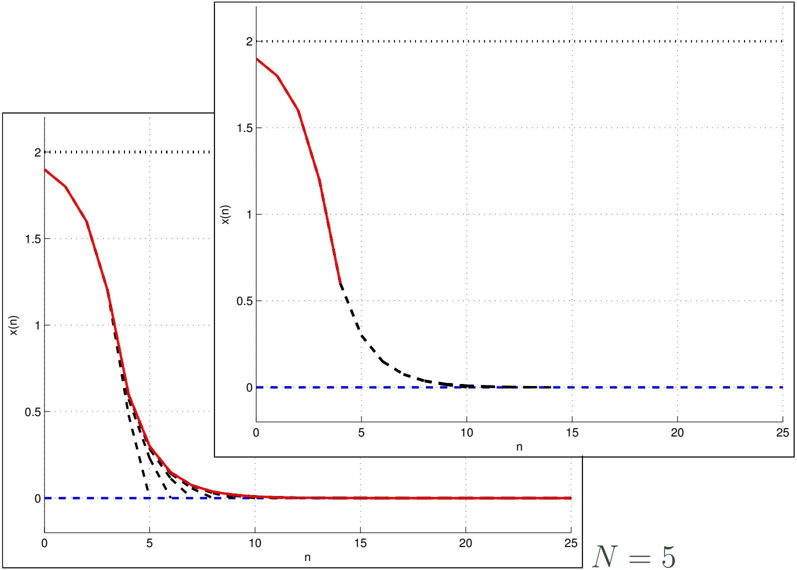

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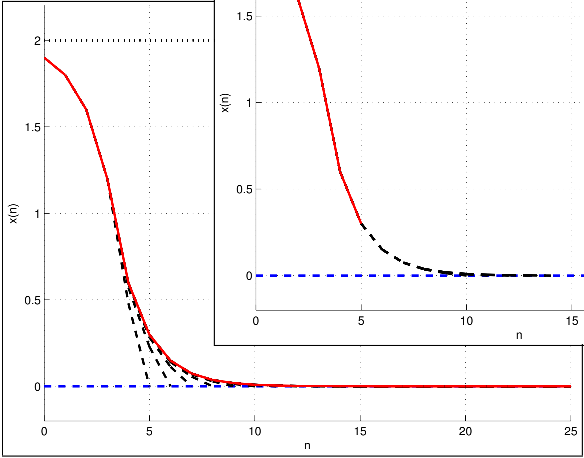

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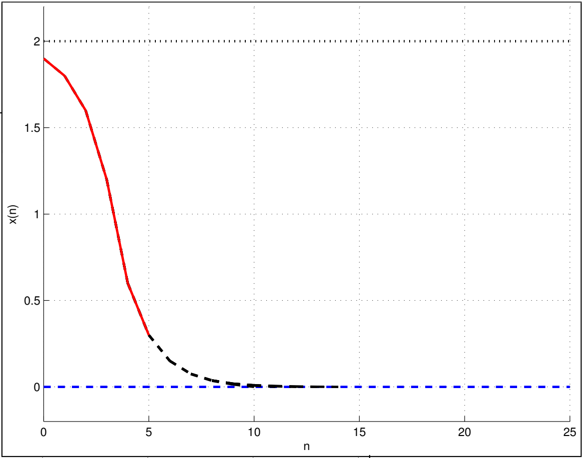

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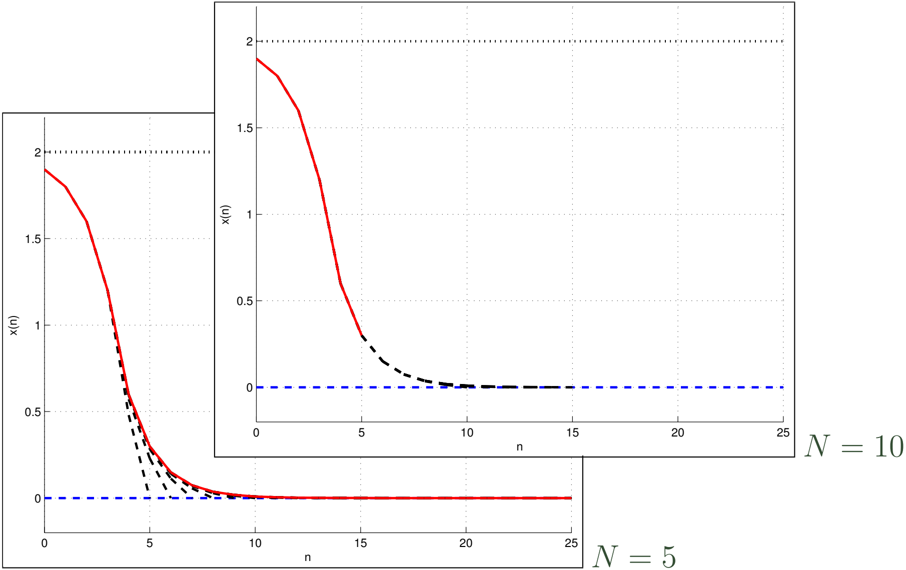

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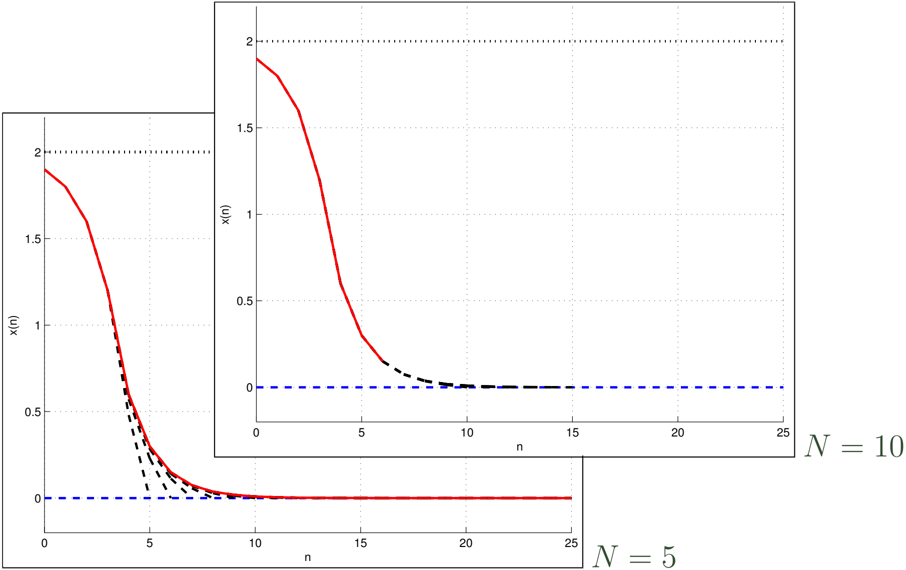

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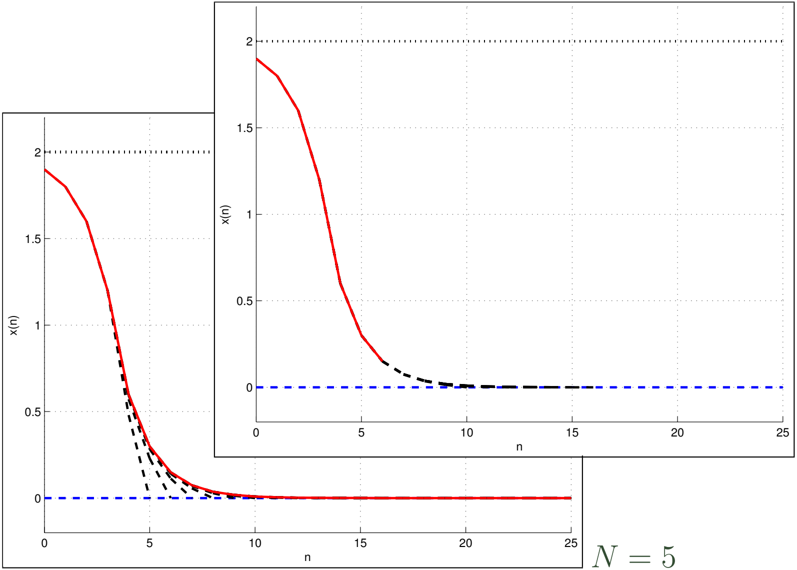

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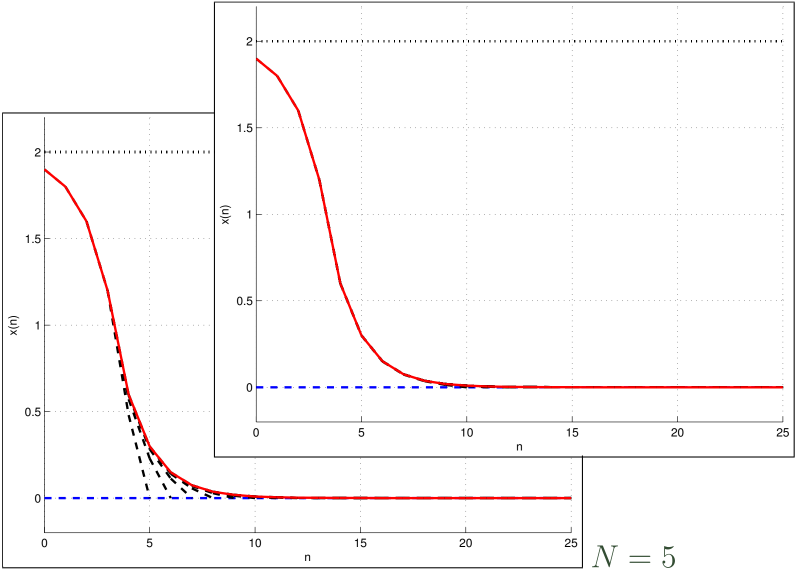

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Example 2: Macroeconomic model

[Brock/Mirman '72]

Minimize the average performance with

$$x(n+1) = \mathbf{u}(n), \quad \ell(x,u) = -\ln(Ax^{\alpha} - u)$$

with $A = 5, \alpha = 0.34$ and constraints $\mathbb{X} = [0.1, 10]$, $\mathbb{U} = [0.1, 5]$

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Again, with the terminal constraint $x_\mathbf{u}(N) = x^e$ the closed loop trajectories converge to x^e (and the averaged functional equals $\ell(x^e, u^e)$)

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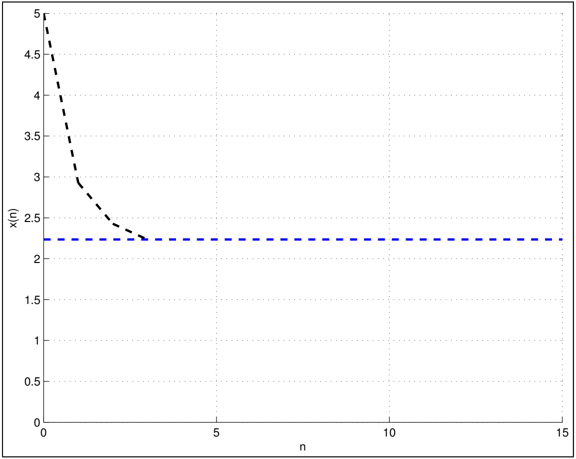

