Nonlinear Model Predictive Control

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(1) Introduction

What is Model Predictive Control (MPC)?

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \ x_{\mathbf{u}}(0) = x_0$$

or, briefly

$$x^+ = f(x, u)$$



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Assume there exists an equilibrium $x_* \in X$ for u = 0, i.e.

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Task: stabilize the system $x^+ = f(x,u)$ at x_* via static state feedback, i.e., find $\mu: X \to U$, such that x_* is asymptotically stable for the feedback controlled system

$$x_{\mu}(n+1) = f(x_{\mu}(n), \mu(x_{\mu}(n))), \ x_{\mu}(0) = x_0$$

Additionally, we impose and

state constraints $x_{\mu}(n) \in \mathbb{X}$ control constraints $\mu(x(n)) \in \mathbb{U}$

for all $n \in \mathbb{N}$ and given sets $\mathbb{X} \subseteq X$, $\mathbb{U} \subseteq U$



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(we will later formalize this property using \mathcal{KL} functions)



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Idea of MPC: use an optimal control problem which minimizes the distance to x_* in order to synthesize a feedback law μ



The idea of MPC

For defining the MPC scheme, we choose a stage cost $\ell(x,u)$ penalizing the distance from x_* and the control effort, e.g., $\ell(x,u) = \|x-x_*\|^2 + \lambda \|u\|^2$ for $\lambda \geq 0$



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The basic idea of MPC is:

- ullet minimize the summed stage cost along trajectories generated from our model over a prediction horizon N
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- repeat this procedure iteratively for all sampling instants $n=0,1,2,\ldots$



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Notation in what follows:

- ullet general feedback laws will be denoted by μ
- ullet the MPC feedback law will be denoted by μ_N



Formal description of the basic MPC scheme:

$$\underset{\mathbf{u} \text{ admissible}}{\mathsf{minimize}} \ J_N(x_{\mu_N}(n), \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \ x_{\mathbf{u}}(0) = x_{\mu_N}(n)$$

(u admissible
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 optimal trajectory $x^*(0), \dots, x^*(N)$ with optimal control $\mathbf{u}^*(0), \dots, \mathbf{u}^*(N-1)$

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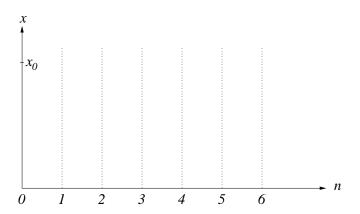
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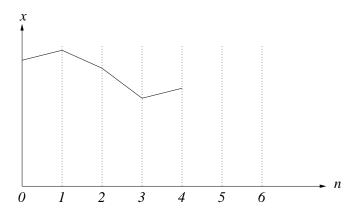
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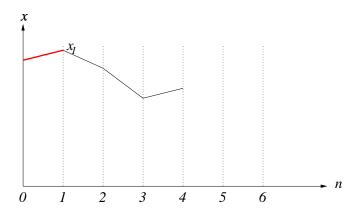




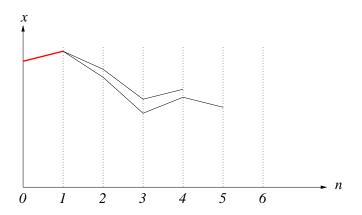


black = predictions (open loop optimization)

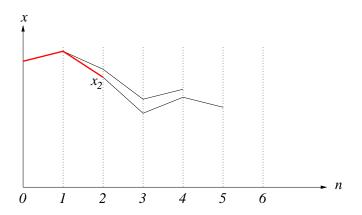




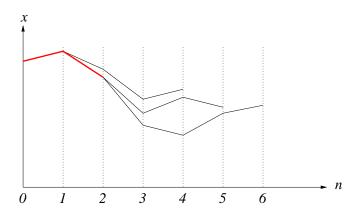




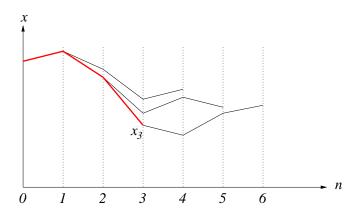




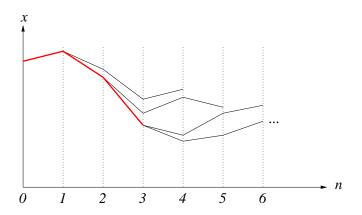




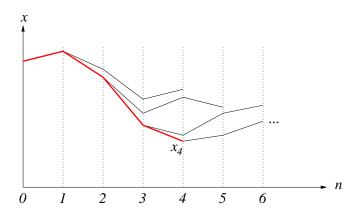




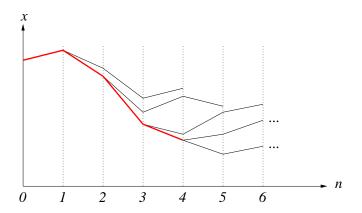




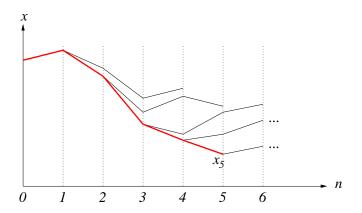






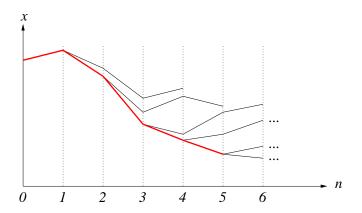








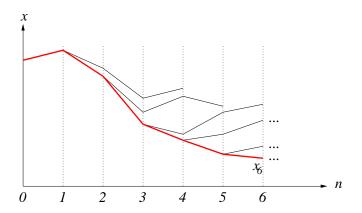
MPC from the trajectory point of view



black = predictions (open loop optimization) red = MPC closed loop, $x_n = x_{\mu_N}(n)$



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Central questions:

• When does MPC stabilize the system?



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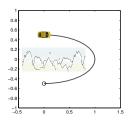
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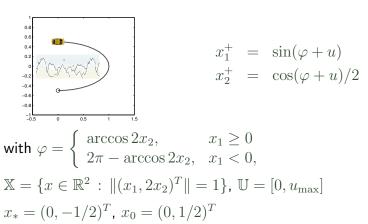
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- When does MPC stabilize the system?
- How good is the performance of the MPC feedback law?
- ullet How long does the optimization horizon N need to be? and, of course, the development of good algorithms (not topic of this course)

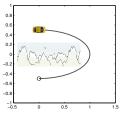












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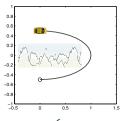
$$\text{with } \varphi = \left\{ \begin{array}{ll} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{array} \right.$$

$$\mathbb{X} = \{x \in \mathbb{R}^2 : \|(x_1, 2x_2)^T\| = 1\}, \ \mathbb{U} = [0, u_{\text{max}}]$$

$$x_* = (0, -1/2)^T$$
, $x_0 = (0, 1/2)^T$

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MPC with $\ell(x,u)=\|x-x^*\|^2+|u|^2$ and $u_{\max}=0.2$ yields asymptotic stability for N=11 but not for $N\leq 10$



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- the feedback value $\mu_N(x_0)$ is the first element of the resulting optimal control sequence
- the example shows that MPC does not always yield an asymptotically stabilizing feedback law



(2a) Background material:

Lyapunov functions

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In this section, we consider discrete time systems without input, i.e.,

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Note: we do not require q to be continuous



Comparison functions

For $\mathbb{R}^+_0 = [0, \infty)$ we use the following classes of comparison functions

$$\mathcal{K} \ := \ \left\{\alpha: \mathbb{R}_0^+ \to \mathbb{R}_0^+ \ \middle| \ \begin{array}{c} \alpha \text{ is continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array}\right\}$$

$$\mathcal{K}_{\infty} := \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,\middle|\, \alpha \in \mathcal{K} \text{ and } \alpha \text{ is unbounded} \right\}$$

$$\mathcal{KL} \ := \ \left\{ \beta: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \middle| \begin{array}{c} \beta(\cdot,t) \in \mathcal{K} \text{ for all } t \in \mathbb{R}_0^+ \\ \text{and } \beta(r,\cdot) \text{ is strictly de-} \\ \text{creasing to } 0 \text{ for all } r \in \mathbb{R}_0^+ \end{array} \right\}$$



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We say that x_* is asymptotically stable for $x^+ = g(x)$ on a forward invariant set Y if there exists $\beta \in \mathcal{KL}$ such that

$$||x(n) - x_*|| \le \beta(||x(0) - x_*||, n)$$

holds for all $x \in Y$ and $n \in \mathbb{N}$

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How can we check whether this property holds?



Lyapunov function

Let $Y \subseteq X$ be a forward invariant set and $x_* \in X$. A function $V: Y \to \mathbb{R}_0^+$ is called a Lyapunov function for $x^+ = g(x)$ if the following two conditions hold for all $x \in Y$:

(i) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ such that

$$\alpha_1(||x - x_*||) \le V(x) \le \alpha_2(||x - x_*||)$$

(ii) There exists $\alpha_V \in \mathcal{K}$ such that

$$V(x^+) \le V(x) - \alpha_V(||x - x_*||)$$



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The bounds $\alpha_1(\|x-x_*\|) \leq V(x) \leq \alpha_2(\|x-x_*\|)$ imply that asymptotic stability holds with $\beta(r,t) = \alpha_1^{-1}(\tilde{\beta}(\alpha_2(r),t))$



Lyapunov functions — discussion

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It is hence sufficient to check the decay of V in one time step

→ it is typically quite easy to check whether a given function is a Lyapunov function



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But it is in general difficult to find a candidate for a Lyapunov function

For MPC, we will use the optimal value functions which we introduce in the next section



(2b) Background material:

Dynamic Programming

Purpose of this section

We define the optimal value functions \mathcal{V}_N for the optimal control problem

minimize
$$J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \ x_{\mathbf{u}}(0) = x_0$$

used within the MPC scheme (with $x_0 = x_{\mu_N}(n)$)



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used within the MPC scheme (with $x_0 = x_{\mu_N}(n)$)

We present the dynamic programming principle, which establishes a relation for these functions and will eventually enable us to derive conditions under which V_N is a Lyapunov function



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An admissible control sequence \mathbf{u}^{\star} is called optimal, if

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An admissible control sequence \mathbf{u}^{\star} is called optimal, if

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Note: an optimal \mathbf{u}^* does not need to exist in general. In the sequel we assume that \mathbf{u}^* exists if x_0 is feasible



