Deterministic
Stabilizing and Economic MPC

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Contents

(1) Introduction

What is Model Predictive Control (MPC)?

Setup

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x_0$$

or, briefly

$$x^+ = f(x, u)$$

with $x \in X$, $u \in U$

- we consider discrete time systems for simplicity of exposition
- continuous time systems can be treated by using the discrete time representation of the corresponding sampled data system or a numerical approximation
- X and U depend on the model. These may be Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ or more general (e.g., infinite dimensional) spaces. For simplicity of exposition we assume that we have a norm $\|\cdot\|$ on both spaces

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Prototype Problem

Assume there exists an equilibrium $x_* \in X$ for $u = 0$, i.e.

$$f(x_*, 0) = x_*$$

Task: stabilize the system

$$x^+ = f(x, u)$$

at x_* via static state feedback, i.e., find $\mu : X \to U$, such that x_* is asymptotically stable for the feedback controlled system

$$x_\mu(n+1) = f(x_\mu(n), \mu(x_\mu(n))), \quad x_\mu(0) = x_0$$

Additionally, we impose state constraints $x_\mu(n) \in \mathbb{X}$ and control constraints $\mu(x(n)) \in \mathbb{U}$

for all $n \in \mathbb{N}$ and given sets $\mathbb{X} \subseteq X$, $\mathbb{U} \subseteq U$

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Prototype Problem

Asymptotic stability means

Attraction: $x_\mu(n) \to x_*$ as $n \to \infty$

plus

Stability: Solutions starting close to x_* remain close to x_*

(we will later formalize this property using $\mathcal{KL}$ functions)

Informal interpretation: control the system to x_* and keep it there while obeying the state and control constraints

Idea of MPC: use an optimal control problem which minimizes the distance to x_* in order to synthesize a feedback law μ

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The idea of MPC

For defining the MPC scheme, we choose a stage cost $\ell(x, u)$ penalizing the distance from x_* and the control effort, e.g., $\ell(x, u) = \|x - x_*\|^2 + \lambda\|u\|^2$ for $\lambda \geq 0$

The basic idea of MPC is:

- minimize the summed stage cost along trajectories generated from our model over a prediction horizon N
- use the first element of the resulting optimal control sequence as feedback value
- repeat this procedure iteratively for all sampling instants $n = 0, 1, 2, \ldots$

Notation in what follows:

- general feedback laws will be denoted by μ
- the MPC feedback law will be denoted by μ_N

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The basic MPC scheme

Formal description of the basic MPC scheme:

At each time instant n solve for the current state $x_{\mu_N}(n)$

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_{\mu_N}(n),\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k),\mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_{\mu_N}(n)$$

($\mathbf{u}$ admissible $\Leftrightarrow$ $\mathbf{u}(k) \in \mathbb{U}$ and $x_{\mathbf{u}}(k) \in \mathbb{X}$)

$\rightsquigarrow$ optimal trajectory $x^\star(0),\ldots,x^\star(N)$

with optimal control $\mathbf{u}^\star(0),\ldots,\mathbf{u}^\star(N-1)$

Define the MPC feedback law $\mu_N(x_{\mu_N}(n)) := \mathbf{u}^\star(0)$

$\rightsquigarrow$ $x_{\mu_N}(n+1) = f(x_{\mu_N}(n),\mu_N(x_{\mu_N}(n))) = f(x_{\mu_N}(n),\mathbf{u}^\star(0)) = x^\star(1)$

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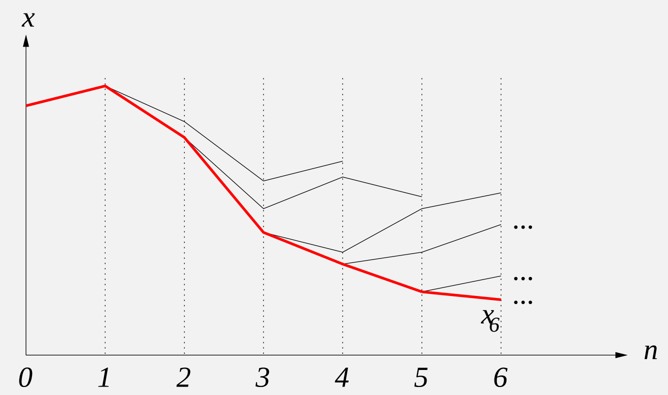
MPC from the trajectory point of view



black = predictions (open loop optimization)

red = MPC closed loop, $x_n = x_{\mu_N}(n)$

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Model predictive control (aka Receding horizon control)

Idea first formulated by A.I. Propoi in 1963, often rediscovered

used in industrial applications since the mid 1970s, mainly for constrained linear systems [Qin & Badgwell, 1997, 2001]

more than 9000 industrial MPC applications in Germany counted in [Dittmar & Pfeifer, 2005]

development of theory since ∼1980 (linear), ∼1990 (nonlinear)

seminal paper for stabilizing MPC: [D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, Automatica, 36(2000), 789–814]

Central questions:

- When does MPC stabilize the system?
- How good is the performance of the MPC feedback law?
- How long does the optimization horizon N need to be?

and, of course, the development of good algorithms (not topic of this course)

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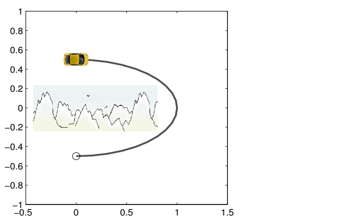
An example



$$x_1^+ = \sin(\varphi + u)$$
$$x_2^+ = \cos(\varphi + u)/2$$

with $\varphi = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$

$\mathbb{X} = \{x \in \mathbb{R}^2 : \|(x_1, 2x_2)^T\| = 1\}$, $\mathbb{U} = [0, u_{\max}]$

$x_* = (0, -1/2)^T$, $x_0 = (0, 1/2)^T$

MPC with $\ell(x,u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$ yields asymptotic stability for $N = 11$ but not for $N \leq 10$

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Summary of Section (1)

- MPC is an online optimal control based method for computing stabilizing feedback laws
- MPC computes the feedback law by iteratively solving finite horizon optimal control problems using the current state $x_0 = x_{\mu_N}(n)$ as initial value
- the feedback value $\mu_N(x_0)$ is the first element of the resulting optimal control sequence
- MPC can considerably reduce the computation time needed for solving infinite horizon optimal control problems, up to real time capability

  $\rightsquigarrow$ model reduction in time
- the car-and-mountain example shows that MPC does not always yield an asymptotically stabilizing feedback law

(2a) Background material:

Lyapunov functions

Purpose of this section

We introduce Lyapunov functions as a tool to rigorously verify asymptotic stability

In the subsequent sections, this will be used in order to establish asymptotic stability of the MPC closed loop

In this section, we consider discrete time systems without input, i.e.,

$$x^+ = g(x)$$

with $x \in X$ or, in long form

$$x(n+1) = g(x(n)), \;\; x(0) = x_0$$

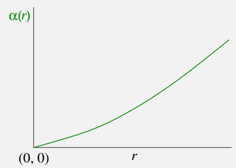
(later we will apply the results to $g(x) = f(x, \mu_N(x))$)

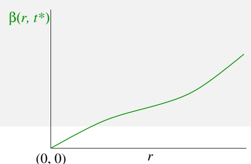
Note: we do not require g to be continuous

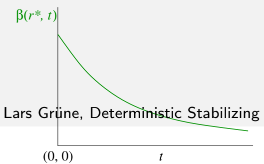
Comparison functions

For $\mathbb{R}_0^+ = [0, \infty)$ we use the following classes of comparison functions

$$\mathcal{K} \;:=\; \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \;\middle|\; \begin{array}{l} \alpha \text{ is continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array} \right\}$$







Asymptotic stability revisited

A point x_* is called an equilibrium of $x^+ = g(x)$ if $g(x_*) = x_*$

A set $Y \subseteq X$ is called forward invariant for $x^+ = g(x)$ if $g(x) \in Y$ holds for each $x \in Y$

We say that x_* is asymptotically stable for $x^+ = g(x)$ on a forward invariant set Y if there exists $\beta \in \mathcal{KL}$ such that

$$\|x(n) - x_*\| \leq \beta(\|x(0) - x_*\|, n)$$

holds for all $x \in Y$ and $n \in \mathbb{N}$

How can we check whether this property holds?

Lyapunov function

Let $Y \subseteq X$ be a forward invariant set and $x_* \in X$. A function $V : Y \to \mathbb{R}_0^+$ is called a Lyapunov function for $x^+ = g(x)$ if the following two conditions hold for all $x \in Y$:

(i) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$$

(ii) There exists $\alpha_V \in \mathcal{K}$ such that

$$V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$$

Stability theorem

Theorem: If the system $x^+ = g(x)$ admits a Lyapunov function V on a forward invariant set Y, then x_* is an asymptotically stable equilibrium on Y

Idea of proof: $V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$ implies that V is strictly decaying along solutions away from x_*

This allows to construct $\tilde{\beta} \in \mathcal{KL}$ with $V(x(n)) \leq \tilde{\beta}(V(x(0)), n)$

The bounds $\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$ imply that asymptotic stability holds with $\beta(r, t) = \alpha_1^{-1}(\tilde{\beta}(\alpha_2(r), t))$

Lyapunov functions — discussion

While the convergence $x(n) \to x_*$ is typically non-monotone for an asymptotically stable system, the convergence $V(x(n)) \to 0$ is strictly monotone

It is hence sufficient to check the decay of V in one time step

$\rightsquigarrow$ it is typically quite easy to check whether a given function is a Lyapunov function

But it is in general difficult to find a candidate for a Lyapunov function

For MPC, we will use the optimal value functions which we introduce in the next section

(2b) Background material: Dynamic Programming

Purpose of this section

We define the optimal value functions V_N for the optimal control problem

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

used within the MPC scheme (with $x_0 = x_{\mu_N}(n)$)

We present the dynamic programming principle, which establishes a relation for these functions and will eventually enable us to derive conditions under which V_N is a Lyapunov function

Optimal value functions

We define the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u}\text{ admissible}} J_N(x_0, \mathbf{u})$$

setting $V_N(x_0) := \infty$ if x_0 is not feasible, i.e., if there is no admissible $\mathbf{u}$ (recall: $\mathbf{u}$ admissible $\Leftrightarrow$ $x_{\mathbf{u}}(k) \in \mathbb{X}$, $\mathbf{u}(k) \in \mathbb{U}$)

An admissible control sequence $\mathbf{u}^\star$ is called optimal, if

$$J_N(x_0, \mathbf{u}^\star) = V_N(x_0)$$

Note: an optimal $\mathbf{u}^\star$ does not need to exist in general. In the sequel we assume that $\mathbf{u}^\star$ exists if x_0 is feasible

Dynamic Programming Principle

Theorem: (Dynamic Programming Principle) For any feasible $x_0 \in \mathbb{X}$ the optimal value function satisfies

$$V_N(x_0) = \inf_{\substack{u \in \mathbb{U} \\ f(x_0,u) \in \mathbb{X}}} \{\ell(x_0, u) + V_{N-1}(f(x_0, u))\}$$

Moreover, if $\mathbf{u}^\star$ is an optimal control, then

$$V_N(x_0) = \ell(x_0, \mathbf{u}^*(0)) + V_{N-1}(f(x_0, \mathbf{u}^\star(0)))$$

holds.

Idea of Proof: Follows by taking infima in the identity

$$\begin{aligned} J_N(x_0, \mathbf{u}) &= \ell(x_{\mathbf{u}}(0), \mathbf{u}(0)) + \sum_{k=1}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \\ &= \ell(x_0, \mathbf{u}(0)) + J_{N-1}(f(x_0, \mathbf{u}(0)), \mathbf{u}(\cdot + 1)) \end{aligned}$$

Corollaries

Corollary: Let $x^\star$ be an optimal trajectory of length N with optimal control $u^\star$ and $x^\star(0) = x$. Then

(i) The "tail"

$$\big(x^\star(k), x^\star(k+1), \ldots, x^\star(N-1)\big)$$

is an optimal trajectory of length $N-k$.

(ii) The MPC feedback μ_N satisfies

$$\mu_N(x) \in \operatorname*{argmin}_{u\in\mathbb{U}} \{\ell(x,u) + V_{N-1}(f(x,u))\}$$

(i.e., $u = \mu_N(x)$ minimizes this expression),

$$V_N(x) = \ell(x,\mu_N(x)) + V_{N-1}(f(x,\mu_N(x)))$$

and

$$u^\star(k) = \mu_{N-k}(x^\star(k)), \quad k = 0, \ldots, N-1$$

Dynamic Programming Principle — discussion

We will see later, that under suitable conditions the optimal value function will play the role of a Lyapunov function for the MPC closed loop

The dynamic programming principle and its corollaries will prove to be important tools to establish this fact

In order to see why this can work, in the next section we briefly look at infinite horizon optimal control problems

Moreover, for simple systems the principle can be used for computing V_N and μ_N — we will see an example in the excercises

(2c) Background material: Relaxed Dynamic Programming

Infinite horizon optimal control

Just like the finite horizon problem we can define the infinite horizon optimal control problem

$$\operatorname*{minimize}_{\mathbf{u}\text{ admissible}} \; J_\infty(x_0,\mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_{\mathbf{u}}(k),\mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

and the corresponding optimal value function

$$V_\infty(x_0) := \inf_{\mathbf{u}\text{ admissible}} J_\infty(x_0,\mathbf{u})$$

If we could compute an optimal feedback μ_∞ for this problem (which is — in contrast to computing μ_N — in general a very difficult problem), we would have solved the stabilization problem

Infinite horizon dynamic programming principle

Recall the corollary from the finite horizon dynamic programming principle

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

The corresponding result which can be proved for the infinite horizon problem reads

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x)))$$

$\rightsquigarrow$ if $\ell(x, \mu_\infty(x)) \geq \alpha_V(\|x - x_*\|)$ holds, then we get

$$V_\infty(f(x, \mu_\infty(x))) \leq V_\infty(x) - \alpha_V(\|x - x_*\|)$$

and if in addition $\alpha_1(\|x - x_*\|) \leq V_\infty(x) \leq \alpha_2(\|x - x_*\|)$ holds, then V_∞ is a Lyapunov function $\rightsquigarrow$ asymptotic stability

Relaxing dynamic programming

Unfortunately, an equation of the type

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x)))$$

cannot be expected if we replace "∞" by "N" everywhere (in fact, it would imply $V_N = V_\infty$)

However, we will see that we can establish relaxed versions of this inequality in which we

- relax "$=$" to "$\geq$"
- relax $\ell(x, \mu(x))$ to $\alpha\ell(x, \mu(x))$ for some $\alpha \in (0, 1]$

$\rightsquigarrow$ $V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$

"relaxed dynamic programming inequality" [Rantzer et al. '06ff]

What can we conclude from this inequality?