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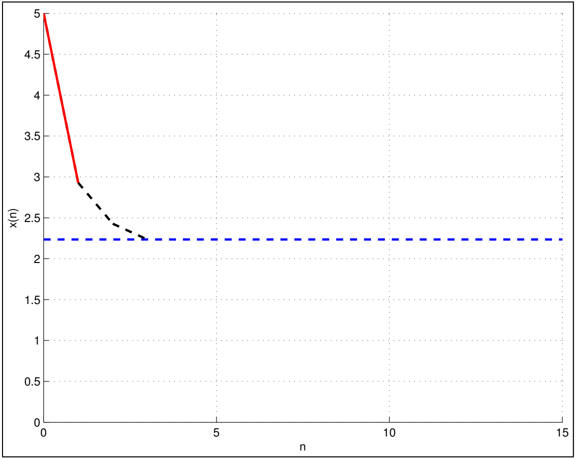

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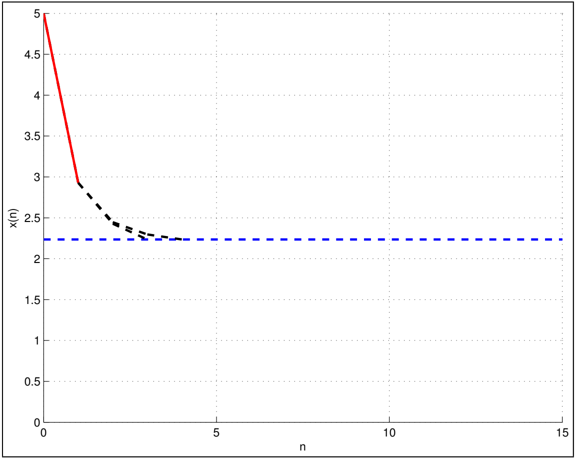

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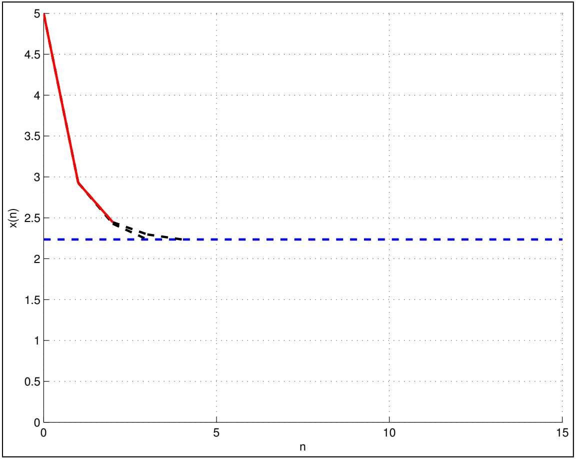

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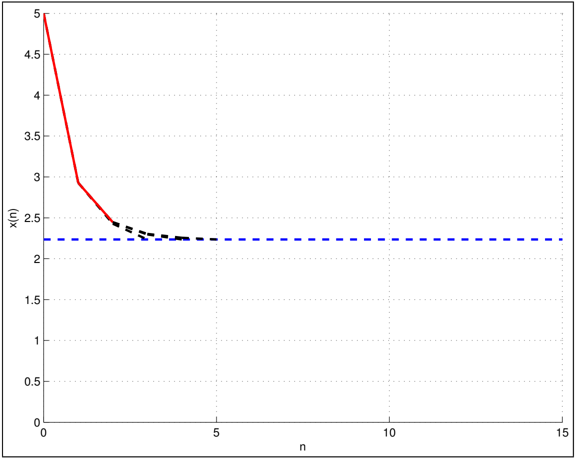

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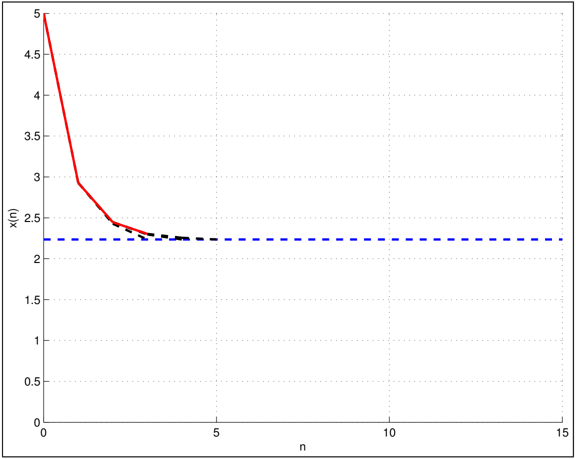

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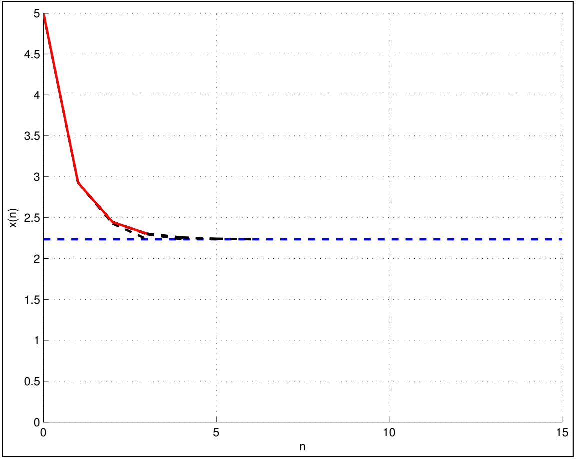

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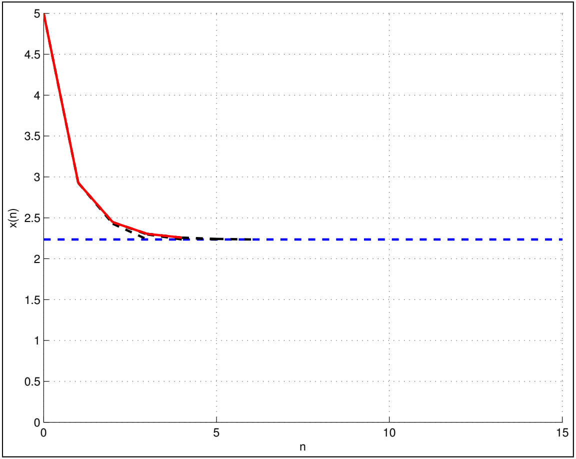

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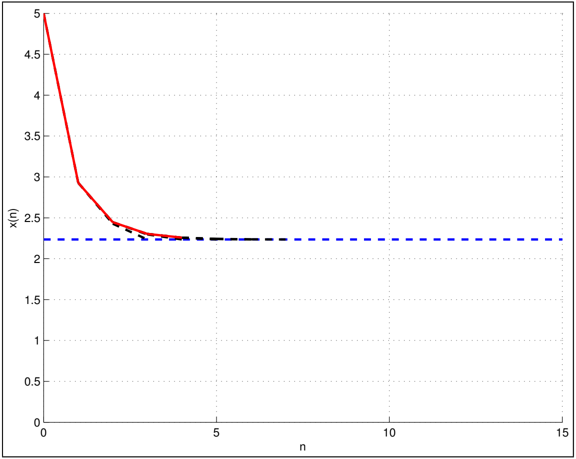

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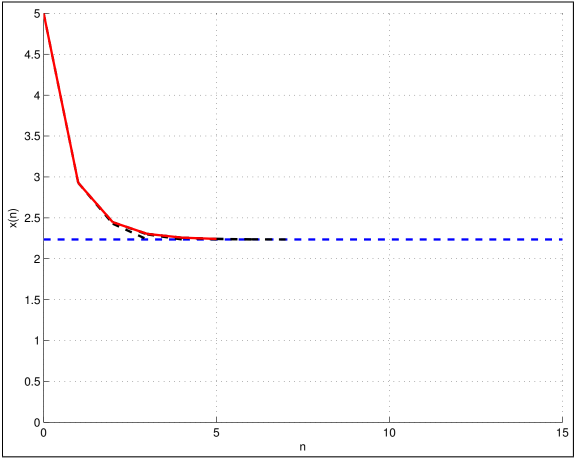

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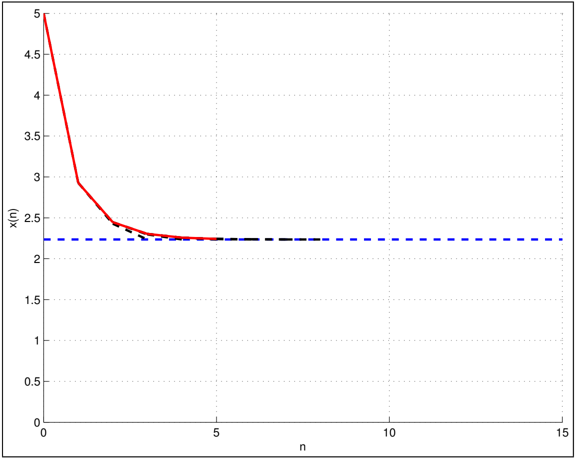

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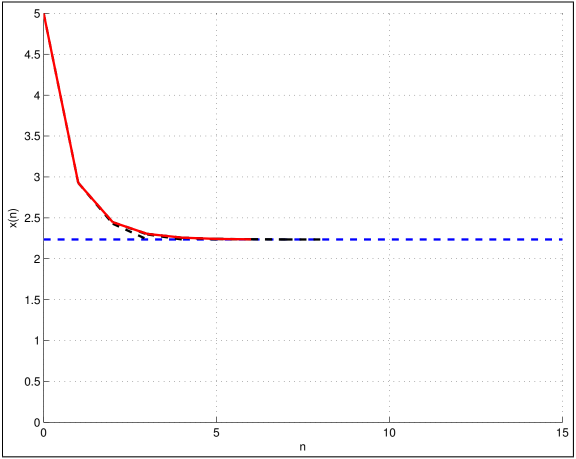

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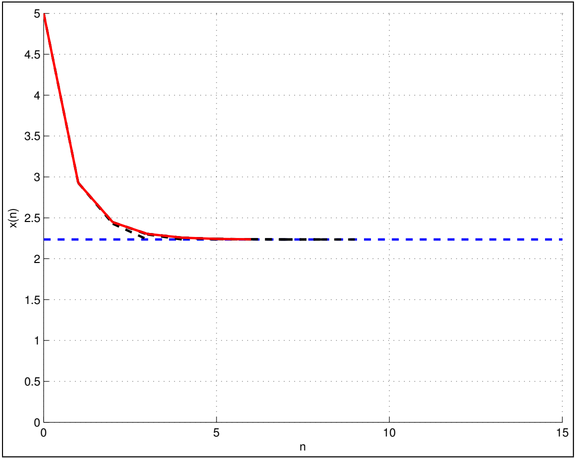

$N = 3$

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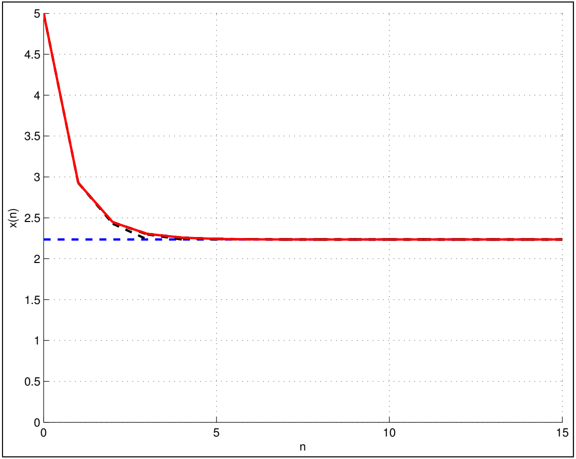