Dynamic Programming Principle

Theorem: (Dynamic Programming Principle) For any feasible $x_0 \in \mathbb{X}$ the optimal value function satisfies

$$V_N(x_0) = \inf_{u \in \mathbb{II}} \left\{ \ell(x_0, u) + V_{N-1}(f(x_0, u)) \right\}$$

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Moreover, if \mathbf{u}^{\star} is an optimal control, then

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Idea of Proof: Follows by taking infima in the identity

$$J_N(x_0, \mathbf{u}) = \ell(x_{\mathbf{u}}(0), \mathbf{u}(0)) + \sum_{k=1}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k))$$

$$= \ell(x_0, \mathbf{u}(0)) + J_{N-1}(f(x_0, \mathbf{u}(0)), \mathbf{u}(\cdot + 1))$$



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(ii) The MPC feedback μ_N satisfies

$$\mu_N(x) = \underset{u \in \mathbb{I}}{\operatorname{argmin}} \left\{ \ell(x, u) + V_{N-1}(f(x, u)) \right\}$$

(i.e., $u = \mu_N(x)$ minimizes this expression)

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and

$$u^{\star}(k) = \mu_{N-k}(x^{\star}(k)), \quad k = 0, \dots, N-1$$



We will see later, that under suitable conditions the optimal value function will play the role of a Lyapunov function for the MPC closed loop



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In order to see why this can work, in the next section we briefly look at infinite horizon optimal control problems



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The dynamic programming principle and its corollaries will prove to be important tools to establish this fact

In order to see why this can work, in the next section we briefly look at infinite horizon optimal control problems

Moreover, for simple systems the principle can be used for computing V_N and μ_N — we will see an example in the excercises



(2c) Background material: Relaxed Dynamic Programming

Infinite horizon optimal control

Just like the finite horizon problem we can define the infinite horizon optimal control problem

$$\label{eq:minimize} \underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \ J_{\infty}(x_0,\mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_\mathbf{u}(k),\mathbf{u}(k)), \ x_\mathbf{u}(0) = x_0$$



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If we could compute an optimal feedback μ_{∞} for this problem (which is — in contrast to computing μ_N — in general a very difficult problem), we would have solved the stabilization problem



Recall the corollary from the finite horizon dynamic programming principle

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$



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Relaxing dynamic programming

Unfortunately, an equation of the type

$$V_{\infty}(x) = \ell(x, \mu_{\infty}(x)) + V_{\infty}(f(x, \mu_{\infty}(x)))$$

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$$V_N(x) \ge \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

"relaxed dynamic programming inequality" [Rantzer et al. '06ff]



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What can we conclude from this inequality?



We define the infinite horizon performance of the MPC closed loop system $x^+=f(x,\mu_N(x))$ as

$$J_{\infty}^{cl}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \ x_{\mu_N}(0) = x_0$$



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Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a forward invariant set for the MPC closed loop and assume that

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Then for all $x \in Y$ the infinite horizon performance satisfies

$$J_{\infty}^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$$



Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_{\infty}$ such that the inequalities

$$V_N(x) \le \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \ge \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N .



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Proof: The assumed inequalities immediately imply that $V=V_N$ is a Lyapunov function for $x^+=g(x)=f(x,\mu_N(x))$ with

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⇒ asymptotic stability



For proving the performance estimate $J_{\infty}^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$, the relaxed dynamic programming inequality implies

$$\alpha \sum_{n=0}^{K-1} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k)))$$

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Since all summands are ≥ 0 , this implies that the limit for $K \to \infty$ exists and we get

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- The performance of the MPC controller can be measured by looking at the infinite horizon value along the MPC closed loop trajectories
- Relaxed dynamic programming gives us conditions under which both asymptotic stability and performance results can be derived



Application of background results

The main task will be to verify the assumptions of the relaxed dynamic programming theorem, i.e.,

$$V_N(x) \ge \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0,1]$, and

$$V_N(x) \le \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \ge \alpha_3(\|x - x_*\|)$$

for all x in a forward invariant set Y for $x^+ = f(x, \mu_N(x))$



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 modify the optimal control problem in the MPC loop by adding terminal constraints and costs



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To this end, we present two different approaches:

- modify the optimal control problem in the MPC loop by adding terminal constraints and costs
- derive assumptions on f and ℓ under which MPC works without terminal constraints and costs



(3) Stability with stabilizing constraints

V_N as a Lyapunov Function

Problem: Prove that the MPC feedback law μ_N is stabilizing



V_N as a Lyapunov Function

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Approach: Verify the assumptions

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of the relaxed dynamic programming theorem for the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$





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- \longrightarrow we have V_{N-1} where we would like to have V_N
- we would get the desired inequality if we could ensure

$$V_{N-1}(f(x,\mu_N(x))) > V_N(f(x,\mu_N(x))) + \text{"small error"}$$

Let us first consider the inequality

$$V_N(x) \ge \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

The dynamic programming principle for V_N yields

$$V_N(x) \ge \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

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(where "small" means that the error can be compensated replacing $\ell(x,\mu_N(x))$ by $\alpha\ell(x,\mu_N(x))$ with $\alpha\in(0,1)$)



Task: Find conditions under which

$$V_{N-1}(f(x,\mu_N(x))) \ge V_N(f(x,\mu_N(x))) +$$
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$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

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→ additional stabilizing constraints were proposed



(3a) Equilibrium terminal constraint

Equilibrium terminal constraint

Optimal control problem

$$\label{eq:minimize} \underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \ J_N(x_0, \mathbf{u}) = \sum_{\mathbf{u}}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \ x_\mathbf{u}(0) = x_0$$

Equilibrium terminal constraint

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Assumption:
$$f(x_*,0) = x_*$$
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Idea: add equilibrium terminal constraint

$$x_{\mathbf{u}}(N) = x_*$$

[Keerthi/Gilbert '88, ...]



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→ we now solve

$$\underset{\mathbf{u} \in \mathbb{U}_{x_*}^N(x_0)}{\text{minimize}} \ J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \ x_{\mathbf{u}}(0) = x_0$$

with $\mathbb{U}_{x_{\bullet}}^{N}(x_{0}):=\{\mathbf{u}\in\mathbb{U}^{N}\text{ admissible and }x_{\mathbf{u}}(N)=x_{*}\}$



Let $\tilde{\mathbf{u}} \in \mathbb{U}_{x_*}^{N-1}(x_0)$



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$$\leadsto$$
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Moreover, since

$$\ell(x_{\mathbf{u}_{*}}(N-1),\mathbf{u}_{N}(N-1)) = \ell(x_{*},0) = 0.$$

the prolongation has zero stage cost



Now, let $\tilde{\mathbf{u}}^{\star} \in \mathbb{U}^{N-1}_{x_*}(x_0)$ be the optimal control for J_{N-1} , i.e.,

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But: the dynamic programming principle remains valid



Relaxed dynamic programming inequality

From the reversed inequality

$$V_{N-1}(x) \ge V_N(x)$$

and the dynamic programming principle

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we immediately get

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→ stability follows if we can ensure the additional inequalities

$$V_N(x) \le \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \ge \alpha_3(\|x - x_*\|)$$



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 \leadsto the feasible set \mathbb{X}_N is the "natural" operating region of MPC with equilbrium terminal constraints



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Stability theorem

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holds.

Note: The constraint $x_{\mathbf{u}}(N) = x_*$ does not imply $x_{u_N}(N) = x_*$



Sketch of proof: All assertions follow from the relaxed dynamic programming theorem if we prove forward invariance of \mathbb{X}_N for the MPC closed loop system $x^+ = f(x, \mu_N(x))$



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The additional condition

$$x(N) = x^*$$



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ensures asymptotic stability in a rigorously provable way, but

• online optimization may become harder



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- online optimization may become harder
- if we want a large feasible set \mathbb{X}_N we typically need a large optimization horizon N (see the car-and-mountains example)
- ullet system needs to be controllable to x^* in finite time
- not very often used in industrial practice



and terminal cost

(3b) Regional terminal constraint

Regional constraint and terminal cost Optimal control problem

minimize
$$J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \ x_{\mathbf{u}}(0) = x_0$$



Optimal control problem

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Idea: add local Lyapunov function $F: \mathbb{X}_0 \to \mathbb{R}_0^+$ as terminal cost

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N))$$

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F is defined on a region \mathbb{X}_0 around x_* which is imposed as terminal constraint $x(N) \in \mathbb{X}_N$

[Chen & Allgöwer '98, Jadbabaie et al. '98 . . .]



We thus change the optimal control problem to

with

$$\mathbb{U}^N_{\mathbb{X}_0}(x_0) := \{\mathbf{u} \in \mathbb{U}^N \text{ admissible and } x_{\mathbf{u}}(N) \in \mathbb{X}_0\}$$

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Which properties do we need for F and X_0 in order to make this work?

Assumptions on $F: \mathbb{X}_0 \to \mathbb{R}_0^+$ and \mathbb{X}_0



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- (ii) F is a Lyapunov function for $x^+ = f(x, \kappa(x))$ on \mathbb{X}_0 which is compatible with the stage cost ℓ in the following sense:

for each $x \in \mathbb{X}_0$ the inequality

$$F(f(x, \kappa(x))) \le F(x) - \ell(x, \kappa(x))$$

holds



Let $\tilde{\mathbf{u}} \in \mathbb{U}_{\mathbb{X}_0}^{N-1}(x_0)$



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By (ii) the stage cost of the prolongation is bounded by

$$\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1)) < F(x_{\mathbf{u}}(N-1)) - F(x_{\mathbf{u}}(N))$$



Let $\tilde{\mathbf{u}}^{\star} \in \mathbb{U}_{\mathbb{X}_0}^{N-1}(x_0)$ be the optimal control for J_{N-1} , i.e.,

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 \rightarrow again we get $V_{N-1} > V_N$

Feasible sets

Define the feasible set

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for some $\alpha_2 \in \mathcal{K}_{\infty}$ under mild conditions, while outside \mathbb{X}_N we get $V_N(x) = \infty$

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Proof: Almost identical to the equilibrium constrained case



Compared to the equilibrium constraint, the regional constraint

• yields easier online optimization problems



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In Section (5) we will see how stability can be proved without stabilizing terminal constraints



• terminal constraints yield that the usual inequality $V_{N-1} < V_N$ is reversed to $V_{N-1} > V_N$



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- this enables us to derive the relaxed dynamic programming inequality (with $\alpha=1$) from the dynamic programming principle
- equilibrium constraints demand more properties of the system than regional constraints but do not require a Lyapunov function terminal cost
- in both cases, the operating region is restricted to the feasible set \mathbb{X}_N