Relaxed dynamic programming

We define the infinite horizon performance of the MPC closed loop system $x^+ = f(x, \mu_N(x))$ as

$$J_\infty^{cl}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \quad x_{\mu_N}(0) = x_0$$

Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a forward invariant set for the MPC closed loop and assume that

$$V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

Then for all $x \in Y$ the infinite horizon performance satisfies

$$J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$$

Relaxed dynamic programming

Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the inequalities

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N.

Proof: The assumed inequalities immediately imply that $V = V_N$ is a Lyapunov function for $x^+ = g(x) = f(x, \mu_N(x))$ with

$$\alpha_1(r) = \alpha_3(r), \quad \alpha_V(r) = \alpha\,\alpha_3(r)$$

$\Rightarrow$ asymptotic stability

Relaxed dynamic programming

For proving the performance estimate $J_\infty^{cl}(x_0,\mu_N) \le V_N(x_0)/\alpha$, the relaxed dynamic programming inequality implies

$$\begin{aligned}
&\alpha \sum_{n=0}^{K-1} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \\
\le\ & \sum_{n=0}^{K-1} \Big(V_N(x_{\mu_N}(n)) - V_N(x_{\mu_N}(n+1)) \Big) \\
=\ & V_N(x_{\mu_N}(0)) - V_N(x_{\mu_N}(K)) \ \le\ V_N(x_{\mu_N}(0))
\end{aligned}$$

Since all summands are ≥ 0, this implies that the limit for $K \to \infty$ exists and we get

$$\alpha J_\infty^{cl}(x_0,\mu_N) = \alpha \sum_{n=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \le V_N(x_{\mu_N}(0))$$

$\Rightarrow$ assertion

Summary of Section (2)

- Lyapunov functions are our central tool for verifying asymptotic stability
- Dynamic programming provides us with equations which will be heavily used in the subsequent analysis
- Infinite horizon optimal control would solve the stabilization problem — if we could compute the feedback law μ_∞
- The performance of the MPC controller can be measured by looking at the infinite horizon value along the MPC closed loop trajectories
- Relaxed dynamic programming gives us conditions under which both asymptotic stability and performance results can be derived

Application of background results

The main task will be to verify the assumptions of the relaxed dynamic programming theorem, i.e.,

$$V_N(x) \ge \alpha\ell(x,\mu_N(x)) + V_N(f(x,\mu_N(x)))$$

for some $\alpha \in (0,1]$, and

$$V_N(x) \le \alpha_2(\|x-x_*\|), \quad \inf_{u\in\mathbb{U}} \ell(x,u) \ge \alpha_3(\|x-x_*\|)$$

for all x in a forward invariant set Y for $x^+ = f(x,\mu_N(x))$

To this end, we present two different approaches:

- modify the optimal control problem in the MPC loop by adding terminal constraints and costs
- derive assumptions on f and ℓ under which MPC works without terminal constraints and costs

(3) Stabilizing Model Predictive Control with stabilizing terminal conditions

V_N as a Lyapunov Function

Problem: Prove that the MPC feedback law μ_N is stabilizing

Approach: Verify the assumptions

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

of the relaxed dynamic programming theorem for the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

Why is this difficult?

Let us first consider the inequality

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

The dynamic programming principle for V_N yields

$$V_N(x) \geq \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

$\rightsquigarrow$ we have V_{N-1} where we would like to have V_N

$\rightsquigarrow$ we would get the desired inequality if we could ensure

$$V_{N-1}(f(x, \mu_N(x))) \geq V_N(f(x, \mu_N(x))) + \text{“small error”}$$

(where “small” means that the error can be compensated replacing $\ell(x, \mu_N(x))$ by $\alpha \ell(x, \mu_N(x))$ with $\alpha \in (0, 1)$)

Why is this difficult?

Task: Find conditions under which

$$V_{N-1}(x) \geq V_N(x) + \text{“small error”}$$

holds

For

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

this appeared to be out of reach until the mid 1990s

Note: $V_{N-1} \leq V_N$ by non-negativity of ℓ; typically with strict “$<$”

$\rightsquigarrow$ additional stabilizing constraints were proposed

Terminal constraints and cost

Optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

We want V_N to become a Lyapunov function

Idea: add local Lyapunov function $F : \mathbb{X}_0 \to \mathbb{R}_0^+$ as terminal cost

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N))$$

F is defined on a region $\mathbb{X}_0$ around x_* which is imposed as terminal constraint $\quad x(N) \in \mathbb{X}_0$

[Chen & Allgöwer '98, Jadbabaie et al. '98 . . .]

Terminal constraints and cost

We thus change the optimal control problem to

$$\underset{\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)}{\text{minimize}} \;\; J_N(x_0,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k),\mathbf{u}(k)) \; + \; F(x_\mathbf{u}(N))$$

with

$$\mathbb{U}^N_{\mathbb{X}_0}(x_0) := \{\mathbf{u}\in\mathbb{U}^N \text{ admissible and } x_\mathbf{u}(N)\in\mathbb{X}_0\}$$

Which properties do we need for F and $\mathbb{X}_0$ in order to ensure

$$V_{N-1}(x) \geq V_N(x) + \text{“small error” ?}$$

With suitable assumptions we can even avoid the “small error”

Terminal constraints and cost

Assumptions on $F:\mathbb{X}_0\to\mathbb{R}^+_0$ and $\mathbb{X}_0$

There exists a controller $\kappa:\mathbb{X}_0\to\mathbb{U}$ with the following properties:

(i) $\mathbb{X}_0$ is forward invariant for $x^+ = f(x,\kappa(x))$:
for each $x\in\mathbb{X}_0$ we have $f(x,\kappa(x))\in\mathbb{X}_0$

(ii) F is a Lyapunov function for $x^+ = f(x,\kappa(x))$ on $\mathbb{X}_0$ which is compatible with the stage cost ℓ in the following sense:
for each $x\in\mathbb{X}_0$ the inequality

$$F(f(x,\kappa(x))) \leq F(x) - \ell(x,\kappa(x))$$

holds

Simplest choice: $\mathbb{X}_0 = \{x_*\}$, $F\equiv 0$, $\kappa\equiv 0$ (if $f(x_*,0)=x_*$)

Prolongation of control sequences

Let $\tilde{\mathbf{u}}\in\mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0) \quad \Rightarrow \quad \tilde{x} := x_{\tilde{\mathbf{u}}}(N-1)\in\mathbb{X}_0$

Define $\mathbf{u}\in\mathbb{U}^N$ as $\mathbf{u}(k) := \begin{cases} \tilde{\mathbf{u}}(k), & k=0,\dots,N-2 \\ \kappa(\tilde{x}), & k=N-1 \end{cases}$

with κ from (i)

$$\Rightarrow \quad x_\mathbf{u}(N) = f(x_{\tilde{\mathbf{u}}}(N-1),\mathbf{u}(N-1)) = f(\tilde{x},\kappa(\tilde{x}))\in\mathbb{X}_0$$

$$\Rightarrow \quad \mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)$$

$\rightsquigarrow$ every $\tilde{\mathbf{u}}\in\mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ can be prolonged to an $\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)$

By (ii) the stage cost of the prolongation is bounded by

$$\ell(x_\mathbf{u}(N-1),\mathbf{u}(N-1)) \leq F(x_\mathbf{u}(N-1)) - F(x_\mathbf{u}(N))$$

Reversal of $V_{N-1}\leq V_N$

Let $\tilde{\mathbf{u}}^\star\in\mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ be the optimal control for J_{N-1}, i.e.,

$$V_{N-1}(x_0) = J_{N-1}(x_0,\tilde{\mathbf{u}}^\star)$$

Denote by $\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)$ its prolongation

$$\begin{aligned} \Rightarrow \quad V_{N-1}(x_0) &= J_{N-1}(x_0,\tilde{\mathbf{u}}^\star) \\ &= \sum_{k=0}^{N-2} \ell(x_{\tilde{\mathbf{u}}^\star}(k),\tilde{\mathbf{u}}^\star(k)) + \underbrace{F(x_{\tilde{\mathbf{u}}^\star}(N-1))}_{\geq \ell(x_\mathbf{u}(N-1),\mathbf{u}(N-1)) + F(x_\mathbf{u}(N))} \\ &\geq \sum_{n=0}^{N-1} \ell(x_\mathbf{u}(k),\mathbf{u}(k)) + F(x_\mathbf{u}(N)) \\ &= J_N(x_0,\mathbf{u}) \;\geq\; V_N(x_0) \end{aligned}$$

$\rightsquigarrow$ as desired, we obtain $V_{N-1}\geq V_N$

Feasible sets

Define the feasible set

$$\mathbb{X}_N := \{x \in \mathbb{X} \,|\, \mathbb{U}^N_{\mathbb{X}_0}(x) \neq \emptyset\}$$

On $\mathbb{X}_N$ one can ensure the inequality

$$V_N(x) \le \alpha_2(\|x - x_*\|)$$

for some $\alpha_2 \in \mathcal{K}_\infty$ under mild conditions, while outside $\mathbb{X}_N$ we get $V_N(x) = \infty$

$\rightsquigarrow$ the MPC control is only defined on $\mathbb{X}_N$!

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Stability theorem

Theorem: Consider the MPC scheme with regional terminal constraint $x_{\mathbf{u}}(N) \in \mathbb{X}_0$ and Lyapunov function terminal cost F compatible with ℓ. Assume that

$$V_N(x) \le \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x,u) \ge \alpha_3(\|x - x_*\|)$$

holds for all $x \in \mathbb{X}_N$.

Then $\mathbb{X}_N$ is forward invariant, the MPC closed loop is asymptotically stable on $\mathbb{X}_N$ and the performance estimate

$$J^{cl}_\infty(x, \mu_N) \le V_N(x)$$

holds.

Proof: Combine dynamic programming with $V_{N-1} \ge V_N$

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Stabilizing terminal conditions — Discussion

Stabilizing terminal conditions

- yield an elegant stability theory
- can indeed improve the stability properties
- can be derived in a systematic way, e.g., by linearization

But:

- large feasible set usually needs a large optimization horizon N
  (see again the car-and-mountains example)
- additional analytical effort for computing F
- hardly ever used in industrial practice

In Section (4) we will see how stability can be proved without stabilizing terminal constraints

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Suboptimality

Recall: For both stabilizing terminal constraints the relaxed dynamic programming theorem yields the estimate

$$J^{cl}_\infty(x_0, \mu_N) \le V_N(x_0)$$

But: How large is V_N ?