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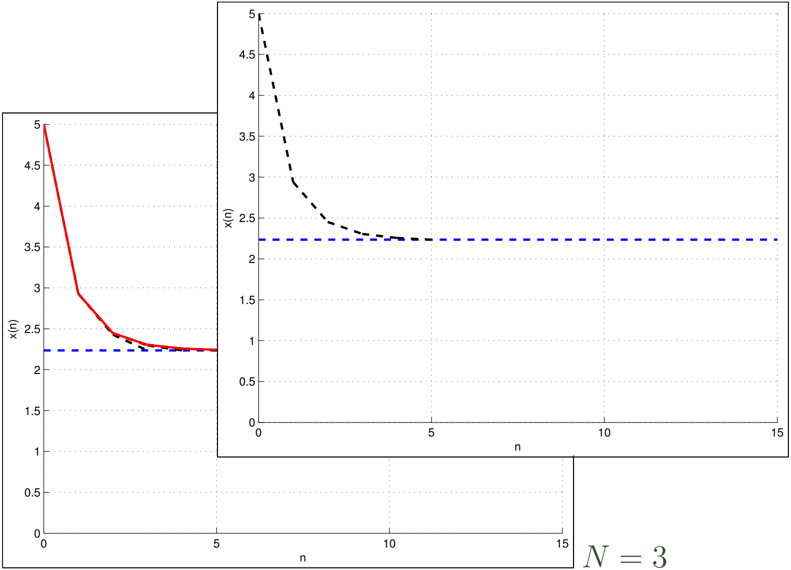

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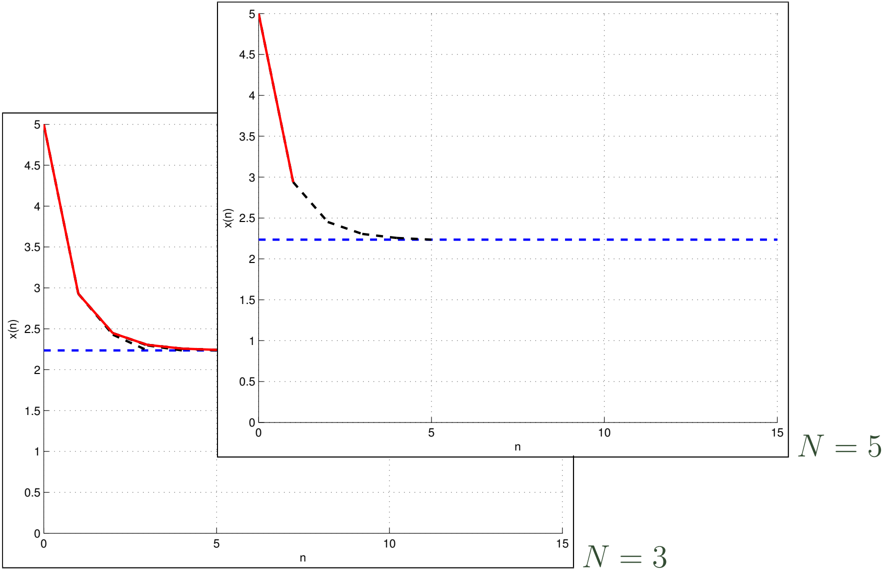

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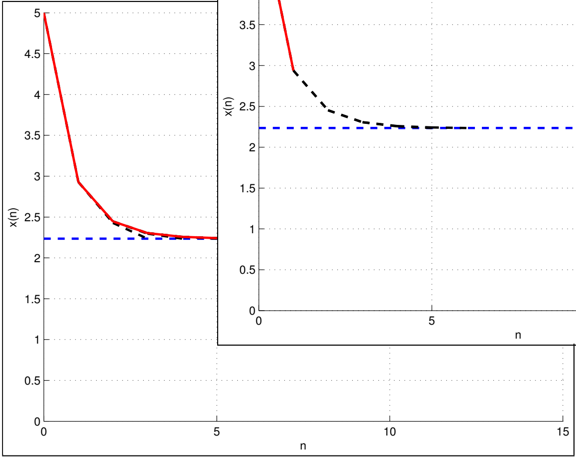

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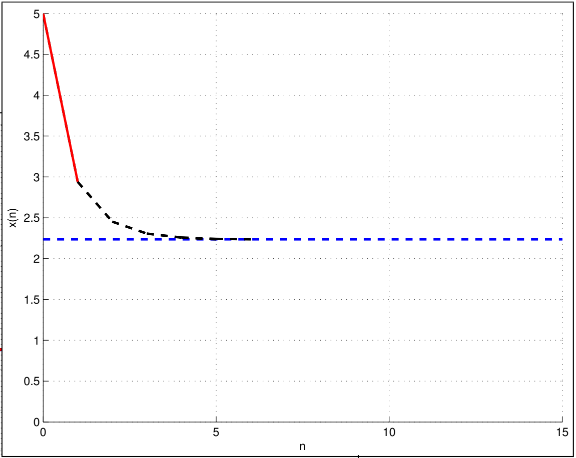

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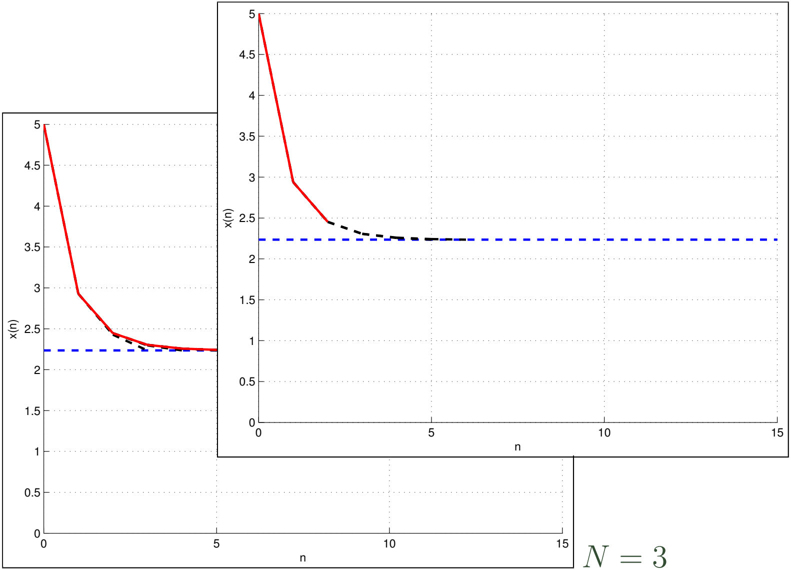

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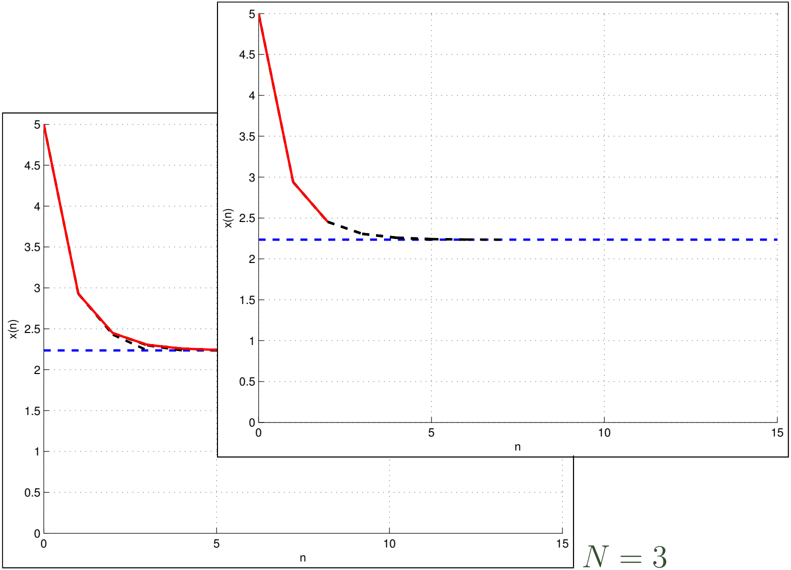

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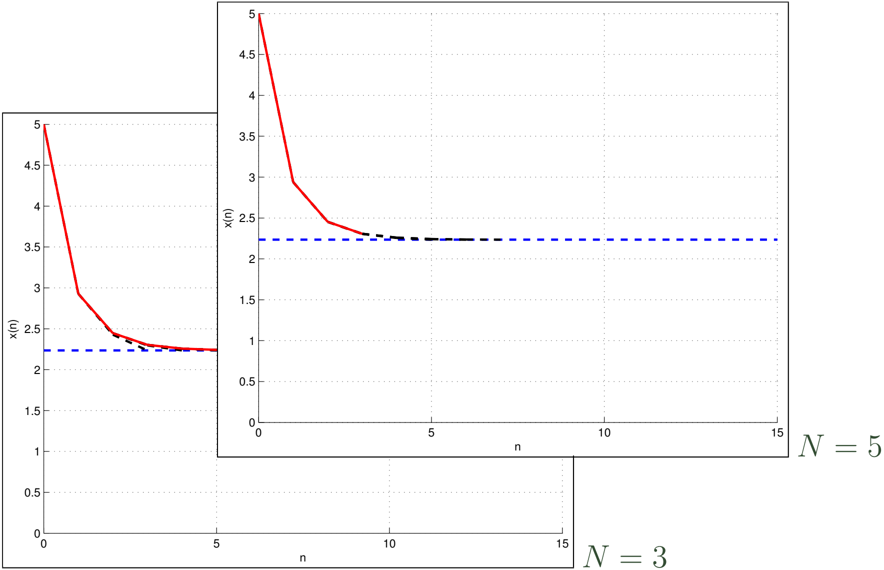

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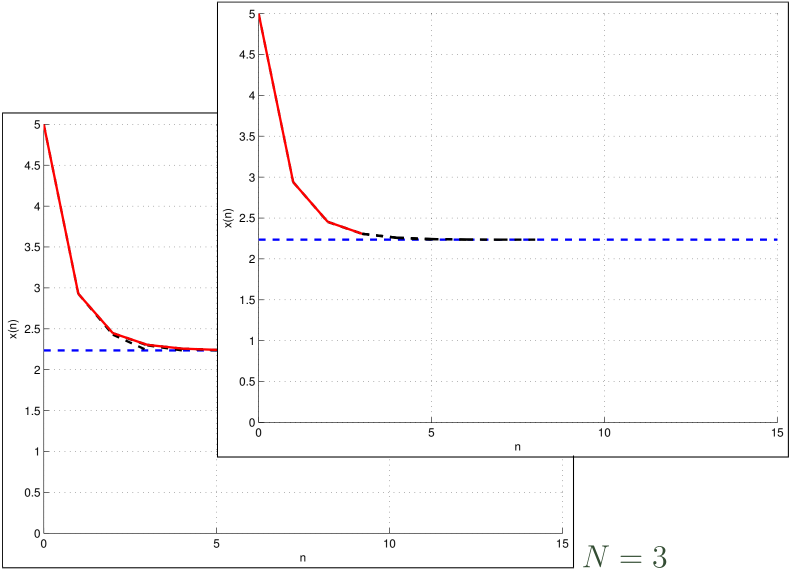