(4) Inverse optimality and suboptimality

Once stability can be guaranteed, we can investigate the performance of the MPC feedback law μ_{N}



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As already mentioned, we measure the performance of the feedback $\mu_N: X \to U$ via the infinite horizon functional

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• Inverse Optimality: show that μ_N is optimal for an altered running cost $\tilde{\ell} \neq \ell$



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In the literature, two different concepts can be found:

- Inverse Optimality: show that μ_N is optimal for an altered running cost $\tilde{\ell} \neq \ell$
- ullet Suboptimality: derive upper bounds for $J^{cl}_{\infty}(x_0,\mu_N)$



Theorem: [Poubelle/Bitmead/Gevers '88, Magni/Sepulchre '97] For both types of terminal constraints, μ_N is optimal for

with
$$\tilde{\ell}(x,u) := \ell(x,u) + V_{N-1}(f(x,u)) - V_N(f(x,u))$$

Theorem: [Poubelle/Bitmead/Gevers '88, Magni/Sepulchre '97] For both types of terminal constraints, μ_N is optimal for

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with
$$\tilde{\ell}(x,u) := \ell(x,u) + V_{N-1}(f(x,u)) - V_N(f(x,u))$$

Note: $\tilde{\ell} \ge \ell$



Theorem: [Poubelle/Bitmead/Gevers '88, Magni/Sepulchre '97]

For both types of terminal constraints, μ_N is optimal for

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$$= \inf_{u \in \mathbb{U}} \{ \tilde{\ell}(x, u) + V_N(f(x, u)) \}$$

and
$$V_N(x) = \ell(x, \mu_N) + V_N(f(x, \mu_N))$$
 $\Rightarrow V_N$ and μ_N satisfy the principle for $\tilde{\ell}$



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• knowing that μ_N is optimal for $\widetilde{J}_{\infty}(x_0,u)$ doesn't give us a simple way to estimate $J^{cl}_{\infty}(x_0,\mu_N)$



Recall: For both stabilizing terminal constraints the relaxed dynamic programming theorem yields the estimate

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But: How large is V_N ?

Without terminal constraints, the inequality $V_N \leq V_{\infty}$ is immediate

However, the terminal constraints also reverse this inequality, i.e., we have $V_N \geq V_{\infty}$ and the gap is very difficult to estimate



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General estimates for fixed N appear difficult to obtain. But we can give an asymptotic result for $N\to\infty$



Asymptotic Suboptimality

Theorem: For both types of terminal constraints the assumptions of the stability theorems ensure

$$V_N(x) \to V_\infty(x)$$

and thus

$$J^{cl}_{\infty}(x,\mu_N) \to V_{\infty}(x)$$

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Idea of proof: uses that any approximately optimal trajectory for J_{∞} converges to x_* and can thus be modified to meet the constraints with only moderately changing its value



 \bullet μ_N is infinite horizon optimal for a suitably altered running cost



- ullet μ_N is infinite horizon optimal for a suitably altered running cost
- the infinite horizon functional along the μ_N -controlled trajectory is bounded by V_N , i.e.,

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(5) Stability and suboptimality without

stabilizing constraints

MPC without stabilizing terminal constraints

We return to the basic MPC formulation

without any stabilizing terminal constraints and costs



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$$\underset{\mathbf{u} \text{ admissible}}{\operatorname{minimize}} \ J_N(x_0,u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k),\mathbf{u}(k)), \ x_{\mathbf{u}}(0) = x_0 = x_{\mu_N}(n)$$

without any stabilizing terminal constraints and costs

In order to motivate why we want to avoid terminal constraints and costs, we consider an example of P double integrators in the plane



Example: [Jahn '10] Consider P 4-dimensional systems

$$\dot{x}_i = f(x_i, u_i) := (x_{i2}, u_{i1}, x_{i4}, u_{i2})^T, \quad i = 1, \dots, P$$

Interpretation: $(x_{i1}, x_{i3})^T = \text{position}, (x_{i2}, x_{i4})^T = \text{velocity}$



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Stage cost:
$$\ell(x, u) = \sum_{i=1}^{P} \|(x_{i1}, x_{i3})^{T} - x_{d}\| + \|(x_{i2}, x_{i4})^{T}\| / 50$$

with $x_d = (0,0)^T$ until t = 20s and $x_d = (3,0)^T$ afterwards



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Constraints: no collision, obstacles, limited speed and control



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Interpretation: $(x_{i1}, x_{i3})^T = \text{position}, (x_{i2}, x_{i4})^T = \text{velocity}$

Stage cost:
$$\ell(x, u) = \sum_{i=1}^{r} \|(x_{i1}, x_{i3})^T - x_d\| + \|(x_{i2}, x_{i4})^T\| / 50$$

with $x_d = (0,0)^T$ until t = 20s and $x_d = (3,0)^T$ afterwards

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The simulation shows MPC for P=128 (\leadsto system dimension 512) with sampling time T=0.02s and horizon N=6



(Some) stability and performance results known in the literature:

[Alamir/Bornard '95]

[Shamma/Xiong '97, Primbs/Nevistić '00]

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Here we explain the last approach



Bounds on the optimal value function

Recall the definition of the optimal value function

$$V_N(x) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k, x), \mathbf{u}(k))$$

Boundedness assumption: there exists $\gamma > 0$ with

$$V_N(x) \le \gamma \ell^*(x)$$
 for all $x \in \mathbb{X}, N \in \mathbb{N}$

where
$$\ell^{\star}(x) := \min_{u \in \mathbb{I}} \ell(x, u)$$

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(sufficient conditions for and relaxations of this bound will be discussed later)

Stability and performance index

We choose ℓ , such that

$$\alpha_3(||x - x_*||) \le \ell^*(x) \le \alpha_4(||x - x_*||)$$

holds for $\alpha_3, \alpha_4 \in \mathcal{K}_{\infty}$ (again, $\ell(x, u) = ||x - x^*||^2 + \lambda ||u||^2$ works)



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Then, the only inequality left to prove in order to apply the relaxed dynamic programming theorem is

$$V_N(f(x,\mu_N(x))) \le V_N(x) - \alpha_N \ell(x,\mu_N(x))$$

for some $\alpha_N \in (0,1)$ and all $x \in \mathbb{X}$

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We can compute α_N from the bound $V_N(x) \leq \gamma \ell^*(x)$



We assume $V_N(x) \leq \gamma \ell^{\star}(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$ (*)

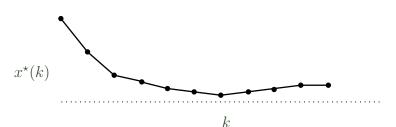
We want $V_N(f(x, \mu_N(x))) \leq V_N(x) - \alpha_N \ell(x, \mu_N(x))$

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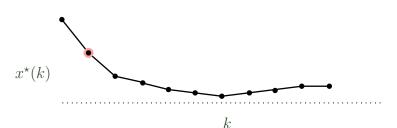
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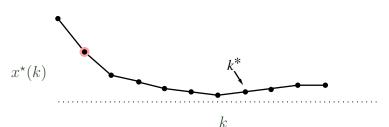




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• use (*) to find $\eta_N > 0$, $k^* \ge 1$ with $\ell^*(x^*(k^*)) \le \eta_N \ell^*(x^*(0))$

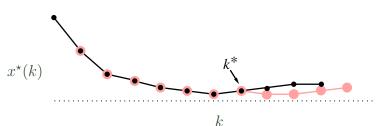




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- concatenate $x^*(1), \dots, x^*(k^*)$ and the optimal trajectory starting in $x^*(k^*)$

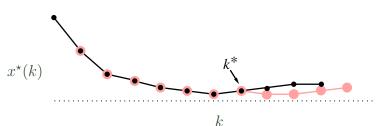




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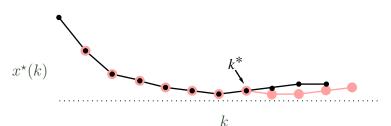




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- $\Rightarrow V_N(x^*(1)) \le J_N(x^*(1), \tilde{\mathbf{u}}) \le V_N(x^*(0)) (1 \gamma \eta_N) \ell(x^*(0), \mathbf{u}^*(0))$



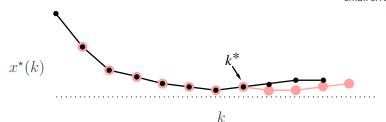


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$$\Rightarrow V_N(x^*(1)) \le J_N(x^*(1), \tilde{\mathbf{u}}) \le V_N(x^*(0)) - (1 - \underbrace{\gamma \eta_N) \ell(x^*(0), \mathbf{u}^*(0))}_{\text{= "small error"}}$$



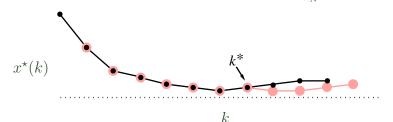


We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$ (*)

We want $V_N(x^*(1)) \leq V_N(x^*(0)) - \alpha_N \ell(x^*(0), \mathbf{u}^*(0))$

- use (*) to find $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$
- concatenate $x^*(1), \dots, x^*(k^*)$ and the optimal trajectory starting in $x^*(k^*) \iff \tilde{x}(\cdot), \ \tilde{\mathbf{u}}(\cdot)$

$$\Rightarrow V_N(x^*(1)) \le J_N(x^*(1), \tilde{\mathbf{u}}) \le V_N(x^*(0)) - \underbrace{(1 - \gamma \eta_N)}_{=\alpha_N} \ell(x^*(0), \mathbf{u}^*(0))$$





We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

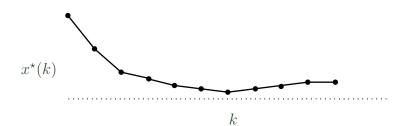
We want $\eta_N > 0$, $k^* \ge 1$ with $\ell^*(x^*(k^*)) \le \eta_N \ell^*(x^*(0))$



We assume
$$V_N(x) \leq \gamma \ell^\star(x)$$
 for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^* \ge 1$ with $\ell^*(x^*(k^*)) \le \eta_N \ell^*(x^*(0))$

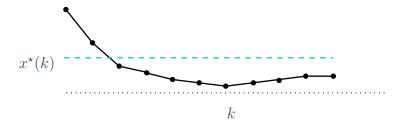
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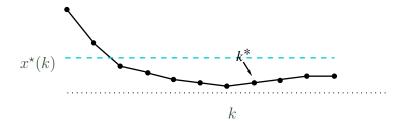
$$V_N(x) \leq \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \leq \gamma \ell^\star(x)/N \text{ for at least}$$
 one k^\star