Without terminal constraints, the inequality $V_N \le V_\infty$ is immediate

However, the terminal constraints also reverse this inequality, i.e., we have $V_N \ge V_\infty$ and the gap is very difficult to estimate

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Suboptimality — example

We consider two examples with $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \mathbb{R}$ for $N = 2$

Example 1: $x^+ = x + u, \quad \ell(x,u) = x^2 + u^2$

Terminal constraints $\quad x_{\mathbf{u}}(N) = x_* = 0$

$V_\infty(x) \approx 1.618x^2, \quad J_\infty^{cl}(x, \mu_2) = 1.625x^2$

Example 2: as Example 1, but with $\quad \ell(x,u) = x^2 + u^4$

$V_\infty(20) \leq 1726, \quad J_\infty^{cl}(x, \mu_2) \approx 11240$

General estimates for fixed N appear difficult to obtain. But we can give an asymptotic result for $N \to \infty$

Asymptotic Suboptimality

Theorem: For both types of terminal constraints the assumptions of the stability theorems ensure

$$V_N(x) \to V_\infty(x)$$

and thus

$$J_\infty^{cl}(x, \mu_N) \to V_\infty(x)$$

as $N \to \infty$ uniformly on compact subsets of the feasible sets, i.e., the MPC performance converges to the optimal one

Idea of proof: uses that any approximately optimal trajectory for J_∞ converges to x_* and can thus be modified to meet the constraints with only moderately changing its value

Summary of Section (3)

- terminal conditions reverse the usual inequality $V_{N-1} \leq V_N$ to $V_{N-1} \geq V_N$
- this enables us to derive the relaxed dynamic programming inequality (with $\alpha = 1$) from the dynamic programming principle
- the operating region is restricted to the feasible set $\mathbb{X}_N$
- $J_\infty^{cl}(x, \mu_N) \leq V_N(x)$ holds and $V_N \to V_\infty$ for $N \to \infty$, but $V_N \gg V_\infty$ is possible

(4) Stabilizing Model Predictive Control without stabilizing terminal conditions

MPC without stabilizing terminal constraints

We return to the basic MPC formulation

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0 = x_{\mu_N}(n)$$

without any stabilizing terminal constraints and costs

In order to motivate why we want to avoid terminal constraints and costs, we consider an example of P double integrators in the plane

Motivation for avoiding terminal conditions

Example: [Annunziato/Borzì '10ff., Fleig '14ff.] The Fokker-Planck Equation

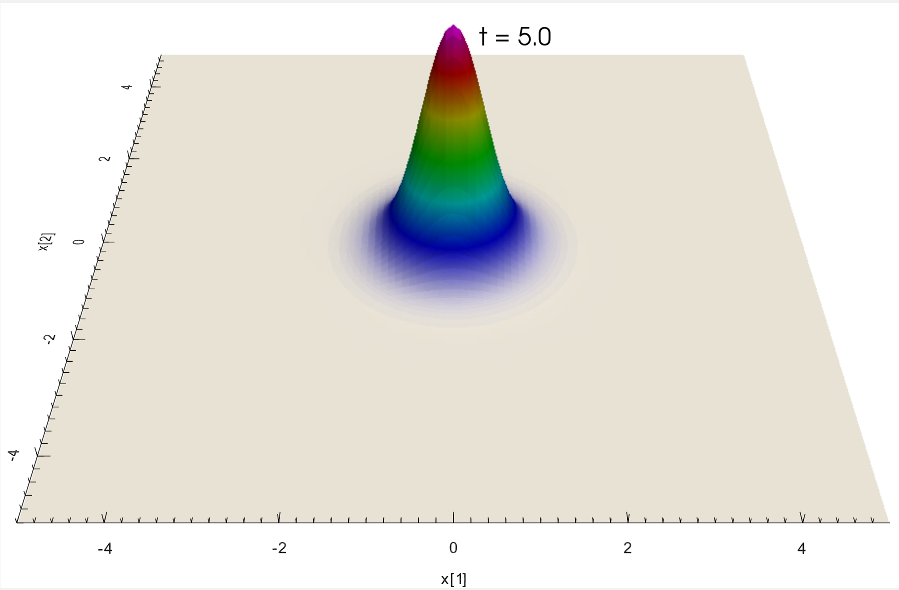
$$\begin{aligned} \partial_t y(x,t) - \sum_{i,j=1}^{d} \partial^2_{x_i x_j}\Big(a_{ij}(x,t)y(x,t)\Big) + \sum_{i=1}^{d} \partial_{x_i}\Big(b_i(x,t;u))y(x,t)\Big) &= 0 \\ y(\cdot,0) &= y_0 \end{aligned}$$

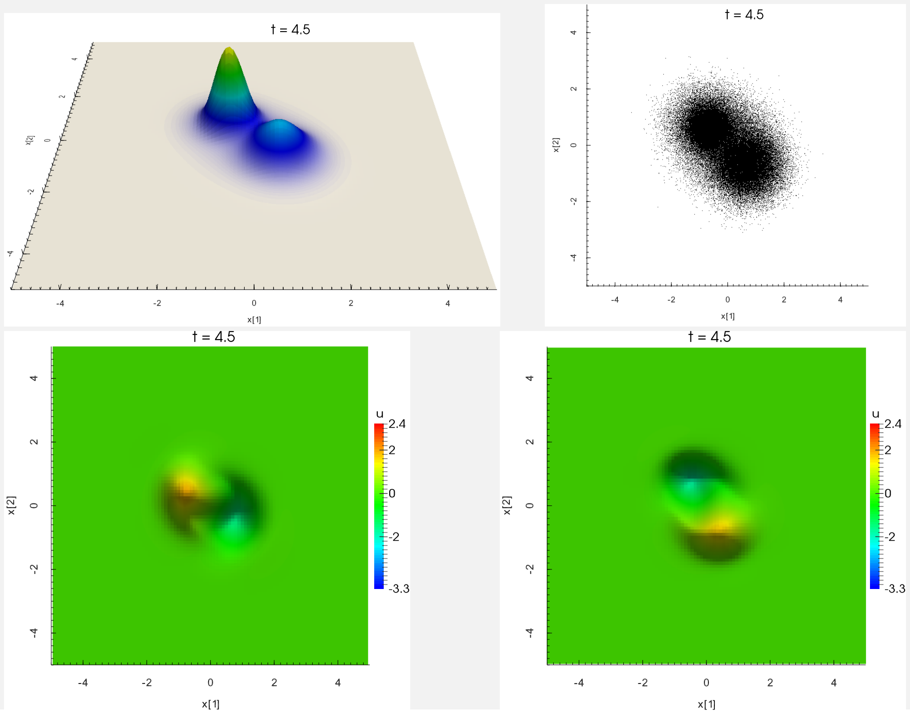
is a parabolic PDE describing the evolution of a probability density function for stochastic control systems

By solving a Fokker-Planck control problem controls for large ensembles of stochastic systems can be computed

Due to the complexity of the problem the derivation of terminal conditions meeting the theoretical assumptions is hardly feasible — but even without MPC works

Reference density function





Stabilizing NMPC without terminal constraint

(Some) stability and performance results known in the literature:

[Alamir/Bornard '95]
use a controllability condition for all $x \in \mathbb{X}$

[Shamma/Xiong '97, Primbs/Nevistić '00]
use knowledge of optimal value functions

[Jadbabaie/Hauser '05]
use controllability of linearization in x_*

[Grimm/Messina/Tuna/Teel '05, Tuna/Messina/Teel '06, Gr./Rantzer '08, Gr. '09, Gr./Pannek/Seehafer/Worthmann '10, Köhler/Müller/Allgöwer '18ff]
use bounds on optimal value functions

Here we explain the last approach

Bounds on the optimal value function

Recall the definition of the optimal value function

$$V_N(x) := \inf_{\mathbf{u}\text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k,x), \mathbf{u}(k))$$

Boundedness assumption: there exists $\gamma > 0$ with

$$V_N(x) \le \gamma \ell^\star(x) \quad \text{for all } x \in \mathbb{X}, N \in \mathbb{N}$$

where $\ell^\star(x) := \min_{u \in \mathbb{U}} \ell(x,u)$

(sufficient conditions for and relaxations of this bound will be discussed later)

Stability and performance index

We choose ℓ, such that

$$\alpha_3(\|x - x_*\|) \le \ell^\star(x) \le \alpha_4(\|x - x_*\|)$$

holds for $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ (again, $\ell(x,u) = \|x - x_*\|^2 + \lambda\|u\|^2$ works)

Then, the only inequality left to prove in order to apply the relaxed dynamic programming theorem is

$$V_N(f(x, \mu_N(x))) \le V_N(x) - \alpha_N \ell(x, \mu_N(x))$$

for some $\alpha_N \in (0,1)$ and all $x \in \mathbb{X}$

We can compute α_N from the bound $V_N(x) \le \gamma \ell^\star(x)$

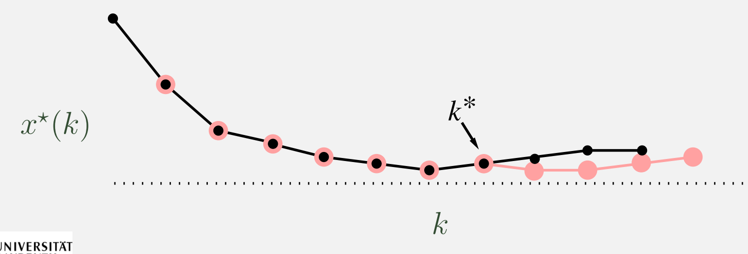
Computing α_N

We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$ $(*)$

We want $V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$

- use $(*)$ to find $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$
- concatenate $x^\star(1), \ldots, x^\star(k^\star)$ and the optimal trajectory starting in $x^\star(k^\star)$ $\rightsquigarrow$ $\tilde{x}(\cdot)$, $\tilde{\mathbf{u}}(\cdot)$

$$\Rightarrow V_N(x^\star(1)) \le J_N(x^\star(1), \tilde{\mathbf{u}}) \le V_N(x^\star(0)) - \underbrace{(1 - \gamma\eta_N)}_{=\alpha_N} \ell(x^\star(0), \mathbf{u}^\star(0))$$



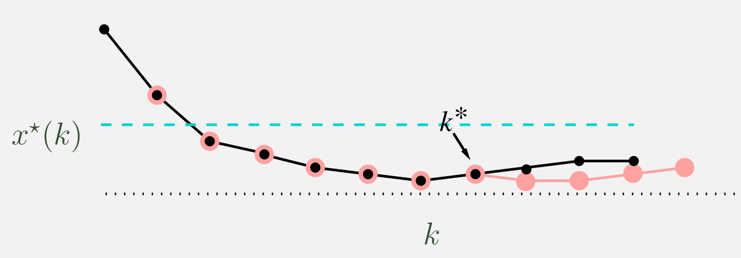
Decay of the optimal trajectory

We assume $\quad V_N(x) \le \gamma \ell^\star(x) \quad$ for all $\quad x \in \mathbb{X},\ N \in \mathbb{N}$

We want $\quad \eta_N > 0,\ k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 1 [Grimm/Messina/Tuna/Teel '05]

$V_N(x) \le \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \le \gamma \ell^\star(x)/N$ for at least one $k^\star \quad \Rightarrow \quad \alpha_N = 1 - \gamma(\gamma-1)/N$



Decay of the optimal trajectory

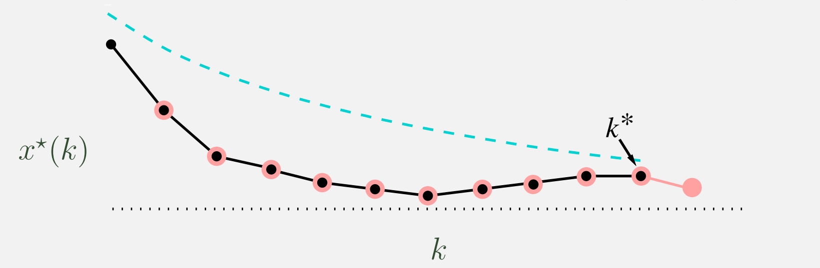
We assume $\quad V_N(x) \le \gamma \ell^\star(x) \quad$ for all $\quad x \in \mathbb{X},\ N \in \mathbb{N}$

We want $\quad \eta_N > 0,\ k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

$V_N(x) \le \gamma \ell^\star(x) \quad \Rightarrow \quad \ell(x^\star(k), u^\star(k)) \le \gamma \left(\frac{\gamma-1}{\gamma}\right)^k \ell^\star(x)$

$\Rightarrow \quad k^\star = N-1 \quad \Rightarrow \quad \alpha_N = 1 - (\gamma-1)^N/\gamma^{N-2}$



Decay of the optimal trajectory

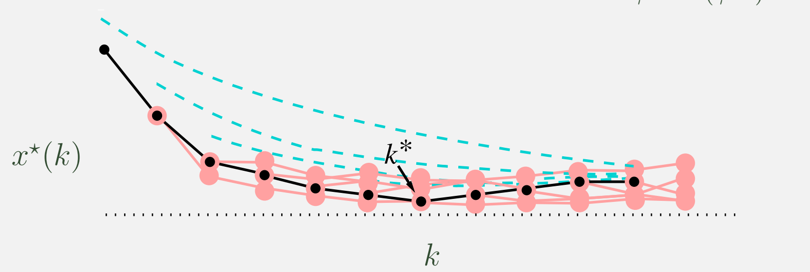
We assume $\quad V_N(x) \le \gamma \ell^\star(x) \quad$ for all $\quad x \in \mathbb{X},\ N \in \mathbb{N}$

We want $\quad \eta_N > 0,\ k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

$V_N(x) \le \gamma \ell^\star(x) \quad \Rightarrow$ formulate all constraints and trajectories

$\Rightarrow$ optimize for $\alpha_N \quad \Rightarrow \quad \alpha_N = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-2}}$



Optimization approach to compute α_N

We explain the optimization approach (Variant 3) in more detail. We want α_N such that

$$V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$$

holds for all optimal trajectories $x^\star(n), \mathbf{u}^\star(n)$ for V_N

The bound and the dynamic programming principle imply:

$$\begin{aligned}
V_N(x^\star(1)) &\le \gamma \ell^\star(x^\star(1)) \\
V_N(x^\star(1)) &\le \ell(x^\star(1), \mathbf{u}^\star(1)) + \gamma \ell^\star(x^\star(2)) \\
V_N(x^\star(1)) &\le \ell(x^\star(1), \mathbf{u}^\star(1)) + \ell(x^\star(2), \mathbf{u}^\star(2)) + \gamma \ell^\star(x^\star(3)) \\
\vdots \quad & \quad \vdots \quad \quad \vdots
\end{aligned}$$

Optimization approach to compute α_N

$\rightsquigarrow$ $V_N(x^\star(1))$ is bounded by sums over $\ell(x^\star(n), \mathbf{u}^\star(n))$