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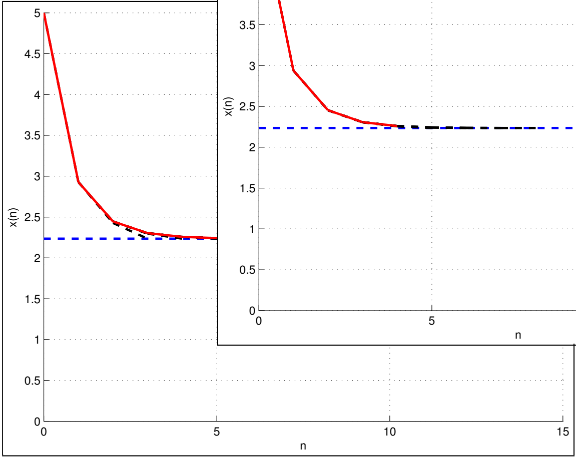

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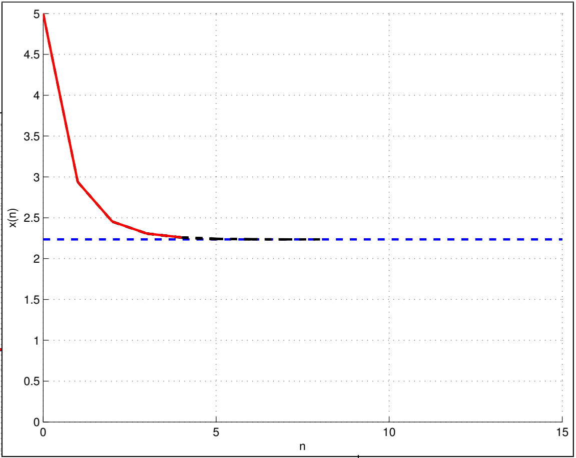

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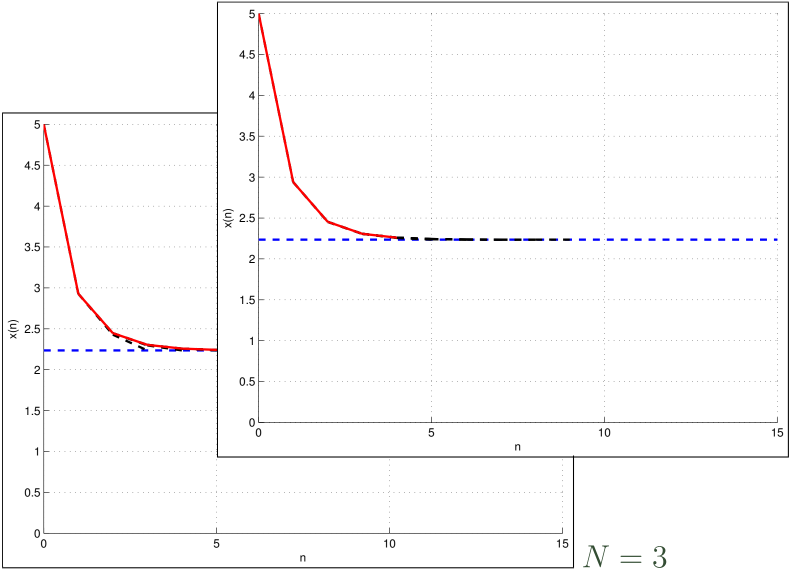

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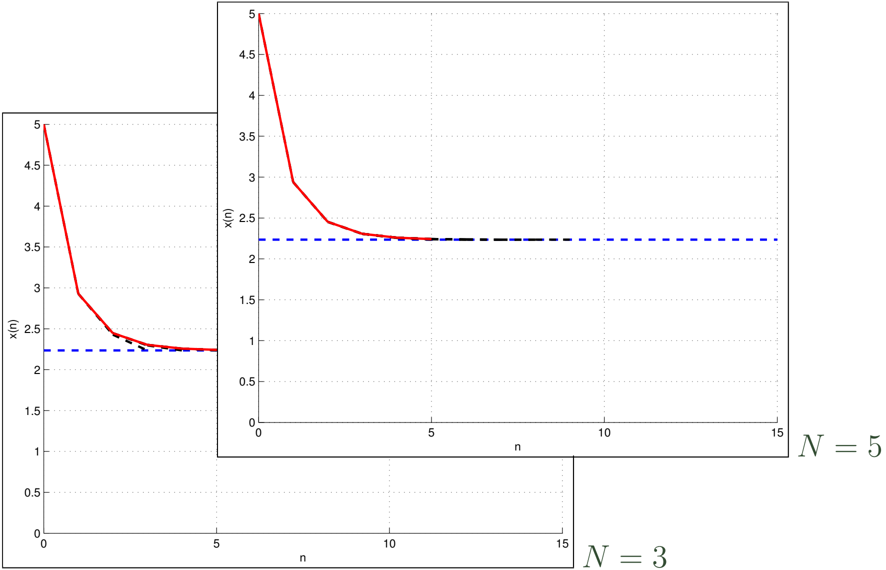

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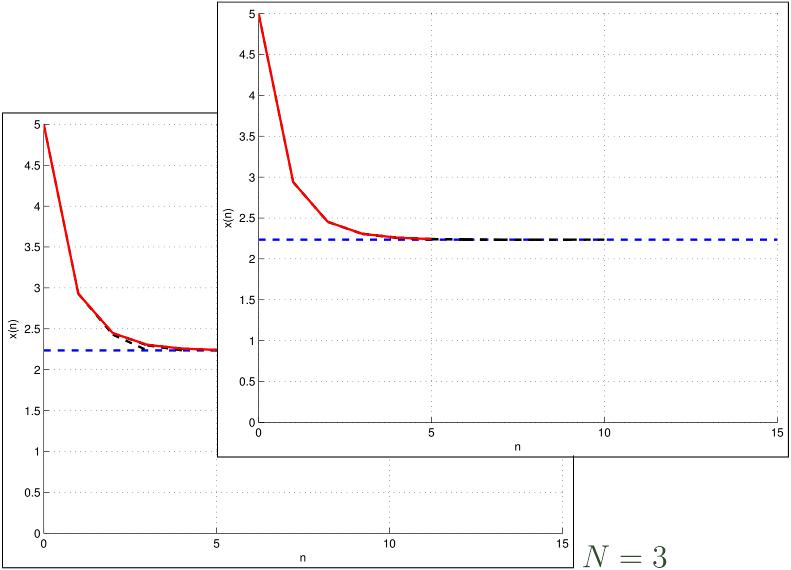

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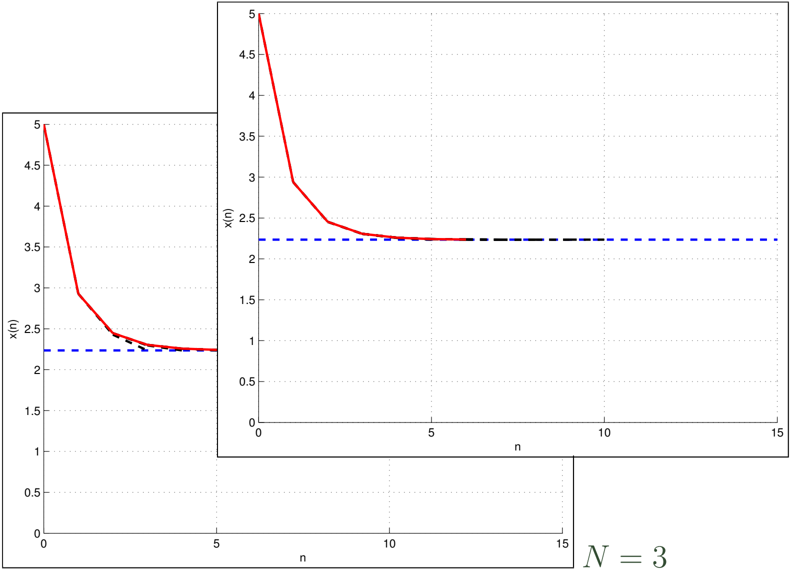

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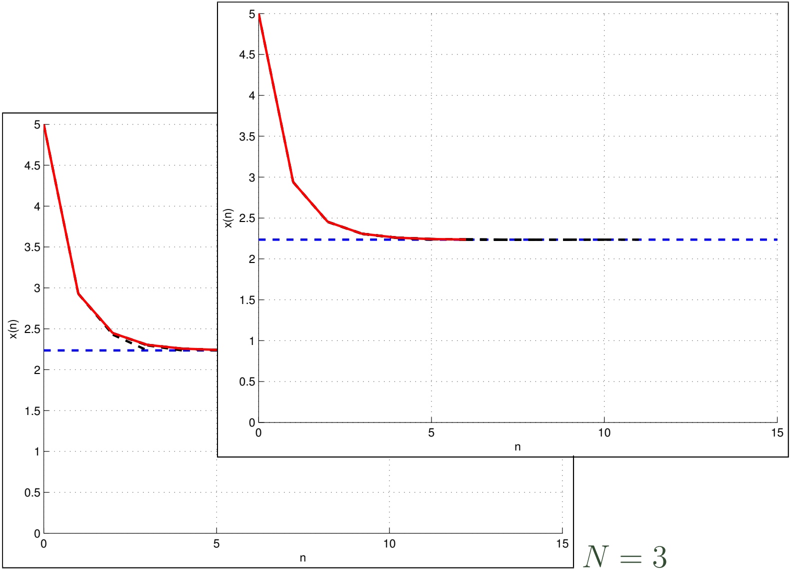

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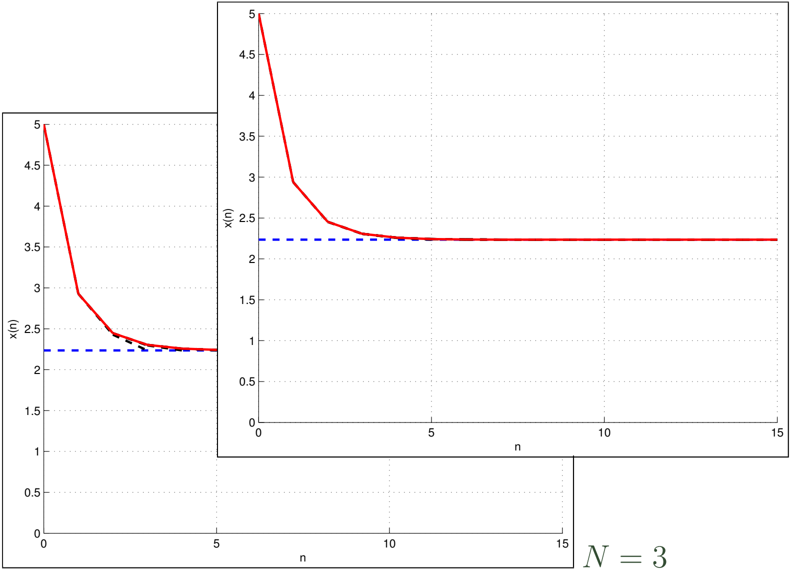

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Regional terminal constraints and Lyapunov-like terminal costs are also possible, but their construction is difficult

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Theorem: [Diehl/Amrit/Rawlings '11, Angeli/Amrit/Rawlings '12] Assume that the optimal control problem is strictly dissipative for the equilibrium (x^e, u^e). Then the MPC closed loop for the scheme with terminal constraint $x_{\mathbf{u}}(N) = x^e$ is asymptotically stable at x^e.