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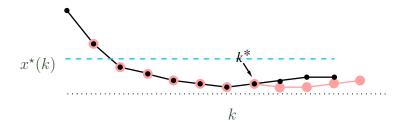
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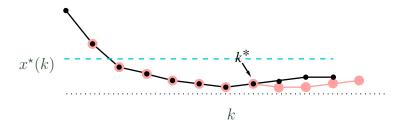
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$$V_N(x) \leq \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \leq \gamma \ell^\star(x)/N \text{ for at least}$$
 one $k^\star \quad \Rightarrow \quad \alpha_N = 1 - \gamma(\gamma - 1)/N$





We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

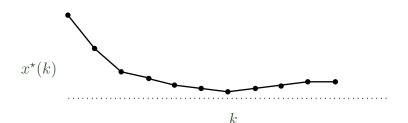
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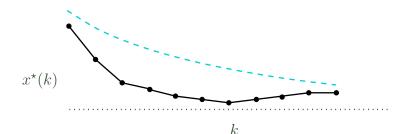
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$$V_N(x) \le \gamma \ell^*(x) \implies \ell(x^*(k), u^*(k)) \le \gamma \left(\frac{\gamma - 1}{\gamma}\right)^k \ell^*(x)$$

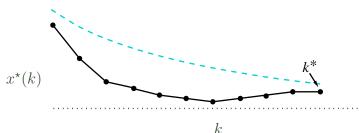


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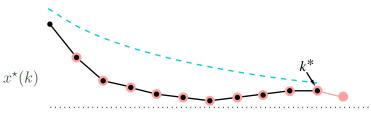


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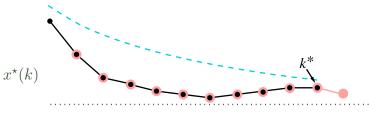
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Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

$$V_N(x) \le \gamma \ell^*(x) \implies \ell(x^*(k), u^*(k)) \le \gamma \left(\frac{\gamma - 1}{\gamma}\right)^k \ell^*(x)$$

$$\implies k^* = N - 1 \implies \alpha_N = 1 - (\gamma - 1)^N / \gamma^{N - 2}$$



k



We assume $V_N(x) \leq \gamma \ell^{\star}(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^\star \geq 1$ with $\ell^\star(x^\star(k^\star)) \leq \eta_N \ell^\star(x^\star(0))$

Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

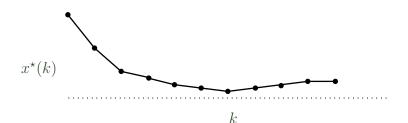


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Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

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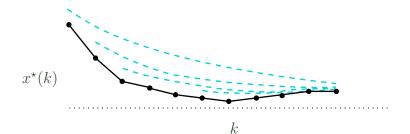


We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want
$$\eta_N > 0$$
, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

$$V_N(x) \le \gamma \ell^*(x) \implies \text{formulate all constraints}$$



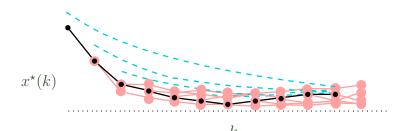


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Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

 $V_N(x) \leq \gamma \ell^{\star}(x) \implies$ formulate all constraints and trajectories





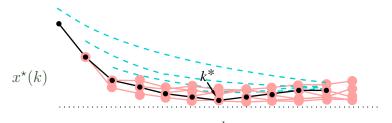
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Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

 $V_N(x) \le \gamma \ell^*(x) \implies$ formulate all constraints and trajectories

 \Rightarrow optimize for α_N





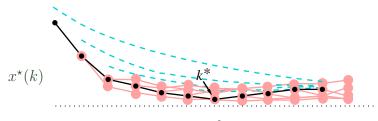
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Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

 $V_N(x) < \gamma \ell^*(x) \implies$ formulate all constraints and trajectories

$$\Rightarrow$$
 optimize for α_N \Rightarrow $\alpha_N = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-2}}$



Optimization approach to compute $lpha_N$

We explain the optimization approach (Variant 3) in more detail. We want α_N such that

$$V_N(x^*(1)) \le V_N(x^*(0)) - \alpha_N \ell(x^*(0), \mathbf{u}^*(0))$$

holds for all optimal trajectories $x^*(n), \mathbf{u}^*(n)$ for V_N



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The bound and the dynamic programming principle imply:

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$$\vdots \qquad \vdots \qquad \vdots$$



 $\longrightarrow V_N(x^*(1))$ is bounded by sums over $\ell(x^*(n), \mathbf{u}^*(n))$



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Find α_N , such that for all optimal trajectories x^* , \mathbf{u}^* :

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The inequalities from the last slides translate to

$$\sum_{n=k}^{N-1} \lambda_n \le \gamma \lambda_k, \quad k = 0, \dots, N-2$$
 (1)

$$\nu \le \sum_{n=1}^{J} \lambda_n + \gamma \lambda_{j+1}, \quad j = 0, \dots, N-2$$
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We call $\lambda_0, \dots, \lambda_{N-1}, \nu > 0$ with (1), (2) admissible



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 \Rightarrow if α_N is such that the inequality

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holds for all admissible λ_n and ν , then the desired inequality will hold for all optimal trajectories



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This is a linear optimization problem whose solution can be computed explicitly (which is nontrivial) and reads

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}}$$



Theorem: [Gr./Pannek/Seehafer/Worthmann '10]: Assume $V_N(x) \leq \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$. If

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then the NMPC closed loop is asymptotically stable with Lyapunov function ${\cal V}_{\cal N}$



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then the NMPC closed loop is asymptotically stable with Lyapunov function V_N and we get the performance estimate $J^{cl}_{\infty}(x,\mu_N) \leq V_{\infty}(x)/\alpha_N$ with

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}} \to 1 \quad \text{as} \quad N \to \infty$$

Conversely, if $N < 2 + \frac{\ln(\gamma - 1)}{\ln \gamma - \ln(\gamma - 1)}$, then there exists a system for which $V_N(x) \le \gamma \ell^*(x)$ holds but the NMPC closed loop is not asymptotically stable.



Horizon dependent γ -values

The theorem remains valid if we replace the bound condition

$$V_N(x) \le \gamma \ell^*(x)$$

by

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This allows for tighter bounds and a refined analysis



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Assume that for each $x_0 \in \mathbb{X}$ there exists an admissible control \mathbf{u} such that

$$\ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \le C\sigma^k \ell^*(x_0), \quad k = 0, 1, 2, \dots$$

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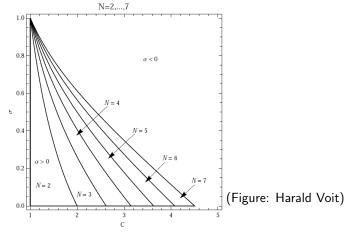
This allows to compute the minimal stabilizing horizon

$$\min\{N \in \mathbb{N} \mid \alpha_N > 0\}$$

depending on C and σ

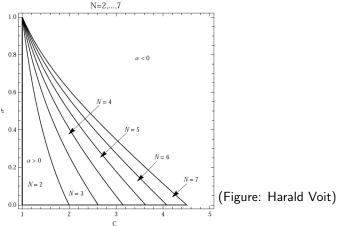


Stability chart for C and σ





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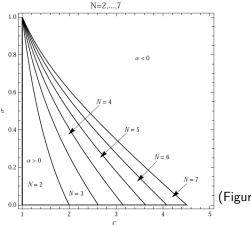
Conclusion: for short optimization horizon N it is

more important: small C ("small overshoot")

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Stability chart for C and σ



(Figure: Harald Voit)

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(we will see in the next section how to use this information)



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- positive definiteness of ℓ can be replaced by a detectability condition [Grimm/Messina/Tuna/Teel '05]



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 - ⇒ semiglobal practical asymptotic stability





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- The γ or γ_N can be computed from controllability properties, e.g., exponential controllability
- The overshoot bound C>0 plays a crucial role or obtaining small stabilizing horizons



(6) Examples for the design of MPC schemes

Design of "good" MPC running costs ℓ

We want small overshoot C in the estimate

$$\ell(x_{\mathbf{u}}(n), \mathbf{u}(n)) \le C\sigma^n \ell^{\star}(x_0)$$



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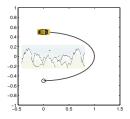
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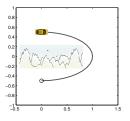
$$\ell(x_{\mathbf{u}}(n), \mathbf{u}(n)) \le C\sigma^n \ell^{\star}(x_0)$$

The trajectories $x_{\mathbf{u}}(n)$ are given, but we can use the running cost ℓ as design parameter



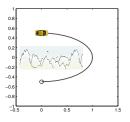






MPC with $\ell(x,u) = \|x - x^*\|^2 + |u|^2$ and $u_{\text{max}} = 0.2$ \longrightarrow asymptotic stability for N = 11 but not for $N \le 10$

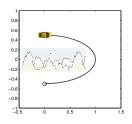




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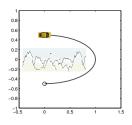
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$$\ell(x, u) = (x_1 - x_1^*)^2 + 5(x_2 - x_2^*)^2 + |u|^2$$





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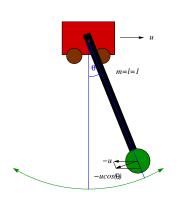
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$$\ell(x,u) = (x_1 - x_1^*)^2 + 5(x_2 - x_2^*)^2 + |u|^2 \longrightarrow \text{as. stab. for } N = 2$$



Example: pendulum on a cart



$$x_1 = \theta = \text{angle}$$

 $x_2 = \text{angular velocity}$
 $x_3 = \text{cart position}$
 $x_4 = \text{cart velocity}$
 $u = \text{cart acceleration}$

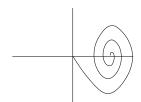
→ control system

$$\dot{x}_1 = x_2(t)
\dot{x}_2 = -g\sin(x_1) - kx_2
-u\cos(x_1)
\dot{x}_3 = x_4
\dot{x}_4 = u$$



Reducing overshoot for swingup of the pendulum on a cart:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = g\sin(x_1) - kx_2 + u\cos(x_1)
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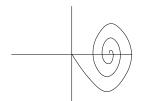
Typical swingup trajectory x_1 and x_2 component



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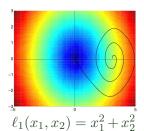
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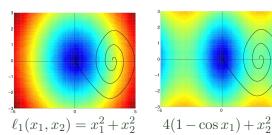
sampling time T = 0.15