For sums of these values, in turn, we get bounds from the dynamic programming principle and the bound:

$$\sum_{n=0}^{N-1} \ell(x^\star(n), \mathbf{u}^\star(n)) = V_N(x^\star(0)) \leq \gamma \ell^\star(x^\star(0))$$

$$\sum_{n=1}^{N-1} \ell(x^\star(n), \mathbf{u}^\star(n)) = V_{N-1}(x^\star(1)) \leq \gamma \ell^\star(x^\star(1))$$

$$\sum_{n=2}^{N-1} \ell(x^\star(n), \mathbf{u}^\star(n)) = V_{N-2}(x^\star(2)) \leq \gamma \ell^\star(x^\star(2))$$

$$\vdots \qquad \vdots$$

Verifying the relaxed Lyapunov inequality

Find α_N, such that for all optimal trajectories $x^\star$, $\mathbf{u}^\star$:

$$V_N(x^\star(1)) \leq V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0)) \qquad (*)$$

Define $\lambda_n := \ell(x^\star(n), \mathbf{u}^\star(n)), \quad \nu := V_N(x^\star(1))$

Then: $(*) \Leftrightarrow \nu \leq \sum_{n=0}^{N-1} \lambda_n - \alpha_N \lambda_0$

The inequalities from the last slides translate to

$$\sum_{n=k}^{N-1} \lambda_n \leq \gamma \lambda_k, \quad k = 0, \dots, N-2 \qquad (1)$$

$$\nu \leq \sum_{n=1}^{j} \lambda_n + \gamma \lambda_{j+1}, \quad j = 0, \dots, N-2 \qquad (2)$$

We call $\lambda_0, \dots, \lambda_{N-1}, \nu \geq 0$ with (1), (2) admissible

Optimization problem

$\Rightarrow$ if α_N is such that the inequality

$$\nu \leq \sum_{n=0}^{N-1} \lambda_n - \alpha_N \lambda_0 \Leftrightarrow \alpha_N \leq \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\lambda_0}$$

holds for all admissible λ_n and ν, then the desired inequality will hold for all optimal trajectories

The largest α_N satisfying this condition is

$$\alpha_N := \min_{\lambda_n, \nu \text{ admissible}} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\lambda_0}$$

This is a linear optimization problem whose solution can be computed explicitly (which is nontrivial) and reads

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}}$$

Stability and performance theorem

Theorem: [Gr./Pannek/Seehafer/Worthmann '10]: Assume $V_N(x) \leq \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$. If

$$\alpha_N > 0 \Leftrightarrow N > 2 + \frac{\ln(\gamma - 1)}{\ln \gamma - \ln(\gamma - 1)} \sim \gamma \ln \gamma$$

then the NMPC closed loop is asymptotically stable with Lyapunov function V_N and we get the performance estimate $J_\infty^{cl}(x, \mu_N) \leq V_\infty(x)/\alpha_N$ with

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}} \to 1 \text{ as } N \to \infty$$

Conversely, if $N < 2 + \frac{\ln(\gamma-1)}{\ln \gamma - \ln(\gamma - 1)}$, then there exists a system for which $V_N(x) \leq \gamma \ell^\star(x)$ holds but the NMPC closed loop is not asymptotically stable.

Horizon dependent γ-values

The theorem remains valid if we replace the bound condition

$$V_N(x) \leq \gamma \ell^\star(x)$$

by

$$V_N(x) \leq \gamma_N \ell^\star(x)$$

for horizon-dependent bounded values $\gamma_N \in \mathbb{R}$, $N \in \mathbb{N}$

$$\rightsquigarrow \quad \alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)}$$

This allows for tighter bounds and a refined analysis

Controllability condition

A refined analysis can be performed if we compute γ_N from a controllability condition, e.g., exponential controllability:

Assume that for each $x_0 \in \mathbb{X}$ there exists an admissible control $\mathbf{u}$ such that

$$\ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \leq C\sigma^k \ell^\star(x_0), \quad k = 0, 1, 2, \ldots$$

for given overshoot constant $C > 0$ and decay rate $\sigma \in (0,1)$

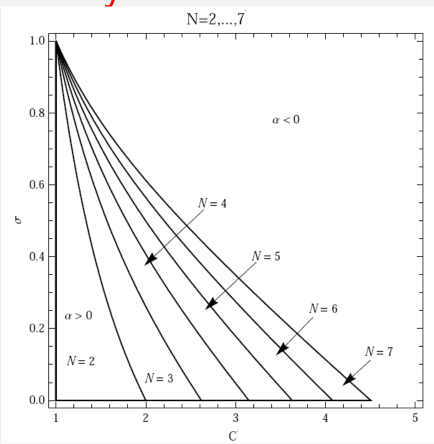
$$\rightsquigarrow \quad V_N(x) \leq \gamma_N \ell^\star(x) \quad \text{for} \quad \gamma_N = \sum_{k=0}^{N-1} C\sigma^k$$

This allows to compute the minimal stabilizing horizon

$$\min\{N \in \mathbb{N} \,|\, \alpha_N > 0\}$$

depending on C and σ

Stability chart for C and σ



(Figure: Harald Voit)

Conclusion: for short optimization horizon N it is
more important: small C ("small overshoot")
less important: small σ ("fast decay")

(we will see at the end of the section how to use this information)

Comments and extensions

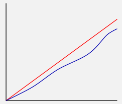
- for unconstrained linear quadratic problems:
  existence of γ $\Leftrightarrow$ (A, B) stabilizable
- additional weights on the last term can be incorporated into the analysis [Gr./Pannek/Seehafer/Worthmann '10]
- instead of using γ, α can be estimated numerically online along the closed loop [Pannek et al. '10ff]
- positive definiteness of ℓ can be replaced by a detectability condition [Grimm/Messina/Tuna/Teel '05]
- under appropriate uniformity assumptions, the results are easily carried over to tracking time variant references $x_{\text{ref}}(n)$ instead of an equilibrium x_* [Gr./Pannek '11, Köhler/Müller/Allgöwer '18ff.]

Comments and extensions

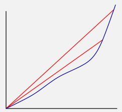
The "linear" inequality $V_N(x) \leq \gamma \ell^\star(x)$ may be too demanding for nonlinear systems under constraints

Generalization: $\quad V_N(x) \leq \rho(\ell^\star(x)), \quad \rho \in \mathcal{K}_\infty$

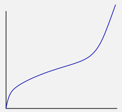
- there is $\gamma > 0$ with $\rho(r) \leq \gamma r$ for all $r \in [0, \infty]$
  $\Rightarrow$ global asymptotic stability



- for each $R > 0$
  there is $\gamma_R > 0$ with $\rho(r) \leq \gamma_R r$ for all $r \in [0, R]$
  $\Rightarrow$ semiglobal asymptotic stability

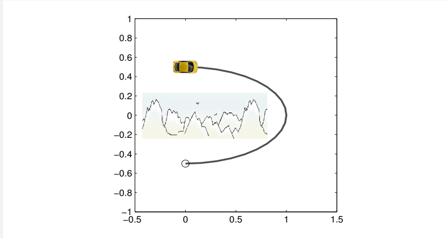


- $\rho \in \mathcal{K}_\infty$ arbitrary
  $\Rightarrow$ semiglobal practical asymptotic stability



[Grimm/Messina/Tuna/Teel '05, Gr./Pannek '11]

The car-and-mountains example reloaded



MPC with $\ell(x,u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$
$\rightsquigarrow$ asymptotic stability for $N = 11$ but not for $N \leq 10$

Reason: detour around mountains causes large overshoot C

Remedy: put larger weight on x_2:

$\ell(x,u) = (x_1 - x_{*,1})^2 + 5(x_2 - x_{*,2})^2 + |u|^2 \rightsquigarrow$ as. stab. for $N = 2$

A PDE example

We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0, 1]$

solution $y = y(t, x)$

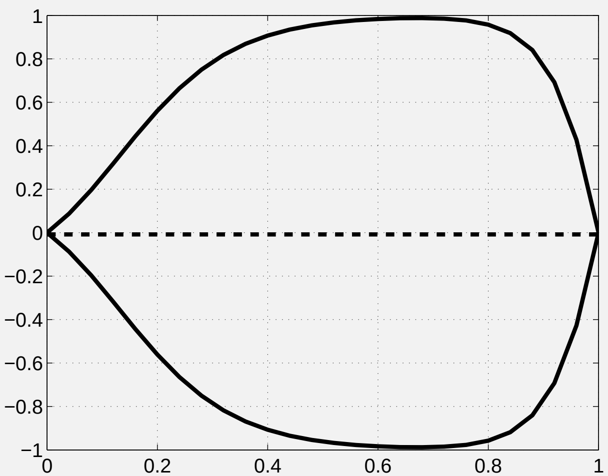
boundary conditions $y(t, 0) = y(t, 1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

and distributed control $u : \mathbb{R} \times \Omega \to \mathbb{R}$

Discrete time system: $y(n) = y(nT, \cdot)$, sampling time $T = 0.025$

The uncontrolled PDE



all equilibrium solutions

MPC for the PDE example

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

Goal: stabilize the sampled data system $y(n)$ at $y \equiv 0$

Usual approach: quadratic L^2 cost

$$\ell(y(n), u(n)) = \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2$$

For $y \approx 0$ the control u must compensate for $y_x \rightsquigarrow u \approx -y_x$

$\rightsquigarrow$ controllability condition

$$\begin{aligned} \ell(y(n), u(n)) &\le C\sigma^n \ell^*(y(0)) \\ \Leftrightarrow \quad \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2 &\le C\sigma^n \|y(0)\|_{L^2}^2 \\ \approx \quad \|y(n)\|_{L^2}^2 + \lambda \|y_x(n)\|_{L^2}^2 &\le C\sigma^n \|y(0)\|_{L^2}^2 \end{aligned}$$

for $\|y_x\|_{L^2} \gg \|y\|_{L^2}$ this can only hold if $C \gg 0$

MPC for the PDE example

Conclusion: because of

$$\|y(n)\|_{L^2}^2 + \lambda \|y_x(n)\|_{L^2}^2 \le C\sigma^n \|y(0)\|_{L^2}^2$$

the controllability condition may only hold for very large C

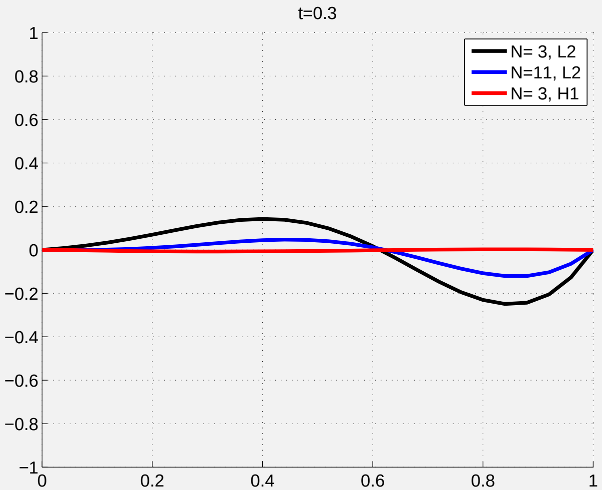
Remedy: use H^1 cost

$$\ell(y(n), u(n)) = \underbrace{\|y(n)\|_{L^2}^2 + \|y_x(n)\|_{L^2}^2}_{=\|y(n)\|_{H^1}^2} + \lambda \|u(n)\|_{L^2}^2.$$

Then an analogous computation yields

$$\|y(n)\|_{L^2}^2 + (1+\lambda)\|y_x(n)\|_{L^2}^2 \le C\sigma^n \Big(\|y(0)\|_{L^2}^2 + \|y_x(0)\|_{L^2}^2 \Big)$$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

domain $\Omega = [0, 1]$

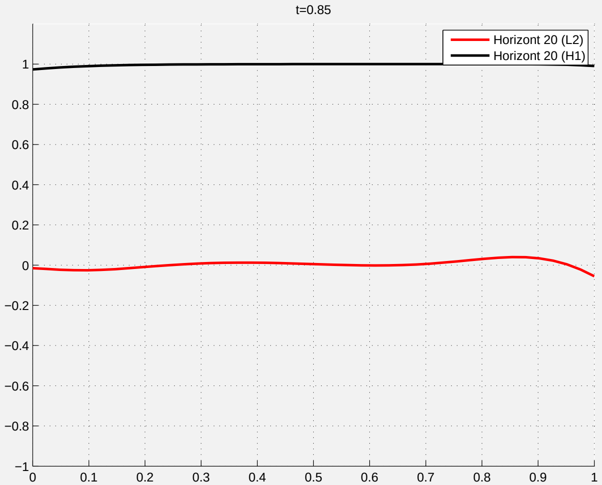
solution $y = y(t, x)$

boundary conditions $y(t, 0) = u_0(t)$, $y(t, 1) = u_1(t)$

parameters $\nu = 0.1$ and $\mu = 10$

with boundary control, stability can only be achieved via large gradients in the transient phase
$\rightsquigarrow$ L^2 should perform better that H^1

Boundary control, L_2 vs. H_1, $N = 20$



Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$
Can be made rigorous for many PDEs [Altmüller et al. '10ff]

Summary of Section (4)

- Stability and performance of MPC without terminal constraints can be ensured by suitable bounds on V_N
- An optimization approach allows to compute the best possible α_N in the relaxed dynamic programming theorem
- The γ or γ_N can be computed from controllability properties, e.g., exponential controllability
- The overshoot bound $C > 0$ plays a crucial role or obtaining small stabilizing horizons
- Computing tight estimates for C is in general a difficult if not impossible task
- But structural knowledge of the system behavior can be sufficient for choosing a "good" ℓ

(5) Economic Model Predictive Control

(5a) Time-invariant problems

Performance

In this section we do not limit ourselves to stabilizing costs as, e.g., $\ell(x,u) = \|x - x_*\|^2 + \lambda\|u - u_*\|^2$