Sketch of proof

(D) $$\ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)) \geq \alpha(\|x - x^e\|)$$

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- Optimality can be proven in a (rather weak) averaged sense, though simulations suggest better optimality properties
- Strict dissipativity ensures both the existence of an optimal equilibrium and asymptotic stability of the closed loop

(9) Economic MPC without terminal constraints

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What happens without terminal constraints? We investigate this for the macroeconomic example [Brock/Mirman '72]

Minimize the average performance with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $x(n+1) = \mathbf{u}(n)$

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Note: now the NMPC algorithm knows neither x^e nor λ

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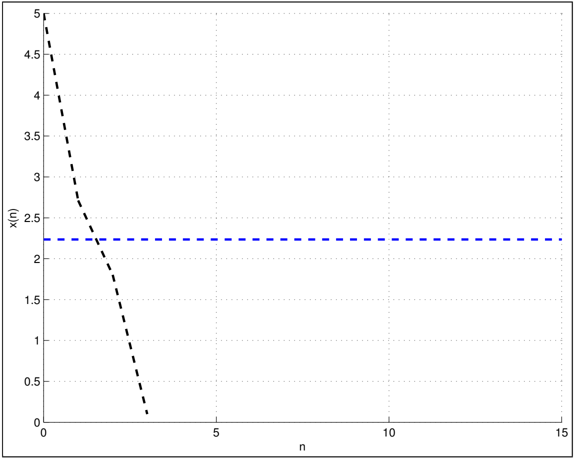

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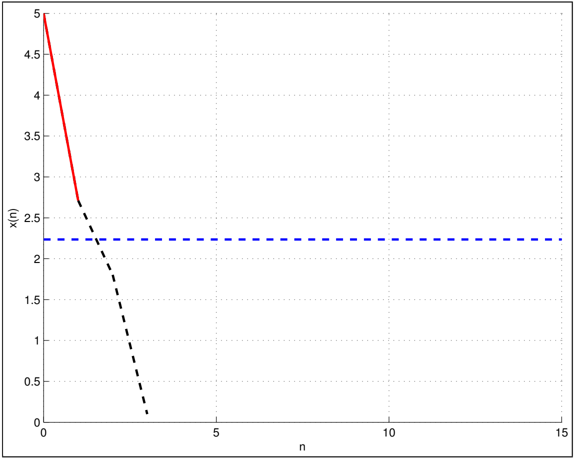

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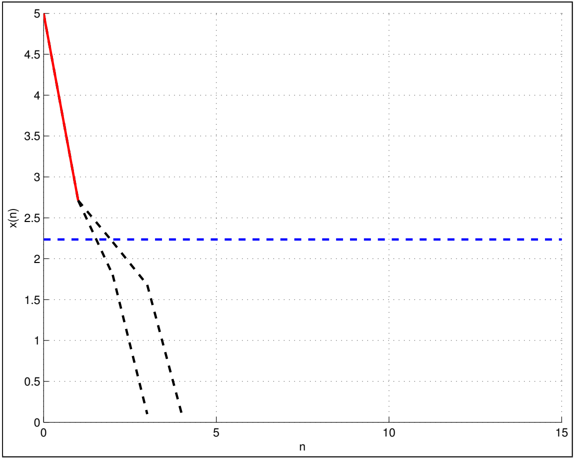

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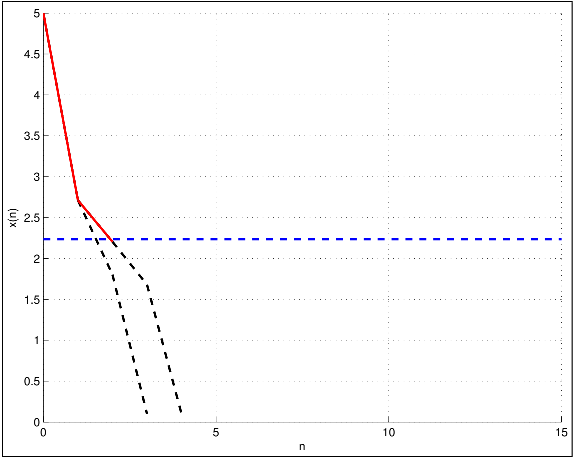

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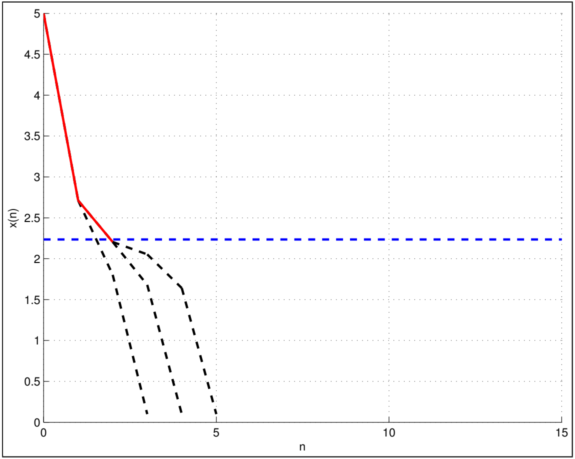

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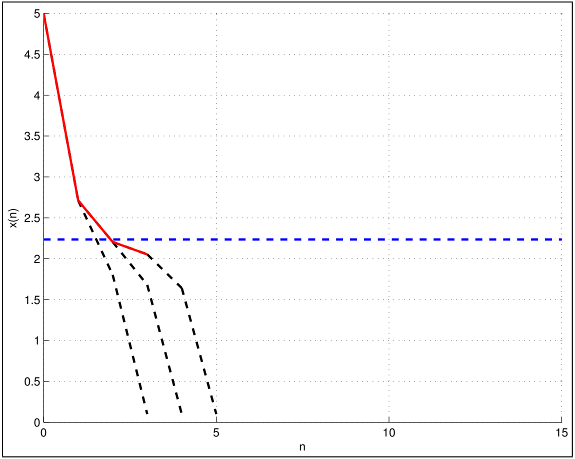

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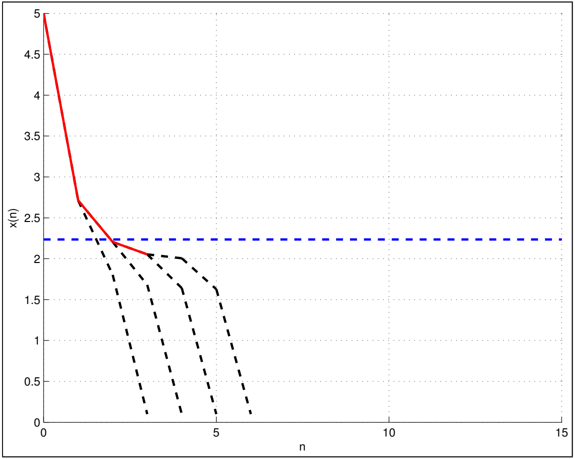

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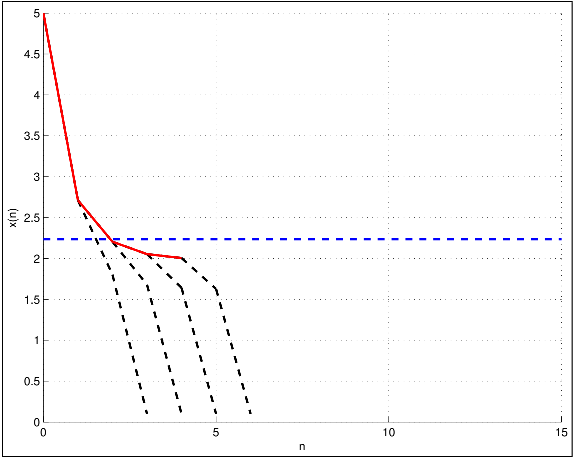

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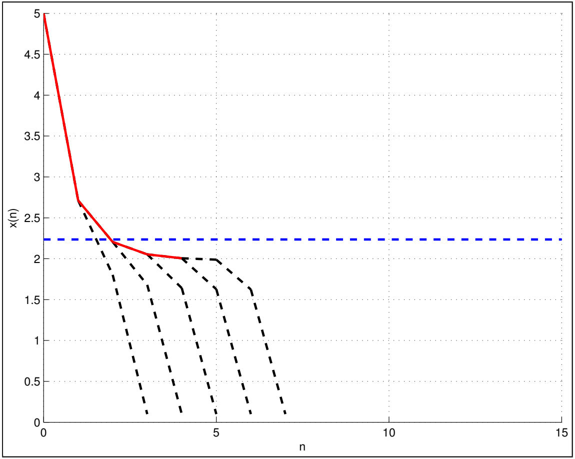

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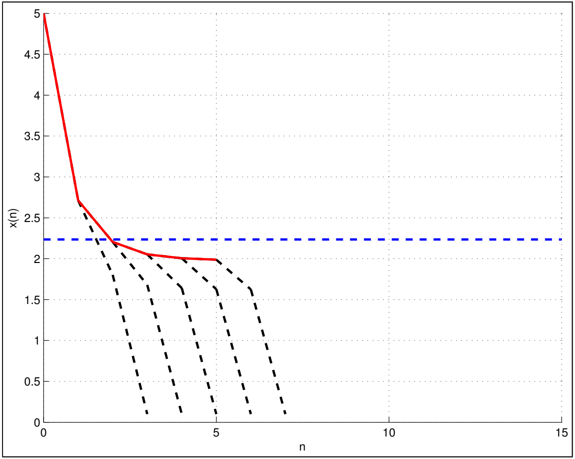

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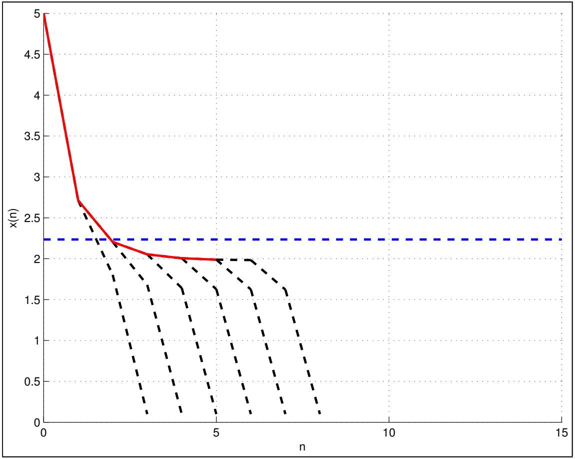