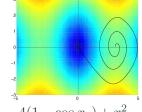


Reducing overshoot for swingup of the pendulum on a cart:

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$$N = 10$$

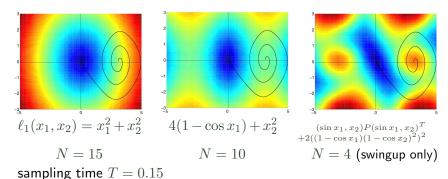
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A PDE example

We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with $\label{eq:sigma} \begin{array}{l} \mbox{domain } \Omega = [0,1] \\ \mbox{solution } y = y(t,x) \\ \mbox{boundary conditions } y(t,0) = y(t,1) = 0 \\ \mbox{parameters } \nu = 0.1 \mbox{ and } \mu = 10 \end{array}$

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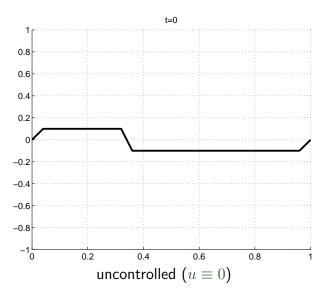
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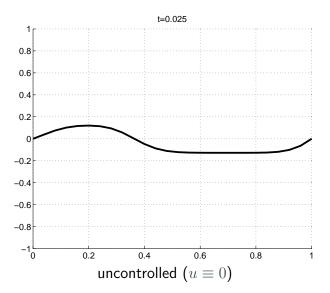
and distributed control $u: \mathbb{R} \times \Omega \to \mathbb{R}$