MPC with more general ℓ is often termed economic MPC. In this setting, performance of μ_N can be measured in two ways

Infinite horizon averaged performance:

$$\overline{J}^{cl}_\infty(x,\mu_N) = \limsup_{K\to\infty} \frac{1}{K}\sum_{n=0}^{K-1} \ell(x_{\mu_N}(n,x),\mu_N(x_{\mu_N}(n,x)))$$

Finite horizon (or transient) performance:

$$J^{cl}_K(x,\mu_N) = \sum_{n=0}^{K-1} \ell(x_{\mu_N}(n,x),\mu_N(x_{\mu_N}(n,x)))$$

Only in special cases $K \to \infty$ makes sense

Example 1: minimum energy control

Example: Keep the state of the system inside the admissible set $\mathbb{X}$ minimizing the quadratic control effort

$$\ell(x,u) = u^2$$

with dynamics

$$x(n+1) = 2x(n) + \mathbf{u}(n)$$

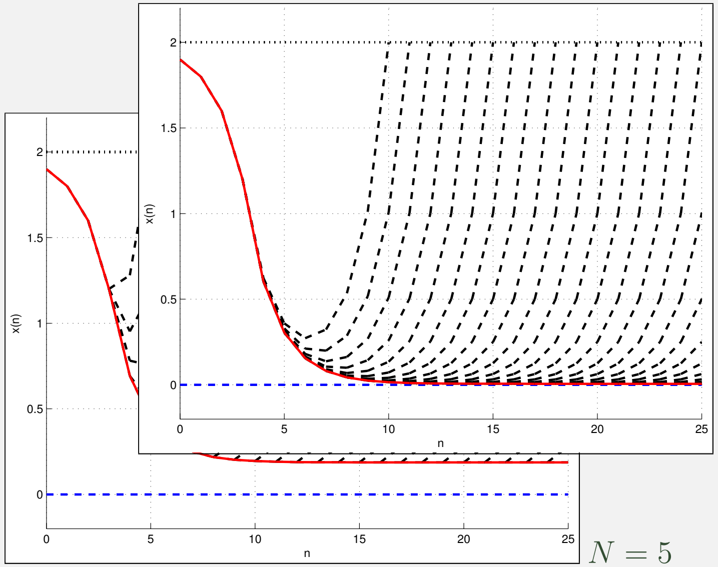
and constraints $\mathbb{X} = [-2,2]$, $\mathbb{U} = [-3,3]$

For this example, the closer the state is to $x^e = 0$, the cheaper it is to keep the system inside X

$\rightsquigarrow$ optimal trajectory should stay near $x^e = 0$

We illustrate MPC for this problem without terminal conditions

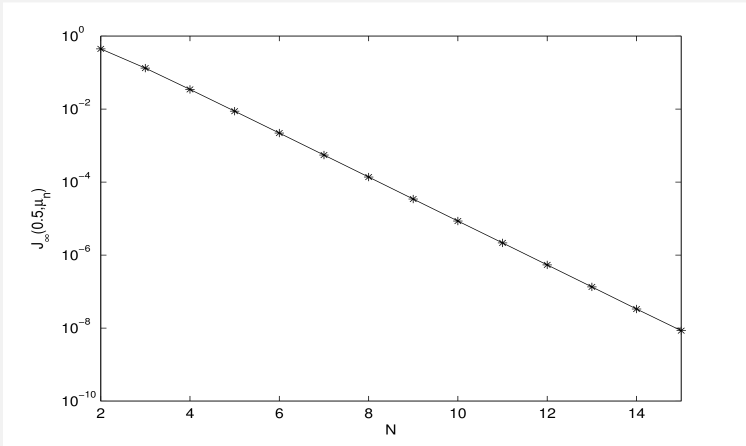
Example 1: trajectories



$N = 10$

$N = 5$

Example 1: averaged closed loop performance



$\overline{J}^{cl}_\infty(0.5,\mu_N) - \ell(x^e,u^e)$ depending on N, logarithmic scale

Example 2: a macroeconomic model

The second example is a 1d macroeconomic model

[Brock/Mirman '72]

Minimize the performance with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $x^+ = u$ and constraints $\mathbb{X} = \mathbb{U} = [0, 10]$

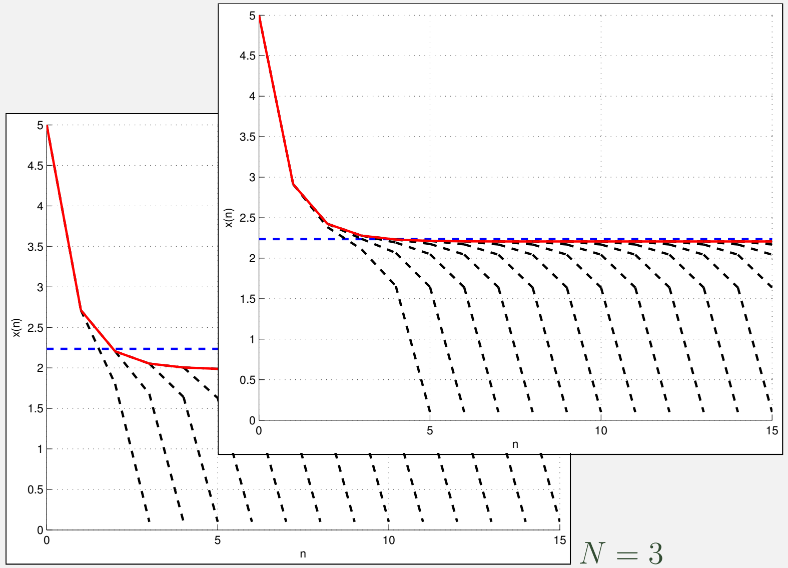
Interpretation:

x	$=$	capital
Ax^α	$=$	capital after one time step before consumption
u	$=$	capital after one time step after consumption
$Ax^\alpha - u$	$=$	consumption
$\ln(Ax^\alpha - u)$	$=$	benefit from consumption (utility function)

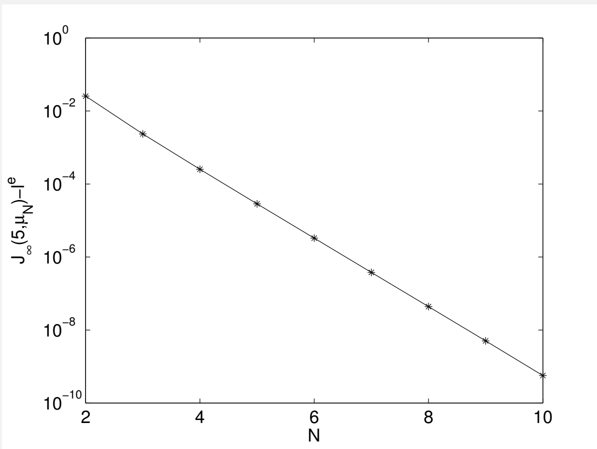
On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

Example 2: trajectories

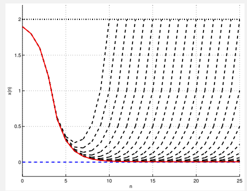


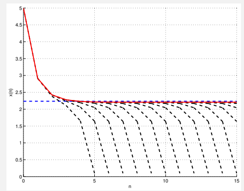
Example 2: averaged closed loop performance



$\overline{J}^{cl}_\infty(5, \mu_N) - \ell(x^e, u^e)$ depending on N, logarithmic scale

Observations





- optimal open loop trajectories approach the optimal equilibrium, stay there for a while, and turn away – "turnpike property"
- closed loop trajectories converge to a neighborhood of the optimal equilibrium whose size tends to 0 as $N \to \infty$
- the averaged closed loop performance satisfies $\overline{J}^{cl}_\infty(x, \mu_N) \to \ell(x^e, u^e)$ as $N \to \infty$ (exponentially fast)

Can we prove this behavior?

We first investigate the turnpike property

The turnpike property

The turnpike property

The turnpike property describes a behaviour of (approximately) optimal trajectories for a finite horizon optimal control problem

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_N(x, \mathbf{u}) = \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(n), \mathbf{u}(n))$$

with state and input constraints $x_{\mathbf{u}}(n) \in \mathbb{X}$, $\mathbf{u}(n) \in \mathbb{U}$

Informal description of the turnpike property: any optimal trajectory stays near an equilibrium x^e most of the time

We illustrate the property by the two examples

Example 1: minimum energy control

Example: Keep the state of the system inside a given interval X minimising the quadratic control effort

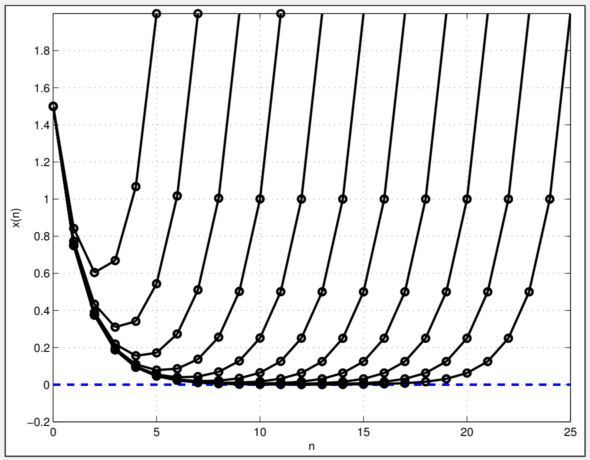
$$\ell(x, u) = u^2$$

with dynamics

$$x^+ = 2x + u$$

and constraints $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

Example 1: optimal trajectories



Optimal trajectories for $N = 5, \ldots, 25$

Example 2: a macroeconomic model

The second example is a 1d macroeconomic model
[Brock/Mirman '72]

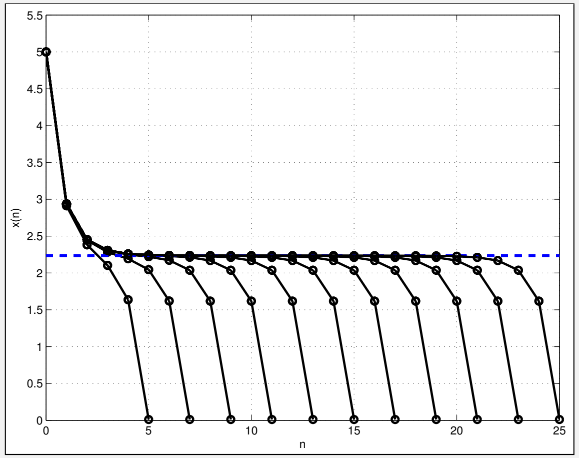
Minimise the finite horizon objective with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

with dynamics $\quad x^+ = u$

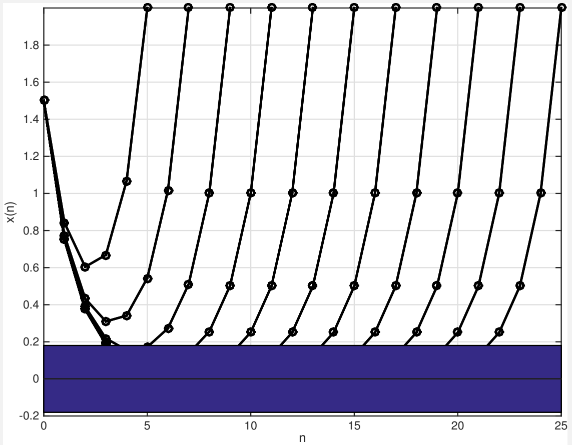
on $\quad \mathbb{X} = \mathbb{U} = [0,10]$

Example 2: optimal trajectories



Optimal trajectories for $N = 5, \dots, 25$

How to formalize the turnpike property?