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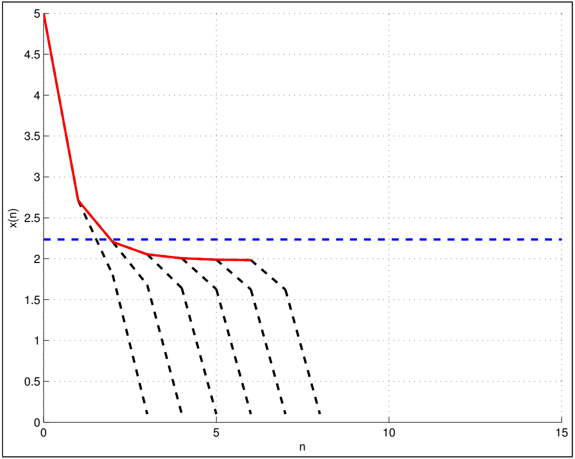

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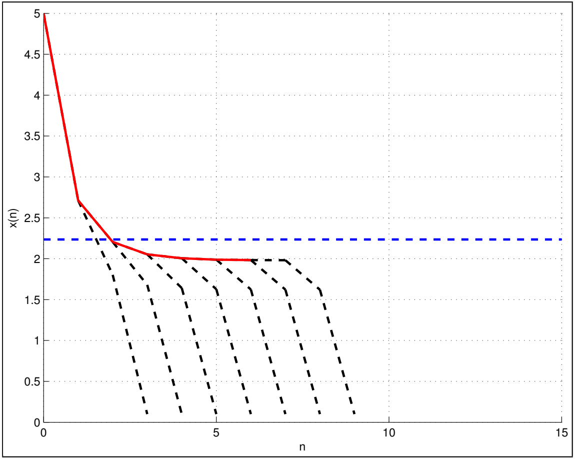

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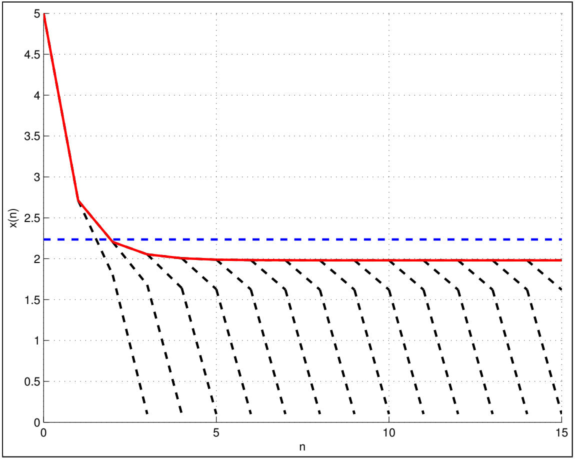

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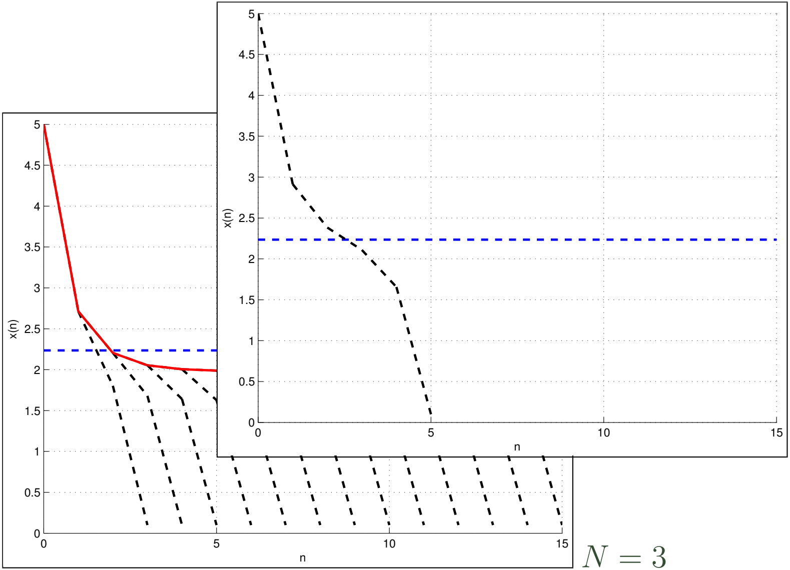

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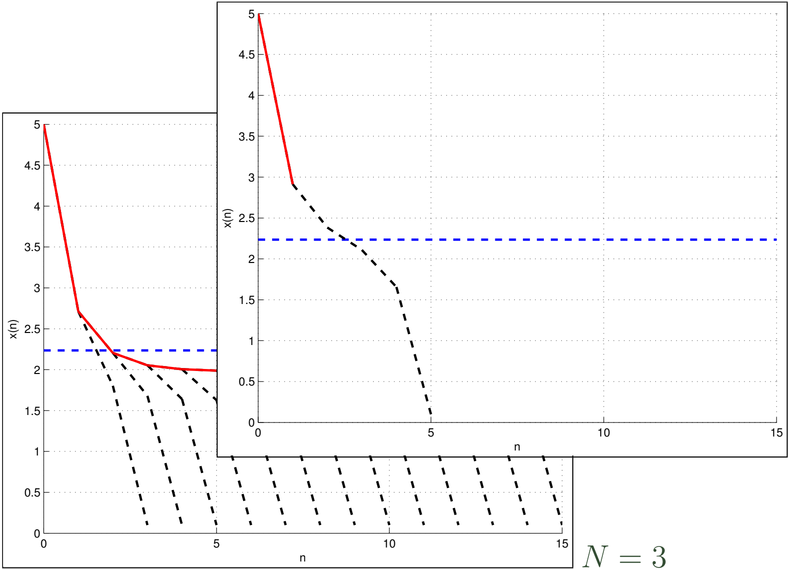

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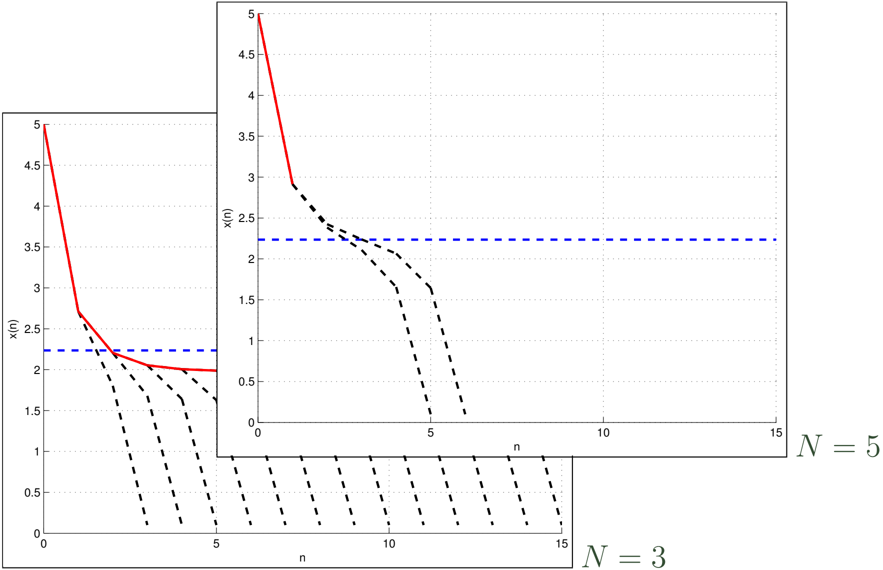

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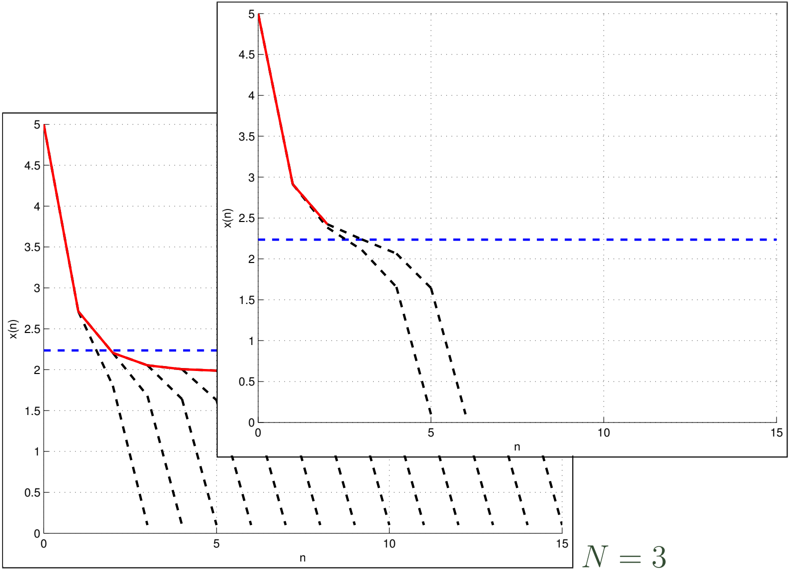

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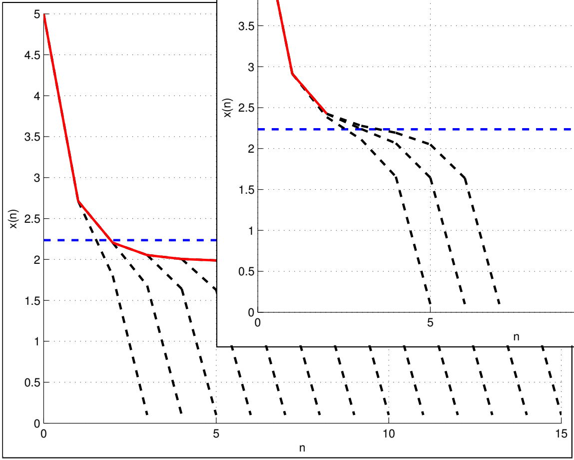

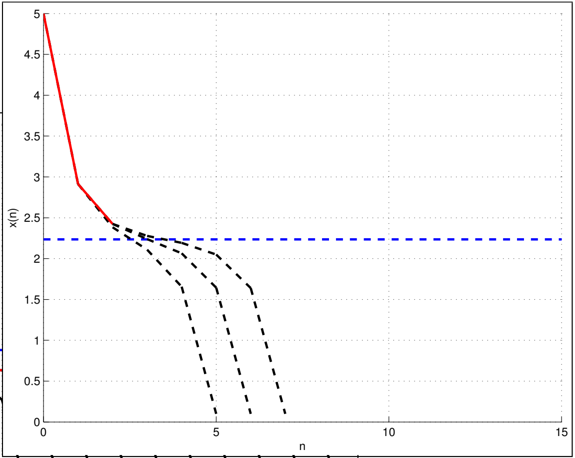

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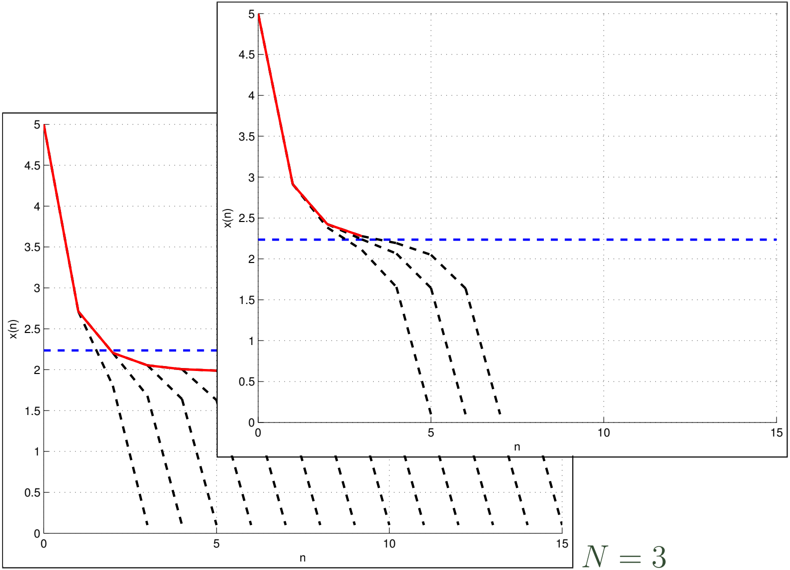