Discrete time system: $y(n) = y(nT, \cdot)$, sampling time T = 0.025

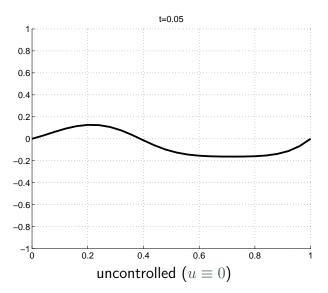




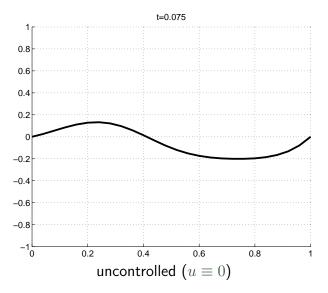




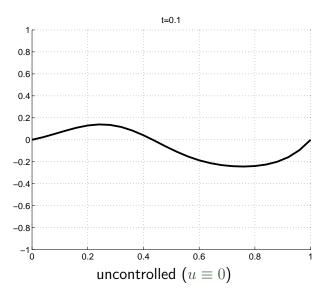




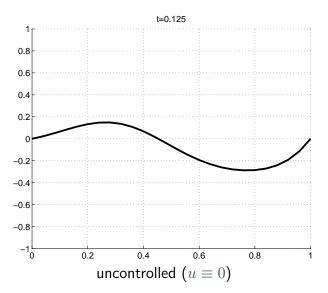




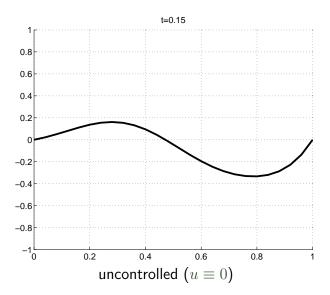




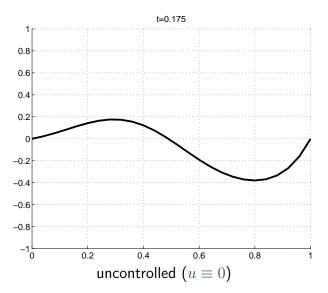




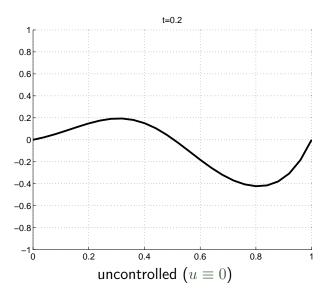




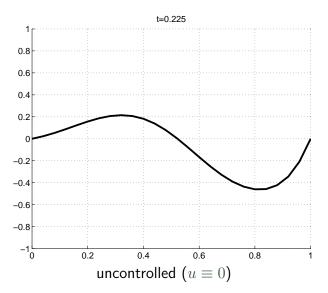




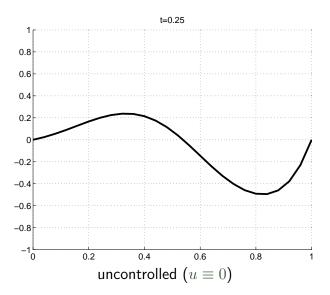




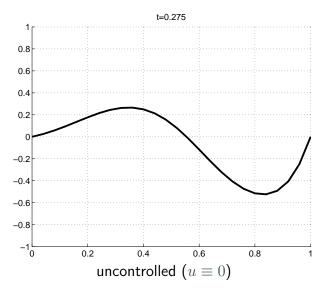




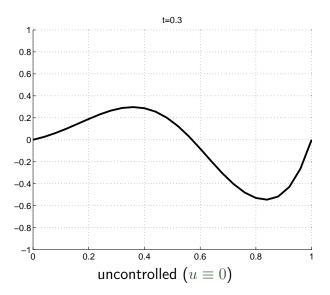




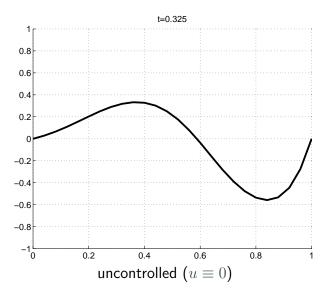




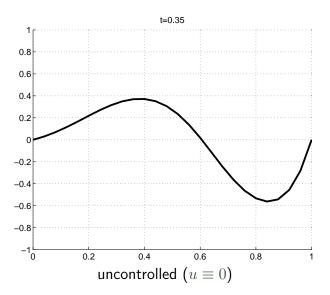




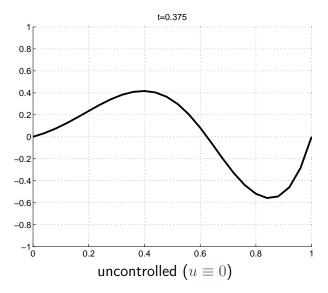




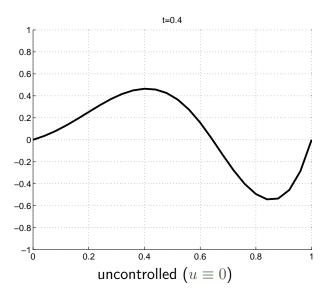




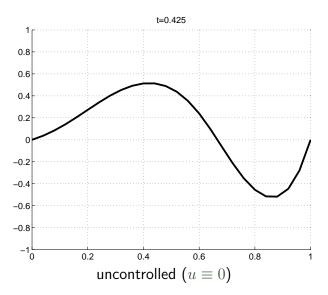




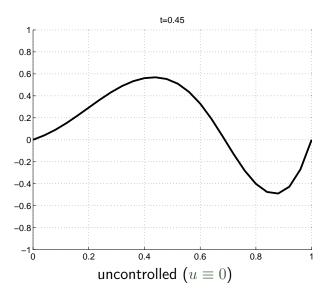




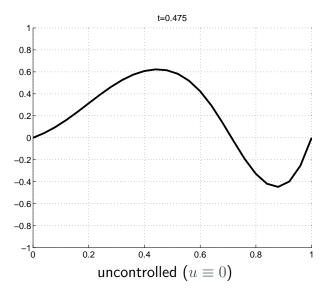




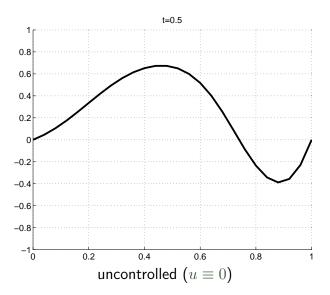




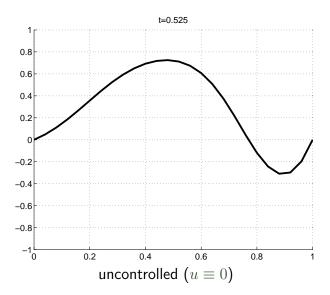




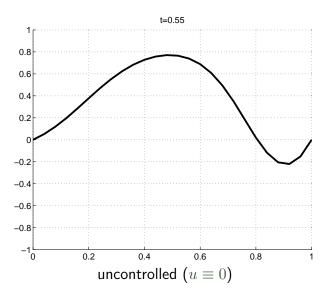




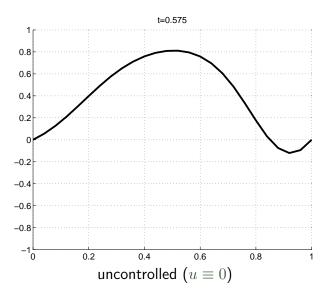




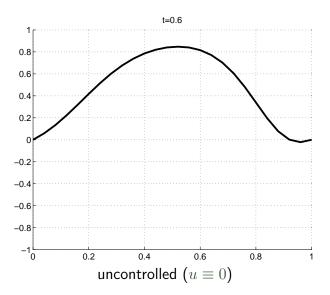




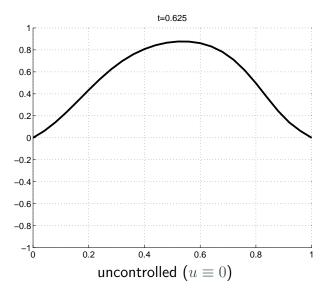




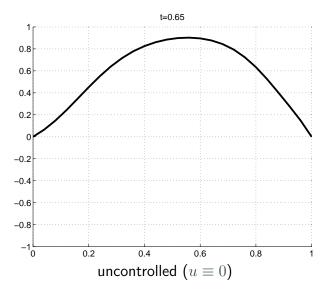




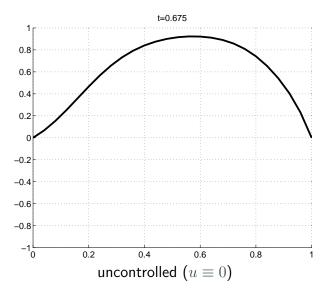




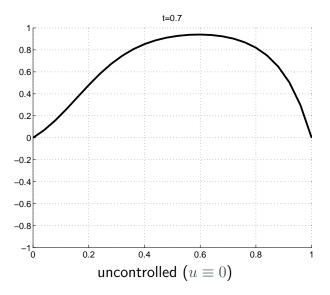




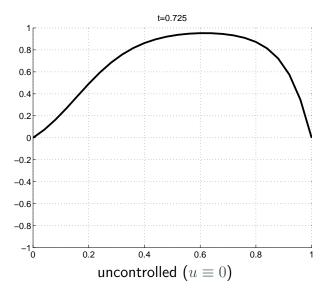




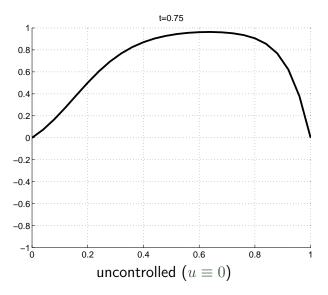




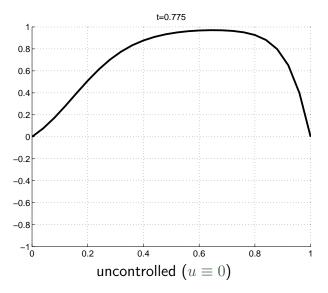




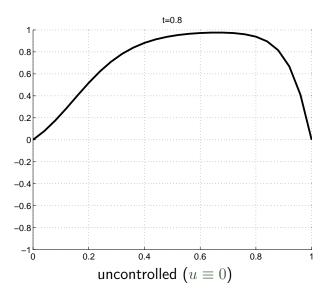




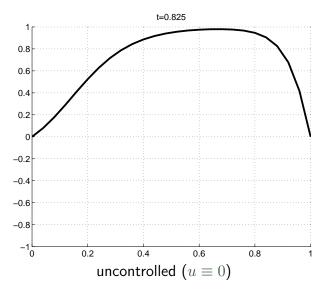




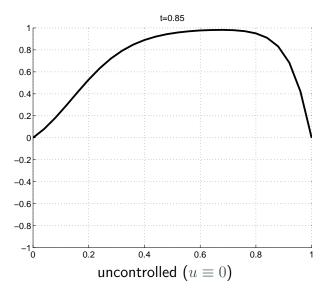




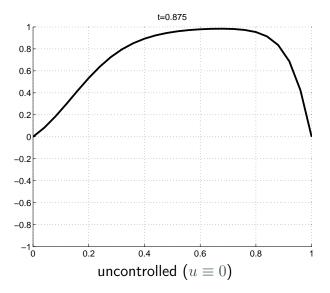




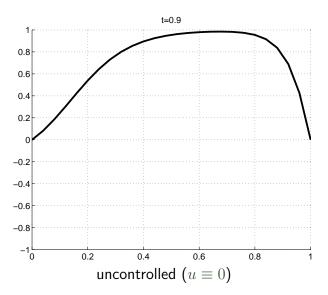




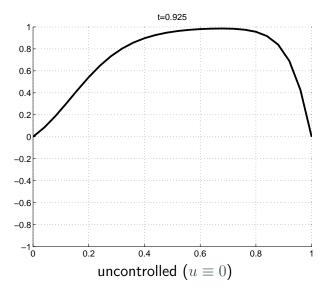




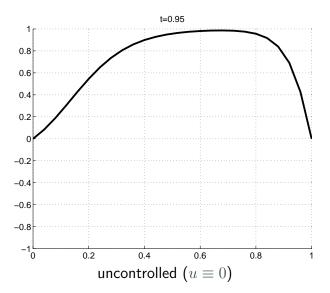




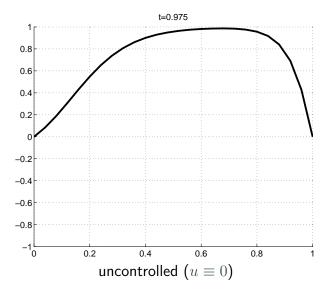




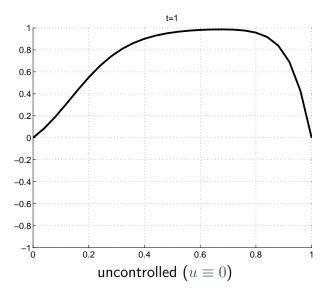






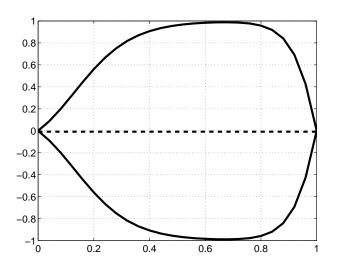








The uncontrolled PDE



all equilibrium solutions



$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$



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Goal: stabilize the sampled data system y(n) at $y \equiv 0$



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Usual approach: quadratic L^2 cost

$$\ell(y(n), u(n)) = ||y(n)||_{L^2}^2 + \lambda ||u(n)||_{L^2}^2$$



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for $||y_x||_{L^2} \gg ||y||_{L^2}$ this can only hold if $C \gg 0$



Conclusion: because of

$$||y(n)||_{L^2}^2 + \lambda ||y_x(n)||_{L^2}^2 \le C\sigma^n ||y(0)||_{L^2}^2$$

the controllability condition may only hold for very large ${\cal C}$



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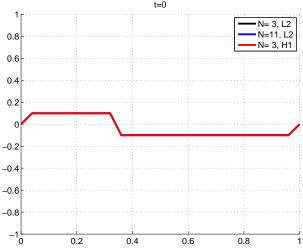
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Then an analogous computation yields

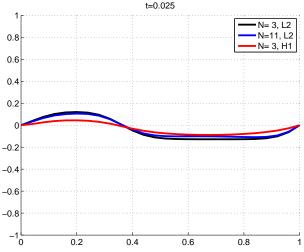
$$||y(n)||_{L^2}^2 + (1+\lambda)||y_x(n)||_{L^2}^2 \le C\sigma^n (||y(0)||_{L^2}^2 + ||y_x(0)||_{L^2}^2)$$





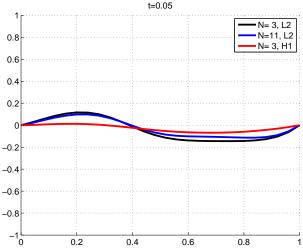
MPC with L_2 and H_1 cost, $\lambda=0.1$, sampling time T=0.025





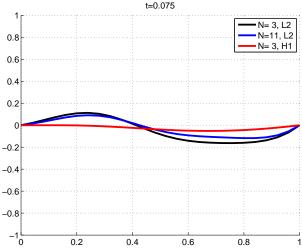
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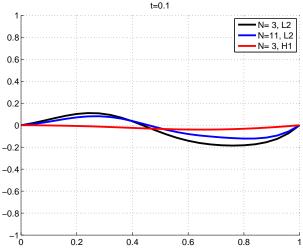
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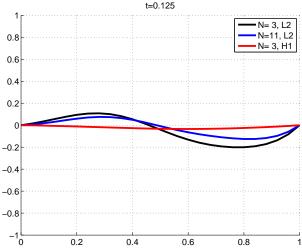
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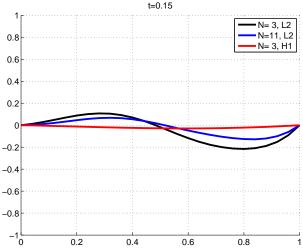
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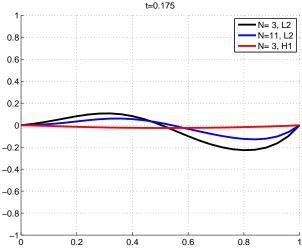
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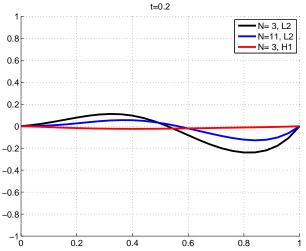
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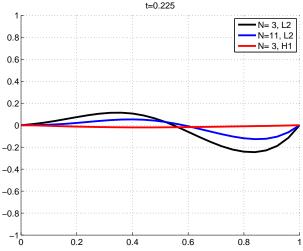
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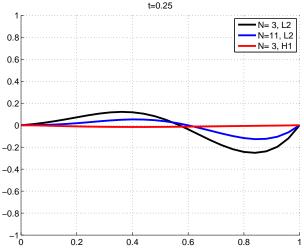
MPC with L_2 and H_1 cost, $\lambda=0.1$, sampling time T=0.025





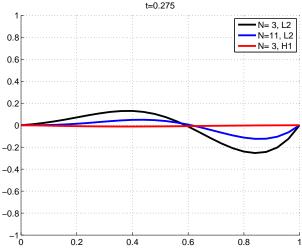
MPC with L_2 and H_1 cost, $\lambda=0.1$, sampling time T=0.025





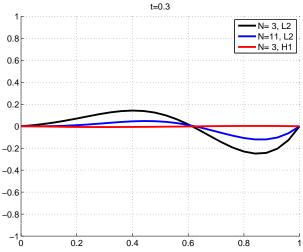
MPC with L_2 and H_1 cost, $\lambda=0.1$, sampling time T=0.025





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Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

$$\operatorname{domain}\,\Omega=[0,1]$$

solution
$$y = y(t, x)$$

boundary conditions
$$y(t,0) = u_0(t)$$
, $y(t,1) = u_1(t)$

parameters
$$\nu=0.1$$
 and $\mu=10$

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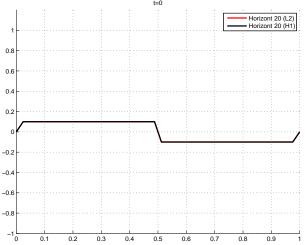
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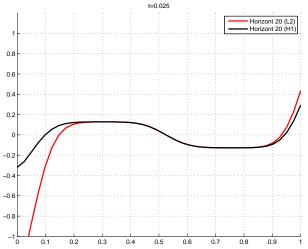
 \longrightarrow L^2 should perform better that H^1





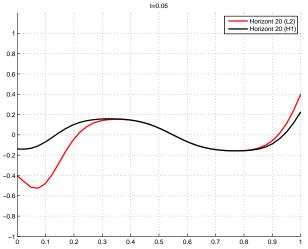
Boundary control, $\lambda=0.001$, sampling time T=0.025





