Lars Grüne
Marleen Stieler
Chair of Applied Mathematics
University of Bayreuth

Nonlinear Model Predictive Control

7^{th} Elgersburg School, March 2015

Exercises - Monday

Exercise 1 (Lyapunov Functions) Consider the two-dimensional difference equation

$$x^+ = (1 - \|x\|) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$$

with $x = (x_1, x_2)^\top \in \mathbb{R}^2$.

(a) Check that $V(x) = x_1^2 + x_2^2$ is a Lyapunov function for the equilibrium $x_* = 0$ on $S = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.

(b) Is V also a Lyapunov function on $S = \mathbb{R}^2$?

(c) Solve (a) and (b) for the difference equation

$$x^+ = \frac{1}{1 + \|x\|} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x.$$

Exercise 2 (Dynamic Programming) Consider the discrete-time control system

$$x^+ = x + u$$

with $x \in \mathbb{X} = \mathbb{R}$ and $u \in \mathbb{U} = \mathbb{R}$ and the stage cost $\ell(x, u) = x^2 + u^2$.

(a) Compute by dynamic programming the optimal value function V_2 and the optimal feedback law μ_2 without stabilizing terminal constraints.

(b) Check whether the resulting MPC closed loop is asymptotically stable.

(c) Compute the value $J_\infty^{cl}(x, \mu_2)$.

(d) Repeat (a)–(c) with the terminal constraint $x_{\mathbf{u}}(N) = 0$. Which feedback law μ_2 yields the better value $J_\infty^{cl}(x, \mu_2)$?

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Exercises - Tuesday

Exercise 3 (MPC Computer Exercise)

(a) Perform experiments with the file `double_integrator.m`, which implements an MPC controller without terminal constraints for the exact discrete-time model of a sampled-data double integrator. Get acquainted with the code and with the way constraints are encoded in MATLAB by changing parameters and constraints.
Note: The MATLAB files are coded directly in MATLAB without using Simulink.

(b) Add equilibrium terminal constraints and compare the solutions to those without terminal constraints for different N.

(c) Use the file as a template in order to implement the "car-and-mountains" example

$$x^+ = \begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \sin(\vartheta(x) + u) \\ \cos(\vartheta(x) + u)/2 \end{pmatrix}$$

with

$$\vartheta(x) = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$$

with equilibrium $x_* = (0, -1/2)^T$, stage cost $\ell(x,u) = \|x - x_*\|^2 + u^2$, control constraints $\mathbb{U} = [0, 0.2]$ and initial value $(0, 1/2)^T$. Determine the minimal stabilizing horizon by experiments for MPC without terminal constraints and with equilibrium terminal constraints.

Exercise 4 (Concept of MPC) After your return from Elgersburg School you explain the basic NMPC idea to a friend. After you finished your explanation, your friend asks:

"If I ride my bicycle and want to make a turn to the left, I first steer a little bit to the right to make my bicycle tilt to the left. Let us assume that this way of making a turn is optimal for a suitable finite horizon optimal control problem. This would mean that the optimal control sequence will initially steer to the right and later steer to the left. If we use this optimal control sequence in an NMPC algorithm, only the first control action will be implemented. As a consequence, we will always steer to the right, and we will make a turn to the right instead of a turn to the left. Does this mean that NMPC does not work for controlling my bicycle?"

What do you respond?

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Exercises - Thursday

Exercise 5 (Terminal constrained vs. unconstrained MPC) Consider again the control system

$$x^+ = x + u$$

with $x \in \mathbb{X} = \mathbb{R}$ and stage cost $\ell(x,u) = x^2 + u^2$, cf. Exercise 2, but now with $\mathbb{U} = [-1, 1]$.

(a) Consider the terminal constraint set $\mathbb{X}_0 = [-a, a]$ for some $a \geq 0$ (note that $a = 0$ coresponds to the equilibrium terminal constraint $x_{\mathbf{u}}(N) = x_* = 0$). Show that for each $N > 0$ and $a \geq 0$ there exists initial values $x_0 \in \mathbb{R}$ which are not contained in the feasible set $\mathbb{X}_N$.

(b) Check that without terminal constraints the MPC closed loop is asymptotically stable for $N = 2$ for arbitrary initial values $x_0 \in \mathbb{R}$. You can do this either by computing μ_2 and V_2 by dynamic programming and checking the conditions of the relaxed dynamic programming theorem or by implementing the closed loop in Matlab (using the code from Exercise 3) and performing numerical experiments.

Exercise 6 (MPC Computer Exercise)

(a) Write a Matlab code simulating an MPC controller for the inverted pendulum on a cart

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g\sin(x_1) - kx_2 + u\cos(x_1) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{aligned}$$

with $g = 9.81$ and $k = 0.1$. Use the sampling time $T = 0.15$, the control constraints $\mathbb{U} = [-10, 10]$ and the stage cost

$$\ell(x,u) = 10x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.01u^2.$$

Determine a horizon $N \in \mathbb{N}$ for which the closed loop is asymptotically stable by running simulations for different initial values. Implement different state constraints of your choice and observe the effect on the closed-loop behavior.

(b) Reduce the control constraints to $\mathbb{U} = [-3, 3]$ (removing all state constraints) and check whether the closed loop is still stable.

(c) Change the stage cost to

$$\ell(x, u) = 20(1 - \cos x_1) + x_2^2 + x_3^2 + x_4^2 + 0.01u^2$$

and observe what happens.

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Exercises - Friday

Exercise 7 (Optimal equilibria) Consider the following economic MPC example from the course:

$$X = U = \mathbb{R},\ \mathbb{X} = [0.1, 10],\ \mathbb{U} = [0.1, 5], \quad x^+ = u,\ \ell(x,u) = -\ln\left(5x^{0.34} - u\right).$$

(a) Compute the set

$$E := \{(x,u) \in \mathbb{X} \times \mathbb{U} \,|\, f(x,u) = x\},$$

i.e., compute all equilibria of the example.

(b) Determine the equilibrium $(x^e, u^e) \in E$ with the minimal value

$$\ell(x^e, u^e) = \min\{\ell(x,u) \,|\, (x,u) \in E\}.$$

(c) Prove that the problem is strictly dissipative.
Hint: For such a problem we can find a linear storage function λ.

Exercise 8 (MPC Computer Exercise) Implement the economic MPC problem from Exercise 7 in the Matlab MPC code and perform the following simulations.

(a) Impose terminal constraints and verify that the solutions indeed satisfy the convergence

$$\lim_{k\to\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) = \ell(x^e, u^e).$$

(b) In a simulation without terminal constraints, determine the limit

$$\lim_{k\to\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k)))$$

numerically and compute its distance to $\ell(x^e, u^e)$ depending on N. Verify that the distance decreases exponentially fast as N grows.

(c) Define stabilizing stage costs $\ell^{\text{stab}} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_0^+$, that are positive definite w.r.t. x^e. For different N, run a simulation without terminal constraints using ℓ^{stab} and compare the MPC closed-loop trajectory to the closed loop resulting from the original cost ℓ.

(d) Compute μ_N and μ_N^{stab} based on ℓ and ℓ^{stab}, respectively, and compare $J_K(x, \mu_N)$ to $J_K(x, \mu_N^{\text{stab}})$ for fixed K and varying N. What do you observe?
Note: The functional $J_K(x, \mathbf{u})$ refers to the original stage costs ℓ.