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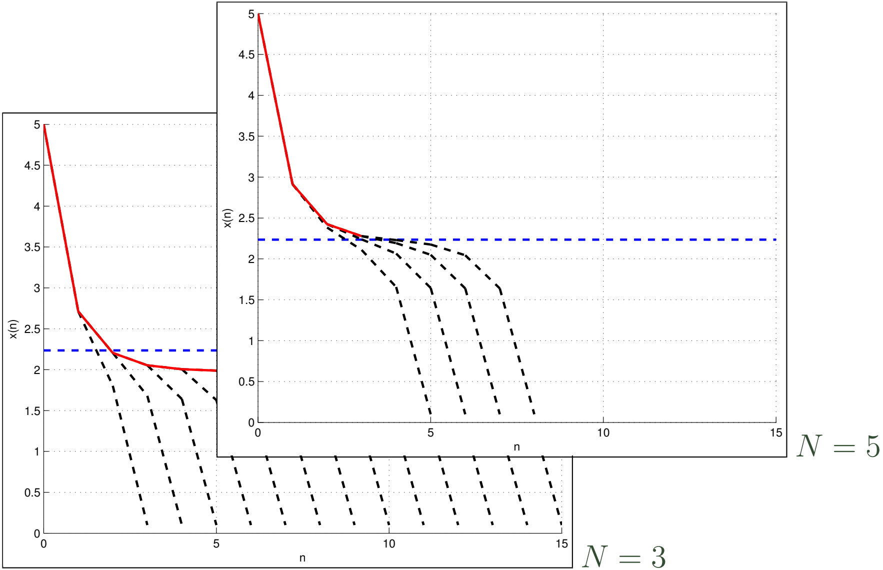

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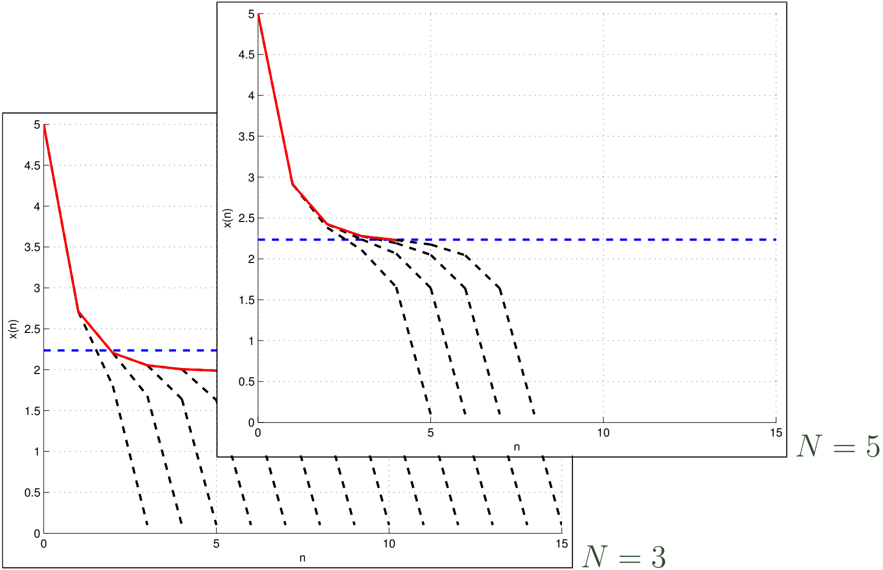

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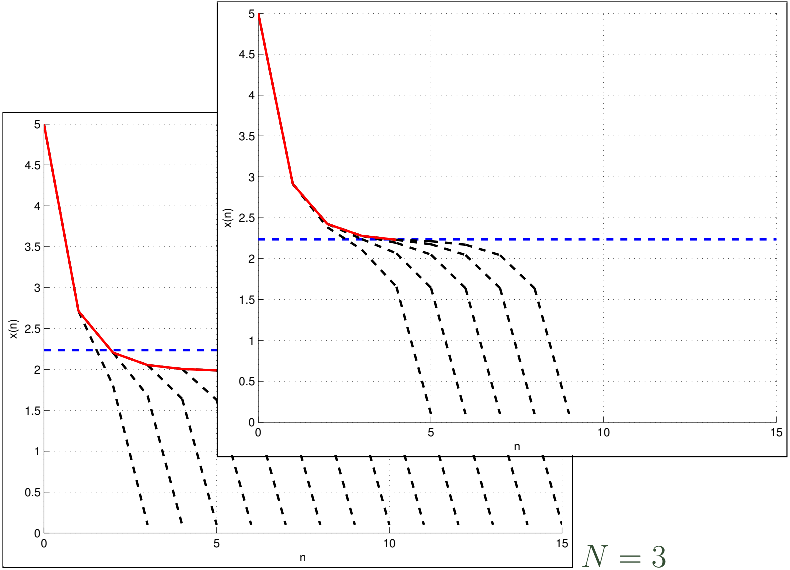

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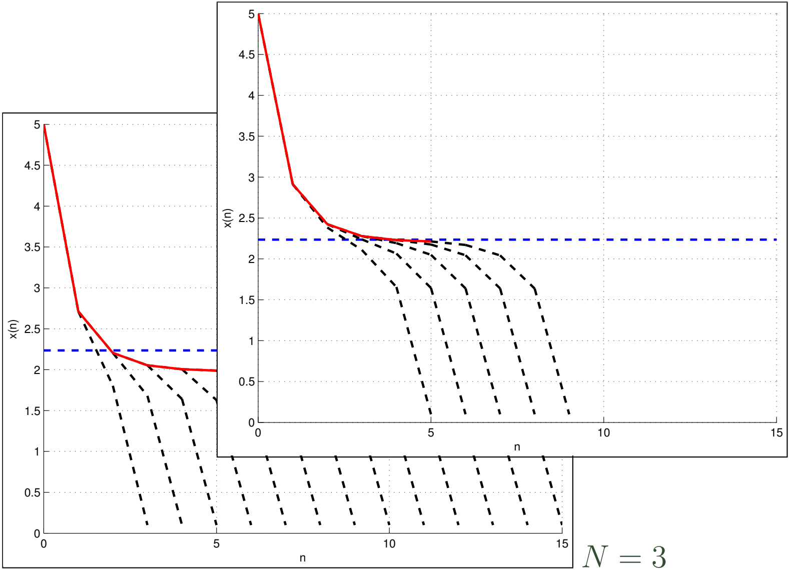
Example: trajectories



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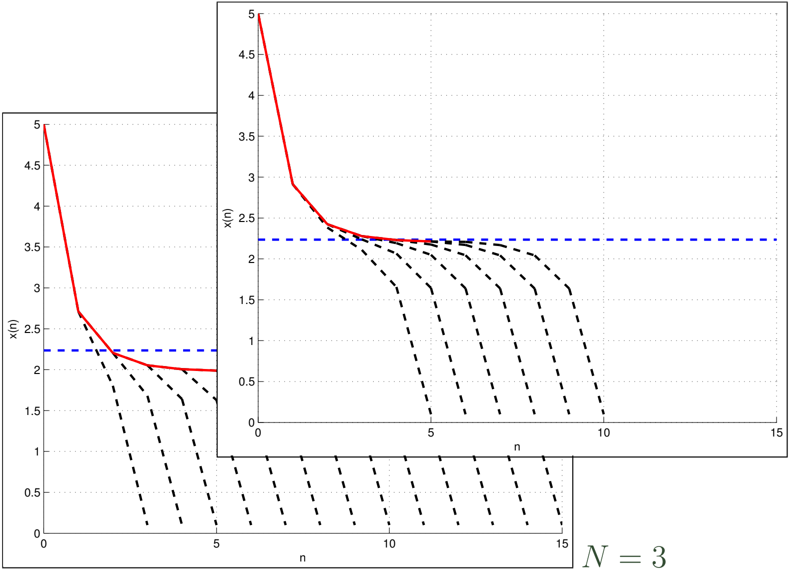

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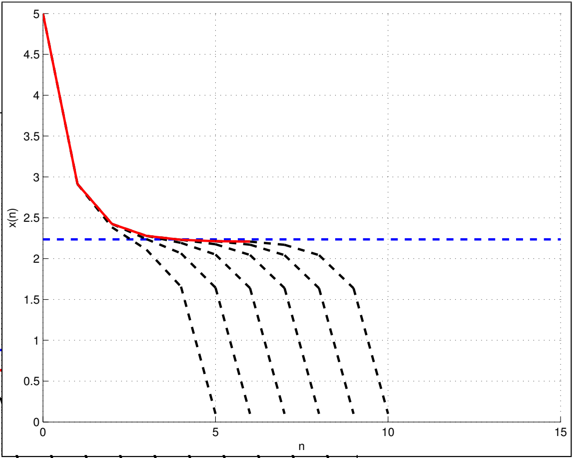

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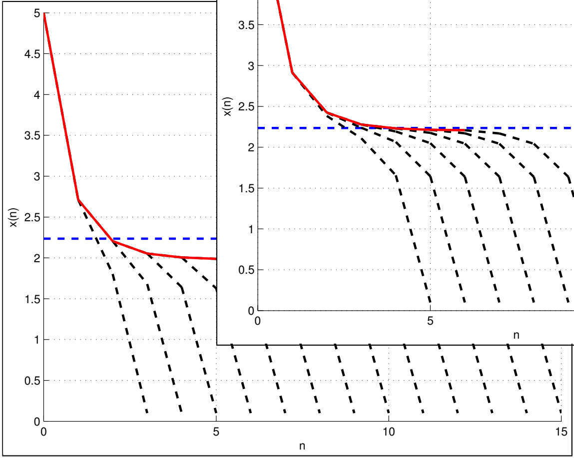

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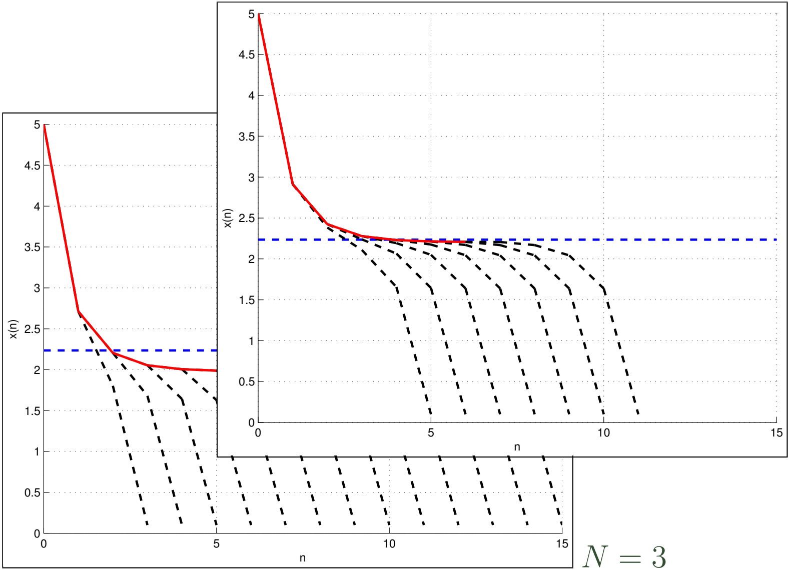