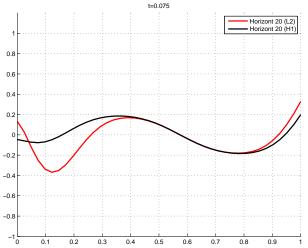
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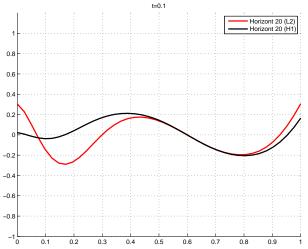
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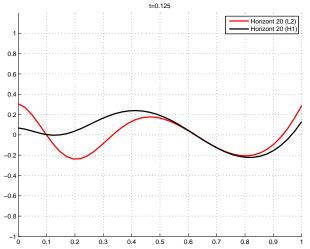
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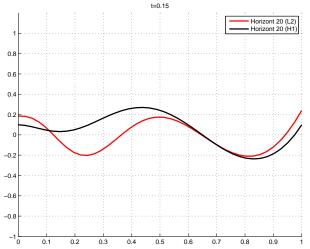
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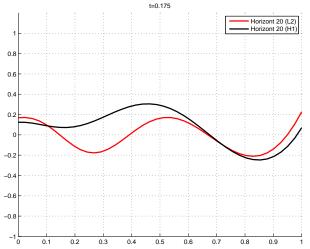
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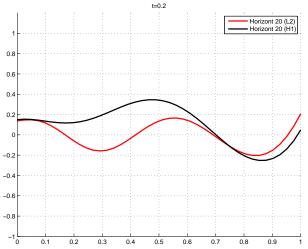
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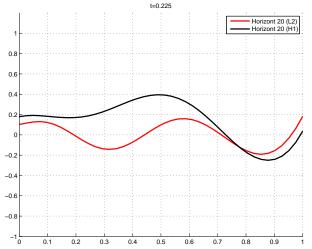
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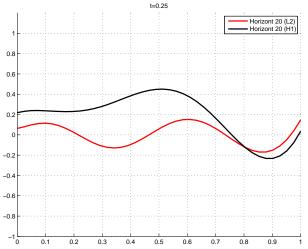
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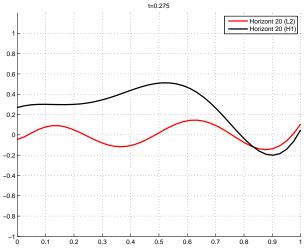
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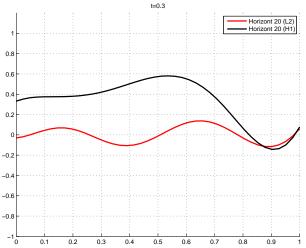
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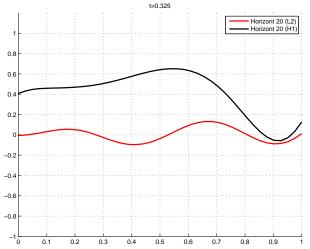
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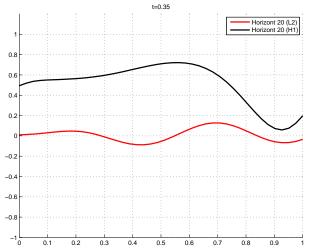
Boundary control, $\lambda=0.001$, sampling time T=0.025





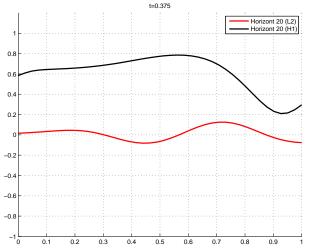
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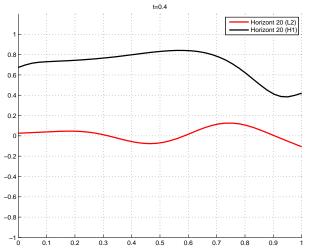
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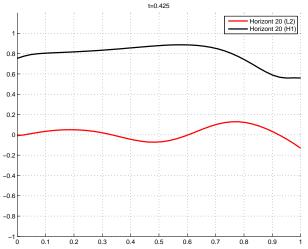
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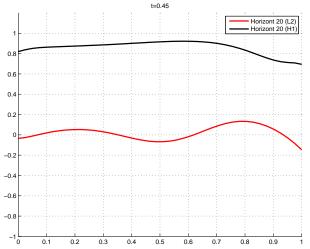
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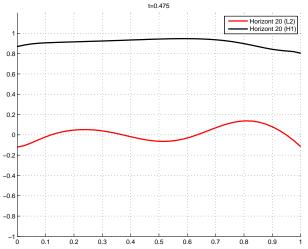
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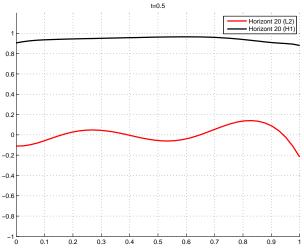
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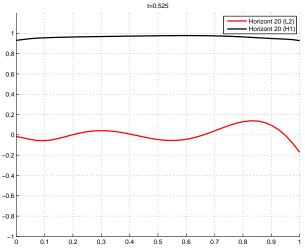
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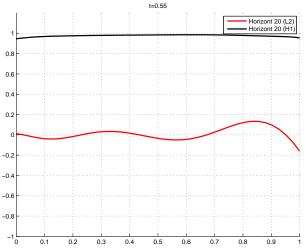
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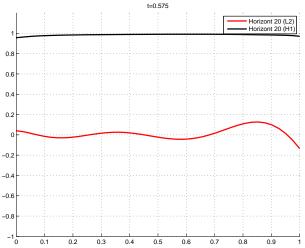
Boundary control, $\lambda=0.001$, sampling time T=0.025





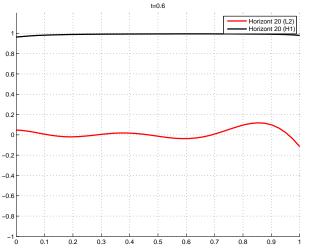
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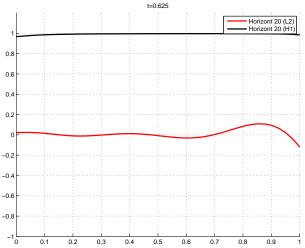
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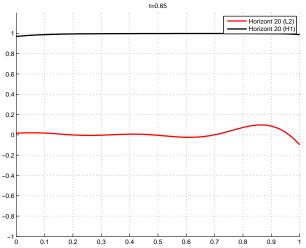
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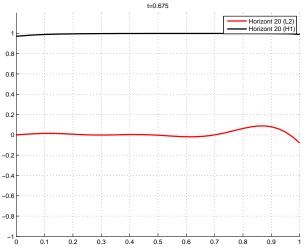
Boundary control, $\lambda=0.001$, sampling time T=0.025





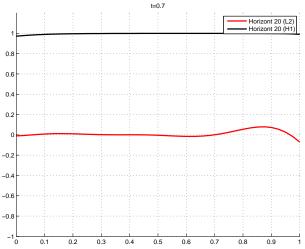
Boundary control, $\lambda=0.001$, sampling time T=0.025





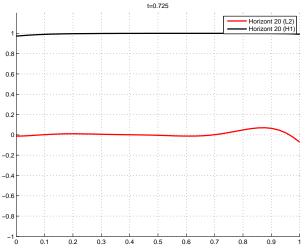
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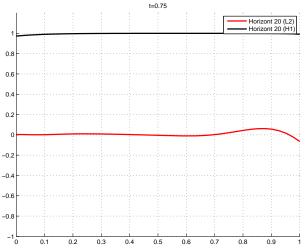
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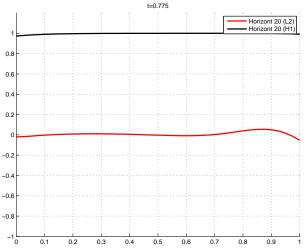
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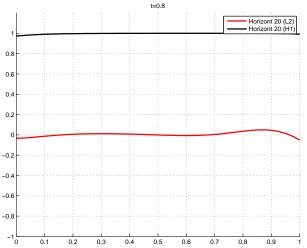
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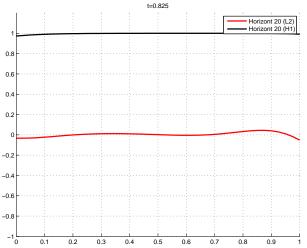
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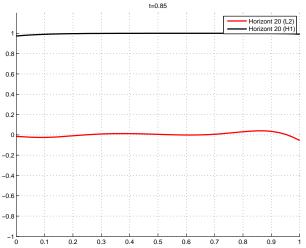
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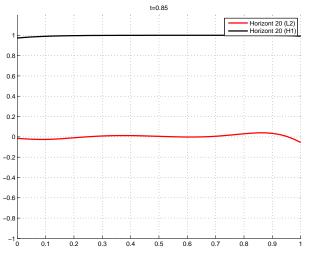
Boundary control, $\lambda=0.001$, sampling time T=0.025





Boundary control, $\lambda=0.001$, sampling time T=0.025





Boundary control, $\lambda=0.001$, sampling time T=0.025 Can be made rigorous for many PDEs [Altmüller et al. '10ff]



Summary of Section (6)

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- Reducing the overshoot constant C by choosing ℓ appropriately can significantly reduce the horizon N needed to obtain stability
- Computing tight estimates for C is in general a difficult if not impossible task
- But structural knowledge of the system behavior can be sufficient for choosing a "good" \(\ell \)



(7) Feasibility

Feasibility

Consider the feasible sets

 $\mathcal{F}_N := \{x \in \mathbb{X} \mid \text{there exists an admissible } \mathbf{u} \text{ of length } N\}$



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 $\mathcal{F}_N := \{x \in \mathbb{X} \mid \text{there exists an admissible } \mathbf{u} \text{ of length } N\}$

So far we have assumed

$$V_N(x) \le \gamma \ell^*(x)$$
 for all $x \in \mathbb{X}$

which implicitly includes the assumption

$$\mathcal{F}_N = \mathbb{X}$$

because
$$V_N(x) = \infty$$
 for $x \in \mathbb{X} \setminus \mathcal{F}_N$



Feasibility

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So far we have assumed

$$V_N(x) \le \gamma \ell^*(x)$$
 for all $x \in \mathbb{X}$

which implicitly includes the assumption

$$\mathcal{F}_N = \mathbb{X}$$

because
$$V_N(x) = \infty$$
 for $x \in \mathbb{X} \setminus \mathcal{F}_N$

What happens if $\mathcal{F}_N \neq \mathbb{X}$ for some $N \in \mathbb{N}$?