Number of points outside the blue neighbourhood is bounded by a number independent of N (here: by 8)

The turnpike property: formal definitions

Let x^e be an equilibrium, i.e., $f(x^e, u^e) = x^e$

Turnpike property: For each $\varepsilon > 0$ and $\rho > 0$ there is $C_{\rho,\varepsilon} > 0$ such that for all $N \in \mathbb{N}$ all optimal trajectories $x^\star$ starting in $B_\rho(x^e)$ satisfy the inequality

$$\#\Big\{k \in \{0,\dots,N-1\} \,\Big|\, \|x^\star(k) - x^e\| \geq \varepsilon\Big\} \leq C_{\rho,\varepsilon}$$

If the infinite horizon problem is well defined, this definition can also be used for $N = \infty$

History

- Apparently first described by [von Neumann 1945]
- Name "turnpike property" coined by [Dorfman/Samuelson/Solow 1957]
- Extensively studied in the 1970s in mathematical economy, cf. survey [McKenzie 1983]
- Renewed interest in recent years [Zaslavski '14ff, Trélat/Zuazua et al. '15ff, Faulwasser et al. '15ff, ...]
- Many applications, e.g., structural insight in economic equilibria; synthesis of optimal trajectories [Anderson/Kokotovic '87]

A first performance estimate

Assumptions

We make the following assumptions:

- The turnpike property holds at an equilibrium (x^e, u^e)
- Without loss of generality we assume $\ell(x^e, u^e) = 0$ — otherwise we replace $\ell(x, u)$ by $\ell(x, u) - \ell(x^e, u^e)$
- The corresponding infinite horizon problem is well defined

We will later explain how to avoid the assumption $\ell(x^e, u^e) = 0$

Towards the first performance estimate

Define the optimal value function $V_N(x) = \inf_{u(\cdot)} J_N(x, u)$ and the optimal control by $u_N^\star$ for $N \in \mathbb{N}$ and $N = \infty$

Assume $V_N(x) \approx V_N(x^e)$ for $x \approx x^e$ and $N = 1, \ldots, \infty$

We can then use the turnpike property to prove

$$|J_P(x_0, u_N^\star) - J_P(x_0, u_\infty^\star)| \leq \varepsilon$$

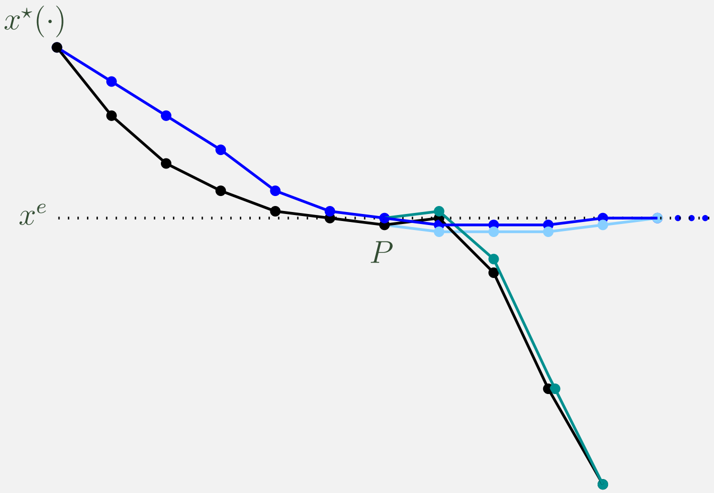
and

$$|J_P(x_0, u_\infty^\star) + V_\infty(x^e) - V_\infty(x_0)| \leq \varepsilon$$

for an error term $\varepsilon > 0$

Proving $|J_P(x_0,u_N^\star) - J_P(x_0,u_\infty^\star)| \le \varepsilon$

Sketch of the idea:



Towards the first performance estimate

From $|J_P(x_0,u_N^\star) - J_P(x_0,u_\infty^\star)| \le \varepsilon$

and $|J_P(x_0,u_\infty^\star) + V_\infty(x^e) - V_\infty(x_0)| \le \varepsilon$ we obtain

$$|J_P(x_0,u_N^\star) + V_\infty(x^e) - V_\infty(x_0)| \le 2\varepsilon$$

This inequality yields:

$$\begin{aligned} J_K^{cl}(x,\mu_N) &= \sum_{n=0}^{K-1} \ell(x_{\mu_N}(n),\mu_N(x_{\mu_N}(n))) \\ &\le \sum_{n=0}^{K-1} J_P(x_{\mu_N}(n),u_N^\star) - J_{P-1}(x_{\mu_N}(n+1)),u_{N-1}^\star) \\ &\le \sum_{n=0}^{K-1} \Big(V_\infty(x_{\mu_N}(n)) - V_\infty(x_{\mu_N}(n+1)) + 4\varepsilon\Big) \\ &= V_\infty(x_{\mu_N}(0)) - V_\infty(x_{\mu_N}(K)) + 4K\varepsilon \end{aligned}$$

Assumptions needed for this construction

What do we need to make this construction work?

(1) Turnpike property

- ensures that the optimal trajectories stay for a certain time near the optimal equilibrium x^e, such that we can merge the trajectories

(2) Continuity of V_N in x^e (uniform in $N = 1,\ldots,\infty$)

- ensures that the merged trajectories have a value close to the original ones

Under these conditions we obtain [Gr. '11; Gr. '16, Theorem 4.4]

$$J_K^{cl}(x,\mu_N) + V_\infty(x_{\mu_N}(K)) \le V_\infty(x_{\mu_N}(0)) + K\varepsilon(N)$$

"The MPC trajectory on $\{0,\ldots,K\}$ is an initial piece of an approximately optimal infinite horizon trajectory"

Interpretation of the estimate

$$J_K^{cl}(x,\mu_N) + V_\infty(x_{\mu_N}(K)) \le V_\infty(x_{\mu_N}(0)) + K\varepsilon(N)$$

If V_∞ is bounded from below, this estimate ensures the error

$$\overline{J}_\infty^{cl}(x,\mu_N) \le \ell(x^e,u^e) + \varepsilon(N)$$

for the averaged performance

$$\overline{J}_\infty^{cl}(x,\mu_N) = \limsup_{K\to\infty} \frac{1}{K}\sum_{n=0}^{K-1} \ell(x_{\mu_N}(n,x),\mu_N(x_{\mu_N}(n,x)))$$

However, the above estimate is better than pure average approximate optimality, because it also ensures the finite-time error

$$K\varepsilon(N)$$

Strict dissipativity

Strict dissipativity in MPC

Around the same time when we discovered the importance of the turnpike property, in MPC with terminal constraints the notion of strict dissipativity was discovered to be an important structural property

[Diehl/Rawlings '11, Angeli/Amrit/Rawlings '12]

Strict dissipativity

[cf. Willems '72]

$$x^+ = f(x,u)$$

Definition: The system is called strictly dissipative if there are $x^e \in \mathbb{X}$, $s : X \times U \to \mathbb{R}$, $\lambda : X \to \mathbb{R}$ bounded from below, $\alpha \in \mathcal{K}$ such that for all $x \in \mathbb{X}$, $u \in \mathbb{U}$

$$\lambda(x^+) \leq \lambda(x) + s(x,u) - \alpha(\|x - x^e\|)$$

$\lambda =$ storage function $=$ energy stored in the system
$s =$ supply rate $=$ energy supplied/extracted from the outside

For $s(x,u) = \ell(x,u) - \ell(x^e,u^e)$, strict dissipativity has various impacts on the MPC scheme

Strict dissipativity and MPC

$$\tilde{\ell}(x,u) := \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u)) \geq \alpha(\|x - x^e\|)$$

It was proved in [Gr. 13, Gr./Stieler '14] that

- strict dissipativity plus a reachability condition implies the turnpike property (in principle known at least since [Carlson/Haurie/Leizarowitz '91])
- strict dissipativity plus local controllability implies continuity of V_N in x^e uniformly in $N = 1, \ldots, \infty$
- strict dissipativity allows to construct Lyapunov functions for the MPC trajectories
- MPC allows for improved performance estimates

The previous examples are strictly dissipative with $\lambda(x) = -x^2/2$ and $\lambda(x) \approx 0.2306x$, respectively

Stabilizing costs are strictly dissipative with $\lambda \equiv 0$

Strict dissipativity and turnpike property

Question: how restrictive is it to assume strict dissipativity?

Theorem [Gr./Müller '16]: Under suitable controllability conditions, strict dissipativity is equivalent to the turnpike property plus optimality of the equilibrium (x^e, u^e)

Theorem [Gr./Guglielmi '18] For finite dimensional linear quadratic problems without state constraints, strict dissipativity and the turnpike property are equivalent to detectability

With bounded state constraints, strict dissipativity and the turnpike property are equivalent to hyperbolicity of the zero dynamics (weaker than detectability)

The full performance result for time-invariant problems

Economic MPC theorem

Theorem: [Gr./Stieler '14]

Let f and ℓ be Lipschitz, $\mathbb{X}$ and $\mathbb{U}$ be compact and assume

(i) local controllability near x^e } ⇒ uniform continuity of V_N

(ii) strict dissipativity

(iii) reachability of x^e from all $x \in \mathbb{X}$ } ⇒ turnpike property

(iv) polynomial growth conditions for $\tilde{\ell}$

(i)–(iv) ⇒ exponential turnpike
[Damm/Gr./Stieler/Worthmann '14]
(for alternative conditions see also [Porretta/Zuazua '13]
[Trelat/Zuazua '14])

Economic MPC theorem

Under assumptions (i)–(iii), there exist $\varepsilon_1(N), \varepsilon_2(K) \to 0$ as $N \to \infty$ and $K \to \infty$, exponentially fast if additionally (iv) holds, such that the following properties hold

(1) Approximate average optimality:

$$\overline{J}^{cl}_\infty(x, \mu_N) \le \ell(x^e, u^e) + \varepsilon_1(N)$$

(2) Practical asymptotic stability: there is $\beta \in \mathcal{KL}$:

$$\|x_{\mu_N}(k, x) - x^e\| \le \beta(\|x - x^e\|, k) + \varepsilon_1(N) \text{ for all } k \in \mathbb{N}$$

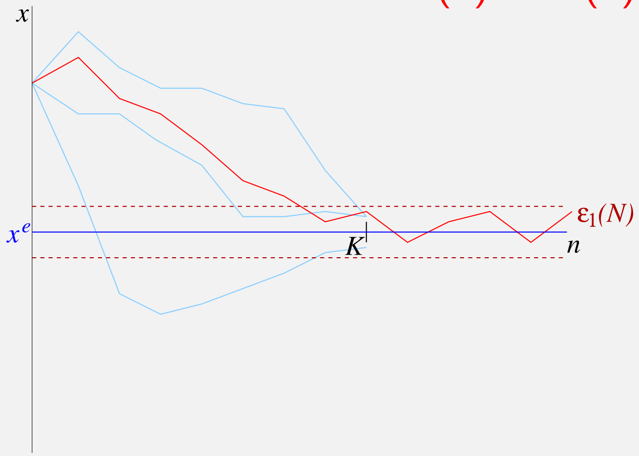
A practical Lyapunov function is given by the optimal value function for the problem with cost $\tilde{\ell}$

(3) Approximate transient optimality: for all $K \in \mathbb{N}$:

$$J^{cl}_K(x, \mu_N(x)) \le J_K(x, \mathbf{u}) + K\varepsilon_1(N) + \varepsilon_2(K)$$

for all admissible $\mathbf{u}$ with $\|x_{\mathbf{u}}(K, x) - x^e\| \le \beta(\|x - x^e\|, K) + \varepsilon_1(N)$

Illustration of (2) and (3)



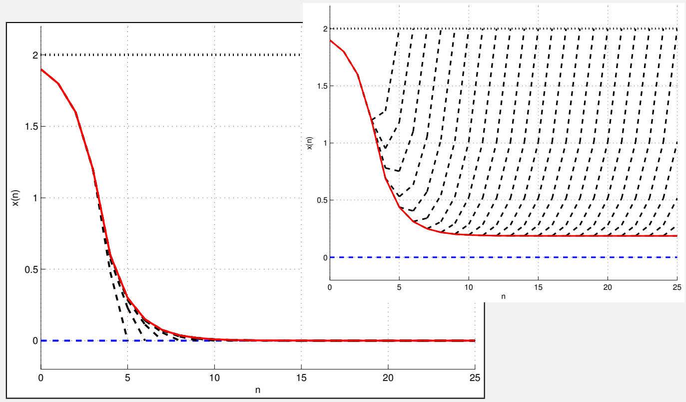
(2): $x_{\mu_N}(n)$ converges to the $\varepsilon_1(N)$-ball around x^e

(3): cost of all other trajectories reaching the ball at time K is higher than that of $x_{\mu_N}(n)$ up to the error $K\varepsilon_1(N)+\varepsilon_2(K)$

Schemes with terminal constraints

If we know the equilibrium x^e, we may use it as a terminal constraint, i.e., in each step of the MPC scheme we optimize only over those trajectories satisfying $x_{\mathbf{u}}(N)=x^e$

Example: $N=5$



Schemes with terminal constraints

Imposing $x_{\mathbf{u}}(N)=x^e$ improves the previous results

Theorem: [Angeli/Amrit/Rawlings '12; Diehl/Rawlings '11]
Under strict dissipativity and controllability, the resulting MPC scheme yields averaged optimal trajectories, i.e,

$$\overline{J}^{cl}_\infty(x,\mu_N) \le \ell(x^e,u^e) + \varepsilon_1(N)$$ ~~$\varepsilon_1(N)$~~

for which x^e is asymptotically stable, i.e.,

$$\|x_{\mu_N}(k,x)-x^e\| \le \beta(\|x-x^e\|,k) + \varepsilon_1(N)$$ ~~$\varepsilon_1(N)$~~.