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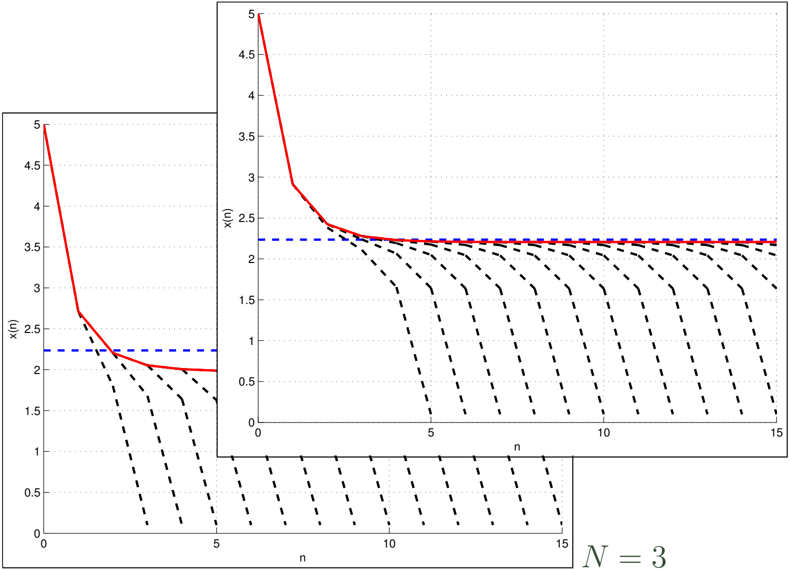

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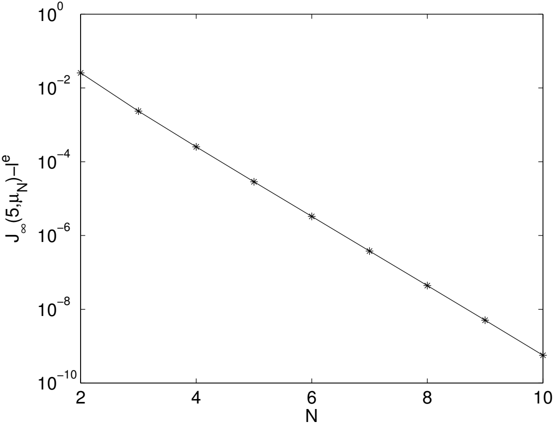
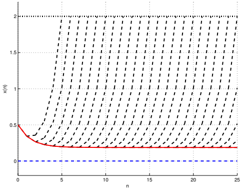
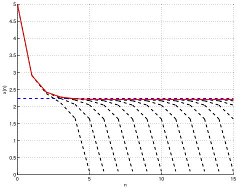
Example: averaged closed loop performance

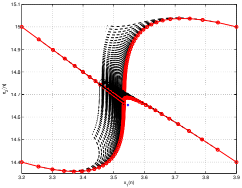


$\overline{J}^{cl}_\infty(5,\mu_N) - \ell(x^e,u^e)$ depending on N, logarithmic scale

Observations







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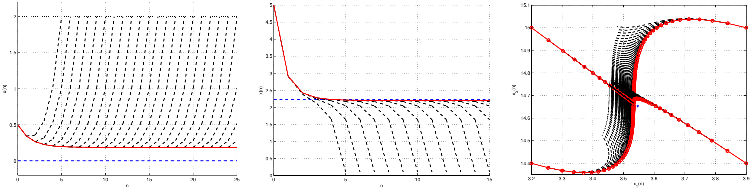

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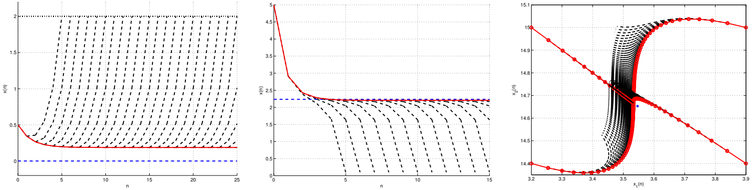

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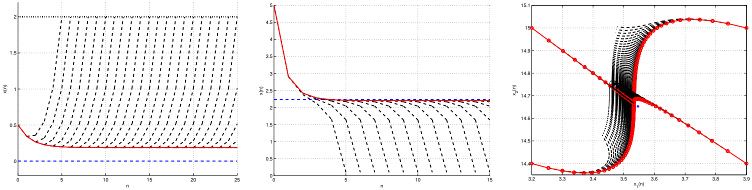

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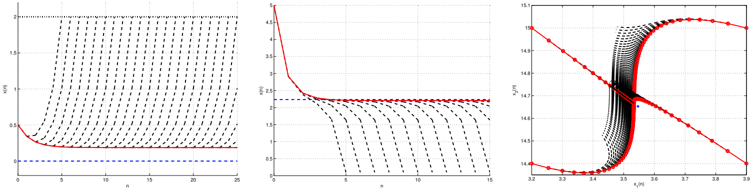

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Can we prove this behavior?

Idea of proof

The following inequality plays the role of the "α_N–inequality" from stabilizing NMPC:

$$V_{N+1}(x) - V_N(x) \leq \ \ell(x^e, u^e) + \text{"error"}$$

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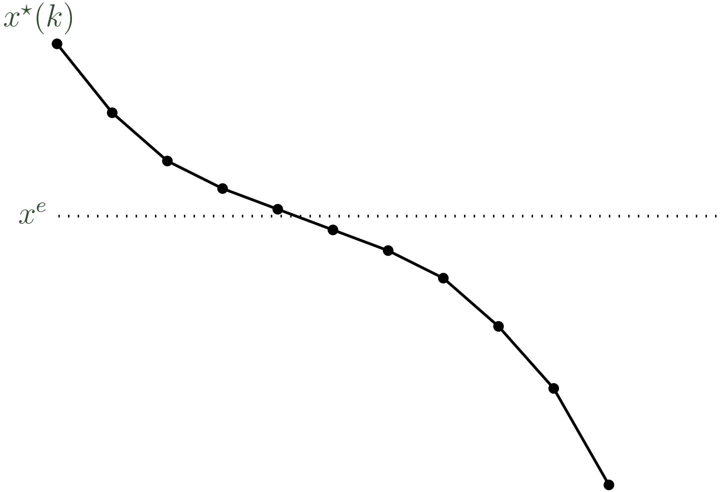
Remedy: prolong the optimal trajectory in the middle

Prolonging in the middle

Sketch of the idea:



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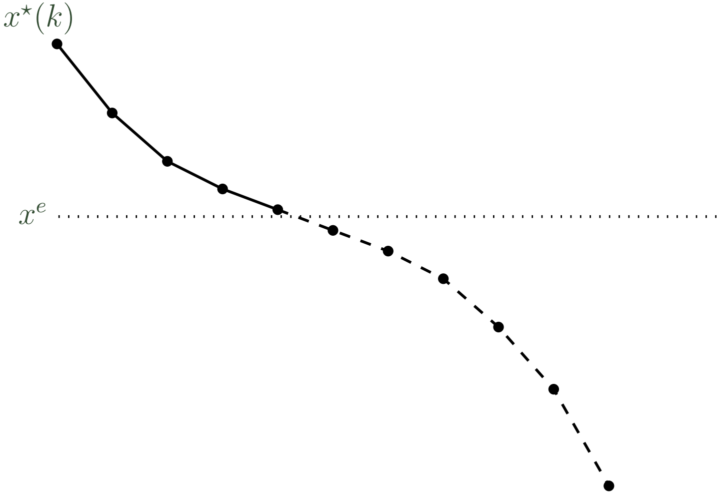

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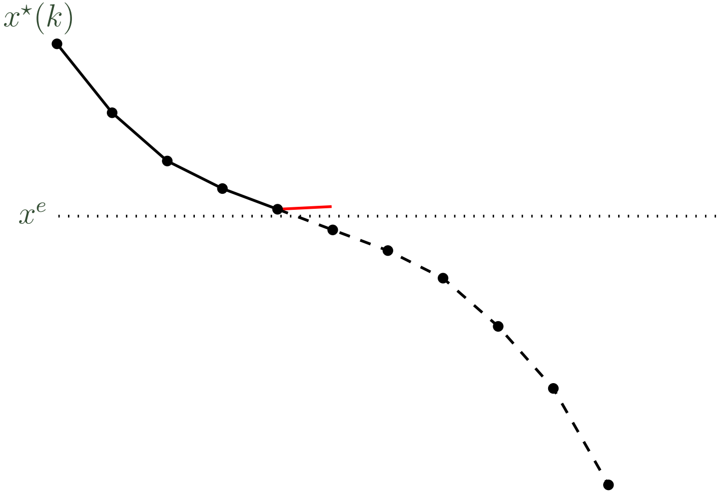

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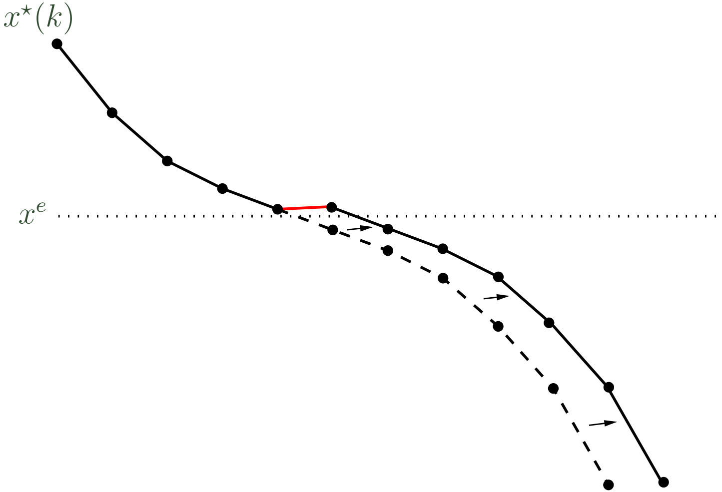

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- can be concluded from strict dissipativity plus (sufficiently fast) controllability towards x^e

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If these two conditions are satisfied, we can show [Gr. '13]

$$\overline{J}^{cl}_{\infty}(x, \mu_N) \to \ell(x^e, u^e) \quad \text{as } N \to \infty$$

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with $\sigma(N) \leq C\theta^N$ for some $\theta \in (0, 1)$

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Recall the modified stage cost

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Theorem: [Gr. '13, Damm/Gr./Stieler/Worthmann '12]

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(iv) appropriate growth conditions for ℓ and $\tilde{\ell}$

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  easily checked for the macroeconomic example

[Damm/Gr./Stieler/Worthmann '12]

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