The MPC feasibility problem

Even though the open-loop optimal trajectories are forced to satisfy $x^{\star}(k) \in \mathbb{X}$, the closed loop solutions $x_{\mu_N}(n)$ may violate the state constraints, i.e., $x_{\mu_N}(n) \not\in \mathbb{X}$ for some n



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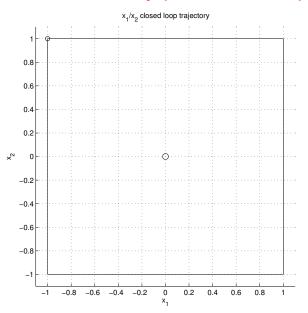
We illustrate this phenomenon by the simple example

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + u/2 \\ x_2 + u \end{pmatrix}$$

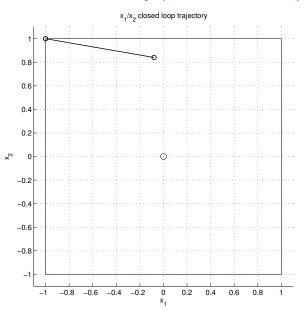
with $\mathbb{X}=[-1,1]^2$ and $\mathbb{U}=[-1/4,1/4].$ For initial value $x_0=(-1,1)^T$, the system can be controlled to 0 without leaving \mathbb{X}

We use MPC with N=2 and $\ell(x,u)=\|x\|^2+5u^2$

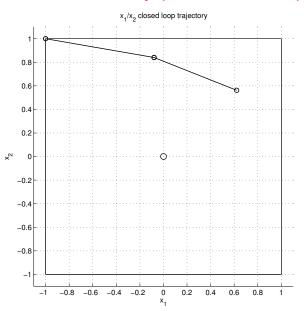




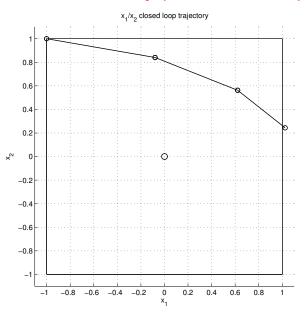




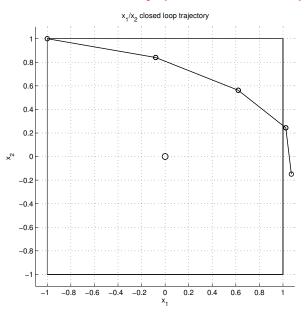




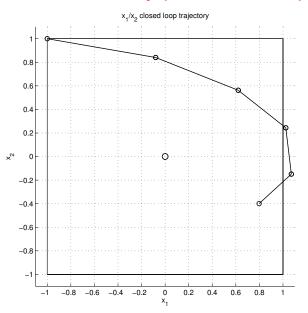




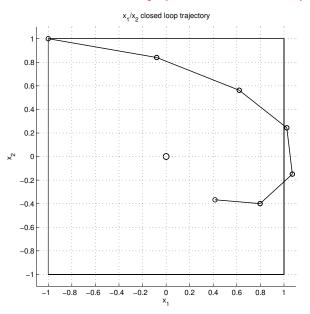




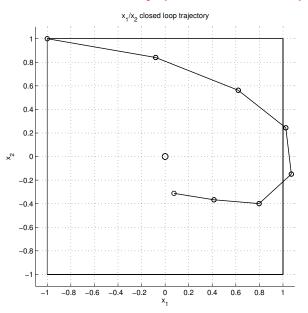




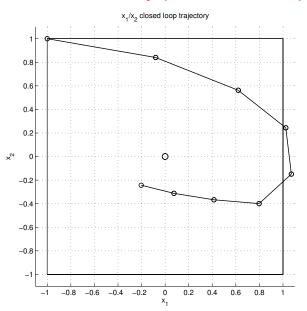




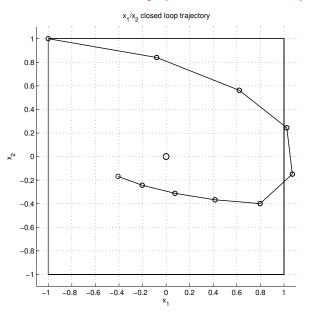




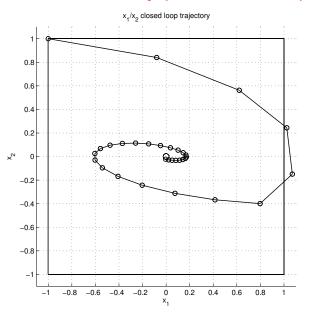














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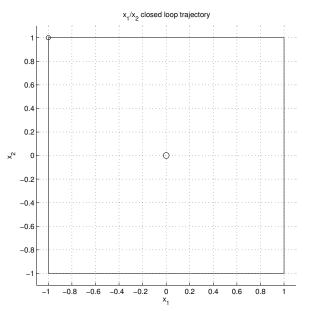
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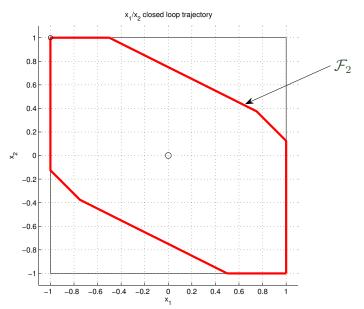
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- $\longrightarrow x_{\mu_N}(n+k)$ is inevitable for some $k \geq 2$

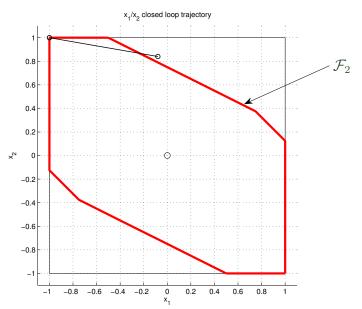




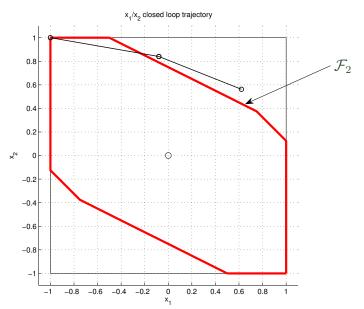




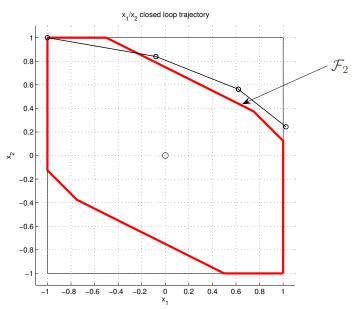




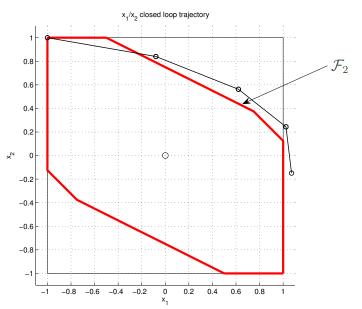




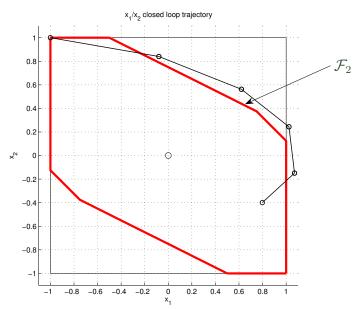




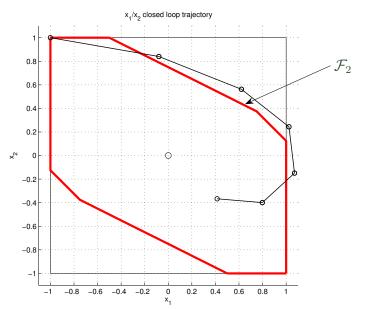




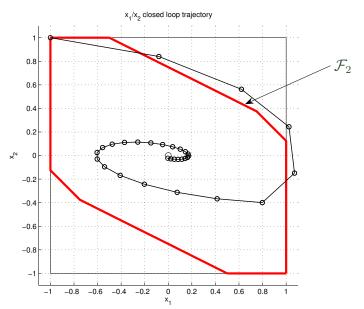














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(this was part of the stability theorem in Section 3)



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Can we find recursively feasible sets for NMPC without terminal constraints?



Theorem: [Kerrigan '00, Gr./Pannek 11] Assume that

$$\mathcal{F}_{N_0} = \mathcal{F}_{N_0-1}$$

holds for some $N_0 \in \mathbb{N}$. Then the set \mathcal{F}_N is recursively feasible for all $N > N_0$.

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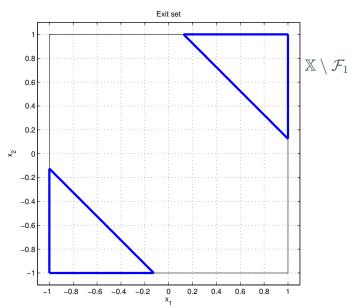
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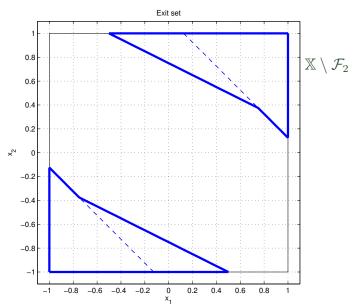


Feasible sets for our example



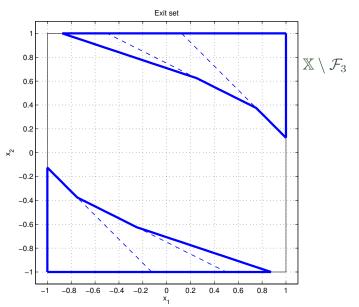


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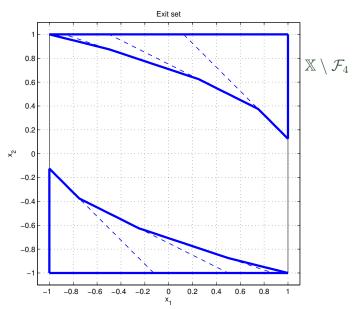


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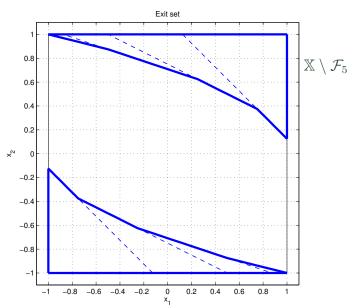


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If \mathbb{X} is compact, then $A_c = \mathcal{F}_{\infty}$ for all sufficiently large N



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If this is desired, a forward invariant terminal constraint \mathbb{X}_0 can be used without terminal cost — the stability proof without terminal constraints also works for this setting





Properties of MPC without terminal constraints compared to terminal constrained MPC

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