In addition [Gr./Panin '15] we get approx. transient optimality

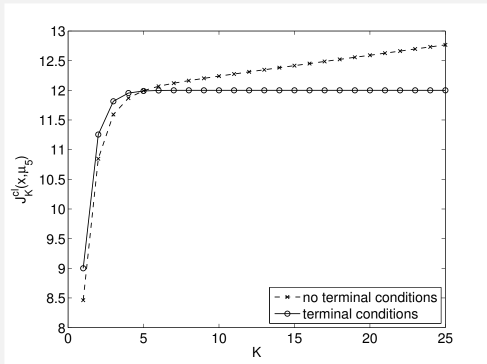
$$J^{cl}_K(x,\mu_N(x)) \le J_K(x,\mathbf{u}) + K\,\varepsilon_1(N)$$ ~~K~~ $\tilde\varepsilon_1(N)+\varepsilon_2(K)$

Example: closed loop cost

$$J^{cl}_K(x,\mu_N(x)) \le J_K(x,\mathbf{u}) + K\varepsilon_1(N) + \varepsilon_2(K)$$

vs.

$$J^{cl}_K(x,\mu_N(x)) \le J_K(x,\mathbf{u}) + \tilde\varepsilon_1(N) + \varepsilon_2(K)$$



But: terminal constraints can cause infeasibility and severe numerical problems

Extensions, further results

- The terminal constraint $x_{\mathbf{u}}(N) = x^e$ can be relaxed to $x_{\mathbf{u}}(N) \in \mathbb{X}_0$ for a neighborhood $\mathbb{X}_0$ of x^e if the functional J_N is appropriately modified [Amrit/Rawlings/Angeli '12, Gr./Panin '15]
- The results can be formulated directly in continuous time [Faulwasser/Bonvin '15, Alessandretti/Aguiar/Jones '15]
- The optimal equilibrium can be replaced by an optimal periodic orbit [Zanon/Gr. '15, Müller/Gr. '15]
- Results for time-varying systems [Zanon/Gros/Diehl '13, Alessandretti/Aguiar/Jones '15, Gr./Pirkelmann '17, Gr./Pirkelmann/Stieler '18]

$\rightsquigarrow$ next section

Summary of Section (5a)

- Economic Model Predictive Control can be seen as a method for splitting up general infinite horizon optimal control problems into the iterative solution of finite horizon problems
- The existence of the turnpike property at an optimal solution is the key ingredient for this approach
- Strict dissipativity is essentially equivalent to this property
- Good news: if MPC works, then it works regardless of whether we checked the conditions — but if we want to be sure we need to check

(5b) Time-varying problems

Setup

We consider nonlinear discrete time time-varying control systems

$$x_u(n+1) = f(n, x_u(n), u(n)), \quad x_u(0) = x_0$$

with $x_u(n) \in X$, $u(n) \in U$, X, U normed spaces

Problem: infinite horizon optimal control

Prototype problem: For a stage cost $\ell : \mathbb{N}_0 \times X \times U \to \mathbb{R}$ solve

$$\text{“minimize''}_{u} \quad J_\infty(x, u) = \sum_{n=0}^{\infty} \ell(n, x_u(n), u(n))$$

subject to state/control constraints $x_u(n) \in \mathbb{X}(n)$, $u(n) \in \mathbb{U}(n)$

with optimal control in feedback form $u(n) = \mu(n, x_u(n))$

Application background

Prototype problem: Keep the temperature in a room in a desired range with mimimal energy consumption for heating and cooling, based on predicted information on outside temperature, room usage etc.

Since there is no terminal time for the process, an infinite horizon formulation with

$\mathbb{X}(n)$ = desired temperature range at time n and

$\ell(n, x, u)$ = energy consumption

is an appropriate mathematical formulation

Application background

The methods are applicable to appropriate PDE models (Boussinesq equations, advection diffusion equations). An example will be given at the end of this section

In order to develop the concepts, we illustrate the results with a simple 1d discrete time model:

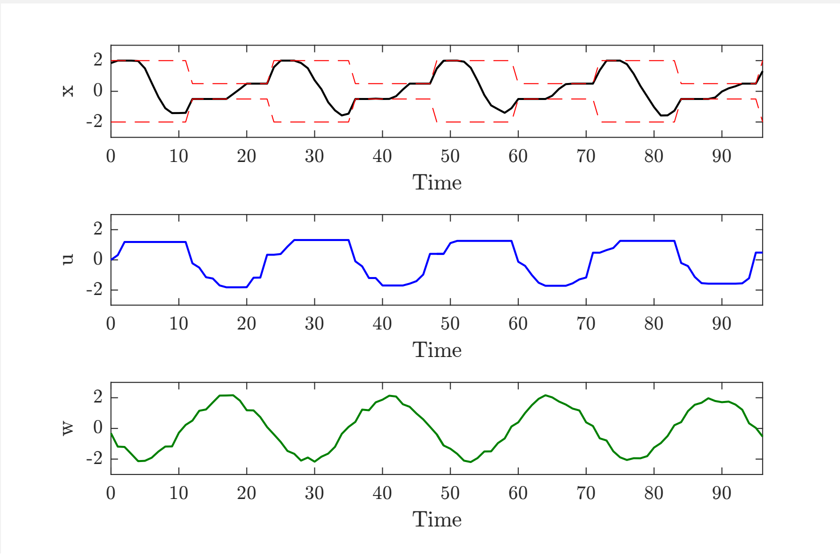
$$x(n+1) = \underbrace{x(n)}_{\text{inside temperature}} + \underbrace{u(n)}_{\text{heating/cooling}} + \underbrace{w(n)}_{\text{outside temperature}}$$

with stage cost

$$\ell(x, u) = u^2$$

and time varying $w(n)$ and $\mathbb{X}(n)$

Example: optimal trajectory



Optimality concept

Already for this simple example, the problem

$$\text{"}\underset{u}{\text{minimize}}\text{"} \quad J_\infty(x, u) = \sum_{n=0}^{\infty} \ell(n, x_u(n), u(n))$$

is not meaningful, because the sum will not converge

Remedy: Overtaking Optimality [Gale '67]

A trajectory $x^\star$ with control $u^\star$ is called overtaking optimal if

$$\limsup_{K\to\infty} \left(\sum_{n=0}^{K-1} \ell(n, x^\star(n), u^\star(n)) - \sum_{n=0}^{K-1} \ell(n, x_u(n), u(n)) \right) \le 0$$

holds for all admissible trajectory-control pairs (x_u, u) with $x_u(0) = x^\star(0)$

Model predictive control

The MPC scheme is straightforwardly generalized to the time-varying setting:

Idea: replace the infinite horizon problem

$$\text{minimize } J_\infty(x,u) = \sum_{n=0}^{\infty} \ell(n, x_u(n), u(n))$$

by the successive solution of finite horizon problems

$$\text{minimize } J_N(n,x,u) = \sum_{k=n}^{n+N-1} \ell(k, x_u(k), u(k))$$

for $n = 0, 1, 2, \dots$ with fixed optimization horizon $N \in \mathbb{N}$ and $x_u(k) \in \mathbb{X}(k)$, $u(k) \in \mathbb{U}(k)$

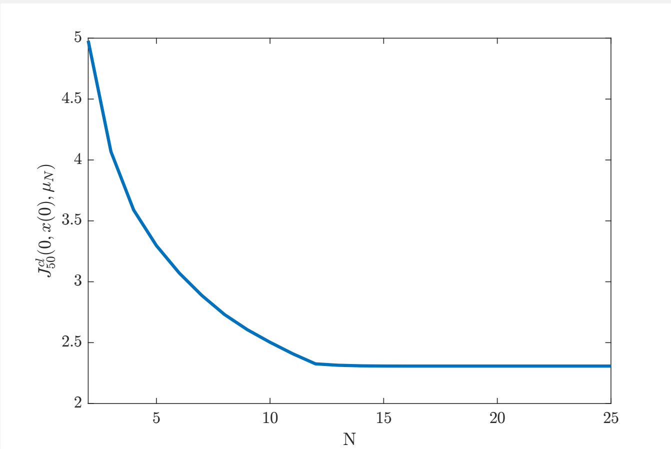
Questions

The questions are as for time-invariant problems:

- Is the MPC closed-loop solution approximately optimal? If yes, for which objective?
- What is the long-time behaviour of the MPC closed loop?

Numerical simulations suggest that the closed-loop cost converges for $N \to \infty$

MPC closed loop cost for different horizon length



But what is the limit?

Remember the time-invariant case

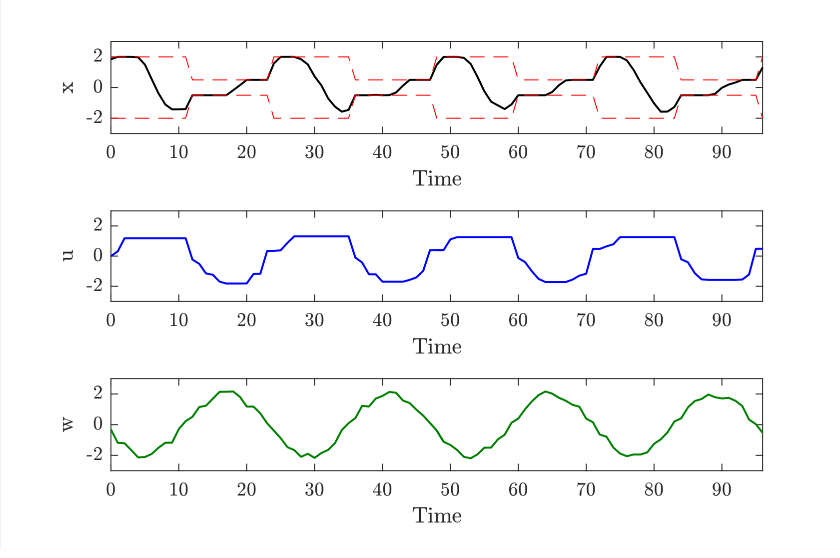
Recall: For time-invariant problems, it is known that MPC approximates infinite horizon optimal trajectories as $N \to \infty$ if

- there exists an optimal equilibrium x^e at which the turnpike property holds, i.e.,
  - finite horizon optimal trajectories stay near x^e most of the time
  - infinite horizon optimal trajectories converge to x^e
- a continuity condition of the optimal value function holds at x^e

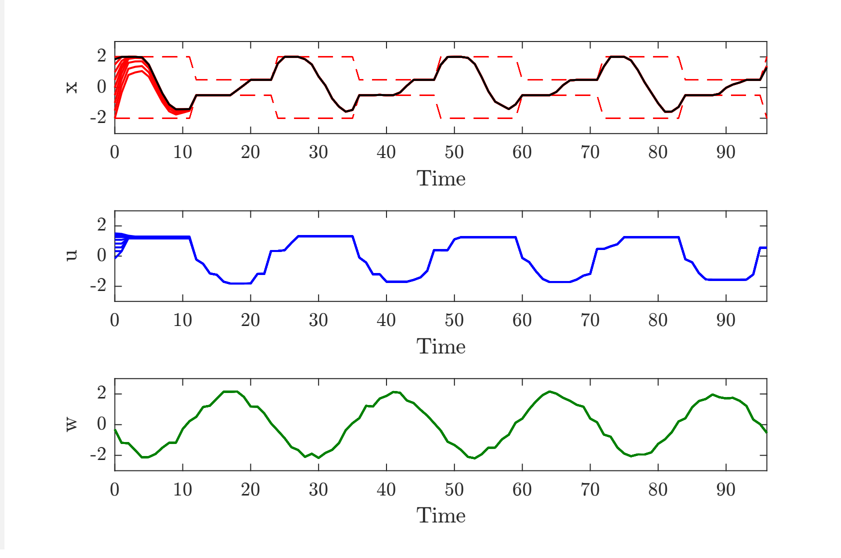
In this case, the closed-loop solutions converge to a neighbourhood of x^e

Question: can we find a counterpart for such an optimal equilibrium x^e for time-varying problems?

MPC closed loop



MPC closed loop for different initial values



A generalized optimal equilibrium

Obviously, the closed loop trajectories converge to the black limit trajectory. How is it characterized?

Idea: generalize the definition of optimal operation at an equilibrium from [Müller '13]:

We say that the system is optimally operated at a trajectory $\hat{x}$ with control $\hat{u}$ if

$$\limsup_{K\to\infty}\left(\sum_{n=0}^{K-1}\ell(n,\hat{x}(n),\hat{u}(n)) - \sum_{n=0}^{K-1}\ell(n,x_u(n),u(n))\right) \le 0$$

holds for all admissible trajectory-control pairs (x_u, u)

Note: this is similar to the definition of overtaking optimality, but now $x_u(0) \neq \hat{x}(0)$ is allowed

Towards the main result

Goal: obtain an approximate optimality result for the MPC closed loop trajectory

Approach: generalize the time-invariant result

To this end, we assume that the problem exhibits a trajectory $\hat{x}$ with control $\hat{u}$ at which the system is optimally operated

Now we have to define suitable time-varying versions of the turnpike property and of the continuity assumption

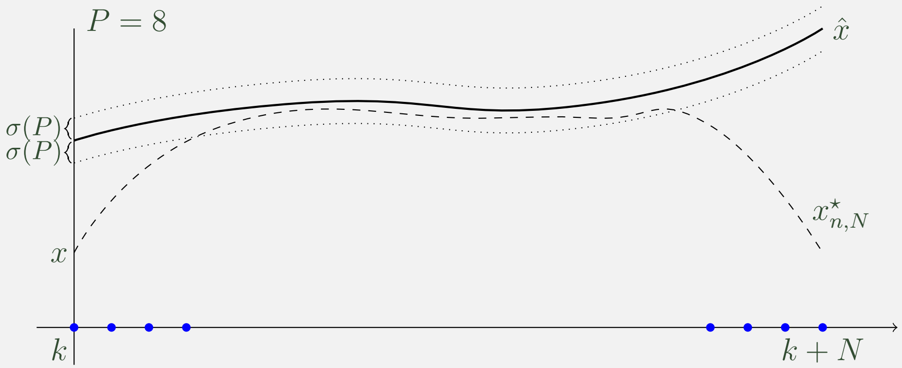
We start with the turnpike property

The turnpike property

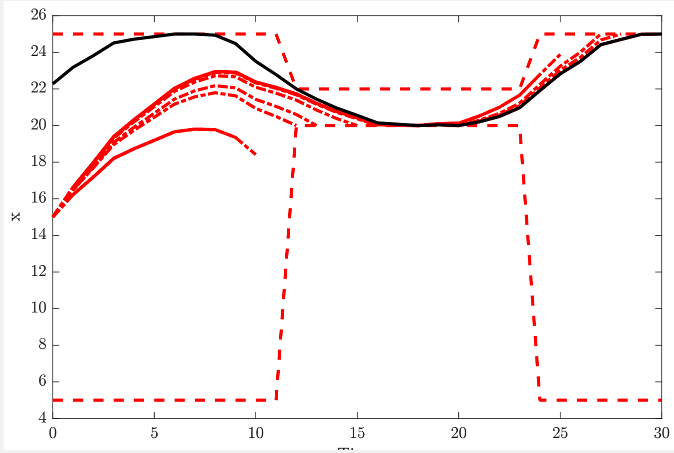
We assume there is a function $\sigma(P) \searrow 0$ as $P \to \infty$ such that for all N the finite horizon optimal pairs $(x^\star_{n,N}, u^\star_{n,N})$ satisfy

$$\|(x^\star_{n,N}(k,x), u^\star_{n,N}(k)) - (\hat{x}(k), \hat{u}(k))\| \leq \sigma(P)$$

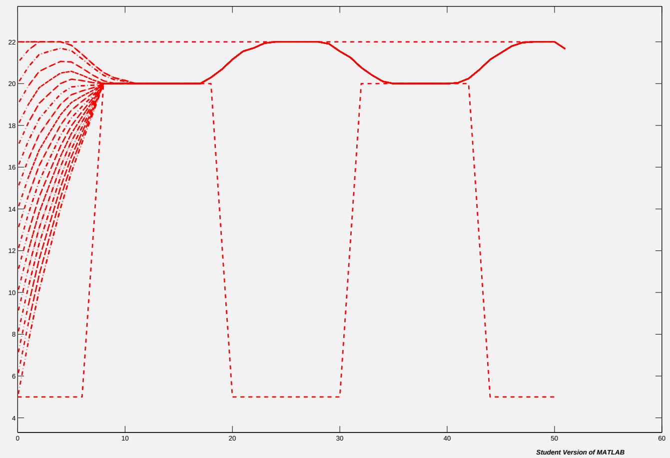
for all but at most P time indices $k \in \{n, \dots, n+N\}$



Example: turnpike property for varying N



Example: turnpike property for varying x_0



Continuity

As a second ingredient for our approximation result we need continuity of the optimal value functions

$$V_N(n,x) := \inf_u J_N(n,x,u) \quad \text{and} \quad V_\infty(n,x) := \inf_u J_\infty(n,x,u)$$

near $x = \hat{x}(n)$

But: V_∞ is not well defined in the framework of overtaking optimality

Remedy: Use the shifted stage cost (cf. [Ramsey 1928])

$$\hat{\ell}(n,x,u) = \ell(n,x,u) - \ell(n,\hat{x}(n),\hat{u}(n))$$

Then, by definition, $\hat{V}_\infty(n,\hat{x}(n)) = 0$

Continuity now formalizes that the shifted problem "behaves well" for x near $\hat{x}(n)$

Approximate continuity

We assume that there is a function $\gamma_V : \mathbb{R}_0^+ \times \mathbb{N} \cup \{\infty\} \to \mathbb{R}_0^+$ such that for all $n \in \mathbb{N}$ and all x from a neighborhood $B_\varepsilon(\hat{x}(n))$ the inequality

$$|\hat{V}_N(n,x) - \hat{V}_N(n,\hat{x}(n))| \le \gamma_V(\|x - \hat{x}(n)\|, N)$$

holds

Here

- $\gamma_V(r,N)$ is continuous, increasing in r, decreasing in N
- $\gamma_V(0,N) \to 0$ if $N \to \infty$, $\gamma_V(0,\infty) = 0$

In order to simplify the presentation we moreover assume there is $C > 0$ with $\hat{V}_N(n,x) \ge -C$ for all $n \in \mathbb{N}$, $x \in \mathbb{X}(n)$

Main performance theorem

Theorem: Assume that the turnpike property and approximate continuity hold. Then there exists an error term $\delta(N) \to 0$ as $N \to \infty$ with

$$\limsup_{K\to\infty} \Big(\sum_{n=0}^{K-1} \ell(n, x_{\mu_N}(n), \mu_N(x_{\mu_N}(n))) - \sum_{n=0}^{K-1} \ell(n, x_u(n), u(n)) - K\delta(N) \Big) \le 0$$

for all admissible (x_u, u) with $x_u(0) = x_{\mu_N}(0)$

In other words: the MPC closed loop trajectory on $\{0,\dots,K\}$ is the initial piece of an overtaking optimal trajectory — up to the error $K\delta(N)$

Note: The factor "K" in the error term usually vanishes when looking at the relative error

Strict dissipativity and convergence theorem

Strict dissipativity generalizes to the time-varying setting

Definition: The optimal control problem is called strictly dissipative at $\hat{x}$ if there exists a storage function $\lambda : \mathbb{N} \times X \times U \to \mathbb{R}$, bounded from below, such that

$$\begin{aligned}\lambda(n+1, f(n,x,u)) \;\le\; & \lambda(n,x) + \ell(n,x,u) - \ell(n,\hat{x}(n),\hat{u}(n)) \\ & - \alpha(\|(x,u) - (\hat{x}(n),\hat{u}(n))\|)\end{aligned}$$

holds for all n, x, u and $\alpha \in \mathcal{K}_\infty$

Theorem: If strict dissipativity holds and $\hat{V}_N$ is bounded from above uniformly in N, then the turnpike property holds

If, in addition, the system is locally controllable around $\hat{x}$, then the approximate continuity property holds

Strict dissipativity

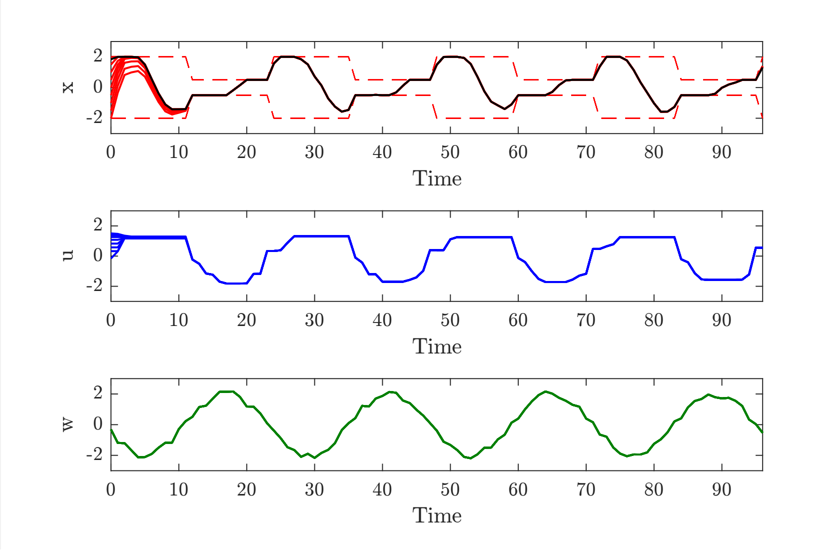
If we assume strict dissipativity, then we can also prove convergence of the MPC closed-loop solution to a neighbourhood of the turnpike trajectory:

Theorem: If the assumptions of the main result and strict dissipativity hold, then there is an error term $\varepsilon(N) \to 0$ as $N \to \infty$ such that

$$\limsup_{n\to\infty} \|x_{\mu_N}(n) - \hat{x}(n)\| \le \varepsilon(N)$$

This is also what we have seen in the example

MPC closed loop for different initial values



Convection-diffusion equation

Consider the partial differential equation

$$y_t - \alpha\Delta y + v\nabla y = 0 \text{ on } Q := \Omega \times [0, T]$$

with boundary conditions

$$\frac{\partial y}{\partial n} + \gamma y = \delta y_{out} \quad \text{on } \Gamma_{out}$$
$$\frac{\partial y}{\partial n} + \gamma y = \delta u \quad \text{on } \Gamma_c$$

where

- $y : Q \to \mathbb{R}$ is the temperature
- $y_{out} : \Sigma_{out} \to \mathbb{R}$ is the time-varying outside temperature
- $u : \Sigma_c \to \mathbb{R}$ is the controlled temperature
- $v : Q \to \mathbb{R}^d$ is the controlled convection
- α, γ, δ are coefficients

Optimal control problem

$$\underset{y,u,v}{\text{minimize}}\ J(y,u,v) = \frac{\varepsilon_u}{2}\int_0^T \|u\|^2_{L^2(\Gamma_c)}dt + \frac{\varepsilon_v}{2}\int_0^T \|v\|^2_{L^2(\Omega)}dt$$

s.t. PDE + boundary conditions

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t) \text{ on } \Gamma_c$$
$$\underline{y}(x,t) \le y(x,t) \le \overline{y}(x,t) \text{ on } \Omega_y$$

where $\Omega_y \subset \Omega$.

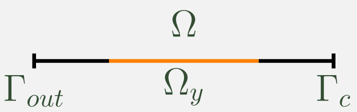


Figure: Example setting.

Numerical solution

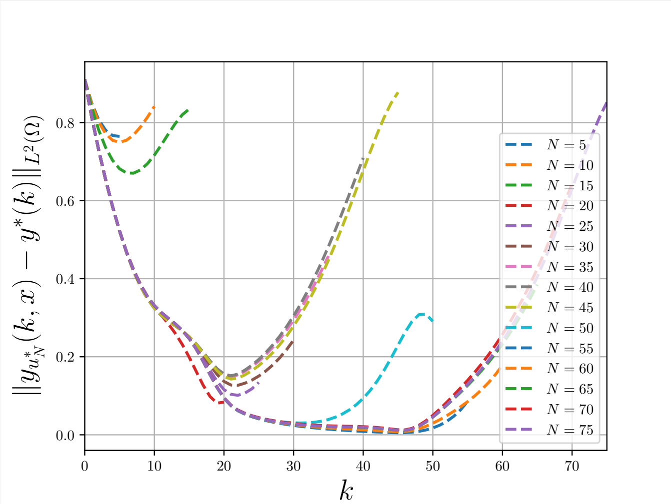
We solve MPC for the sampled-data system with sampling time 0.01

In order to solve the optimal control problem, we use a direct discretization (or early lumping) approach, based on a finite element discretization using FEniCS [Alnæs et al. '15] and using Ipopt[1] for the resulting finite dimensional optimization problem

This is not the most sophisticated and efficient numerical approach but works ok for a one-dimensional state space

[1] https://projects.coin-or.org/Ipopt

Turnpike behaviour



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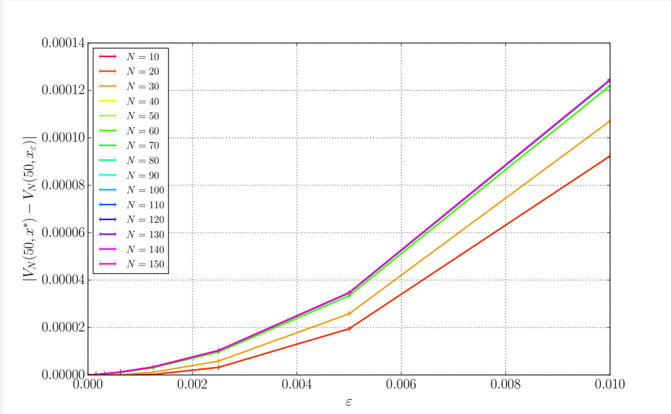
Continuity



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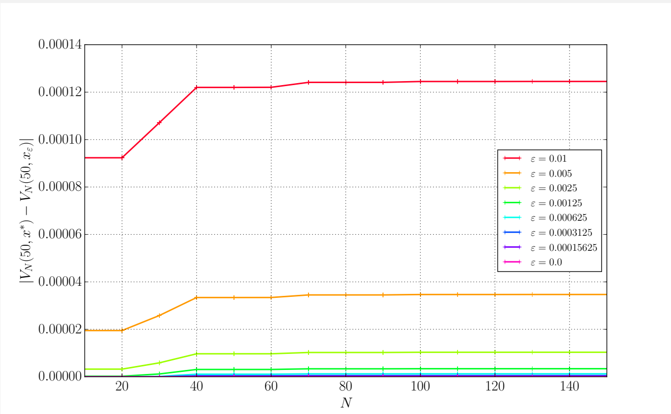
Continuity — Dependence on N



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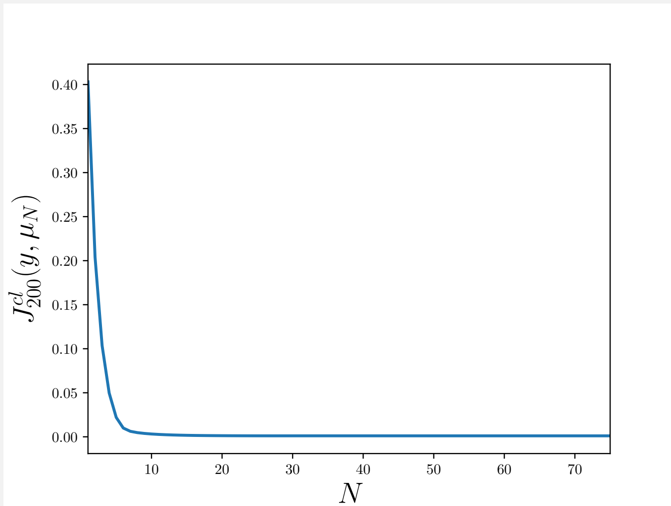
Closed loop cost



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Summary of Section (5b)

- For time-varying problems, an overtaking optimal time-varying solution replaces the optimal equilibrium
- Then, both turnpike theory and strict dissipativity can be generalized
- With this generalization, similar stability and performance results for MPC as in the time-invariant case can be obtained

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L. Grüne, S. Pirkelmann, *Economic model predictive control for time-varying systems: performance and stability results*, Optimal Control Applications and Methods, 41, 42–64, 2019