

Nonlinear Model Predictive Control

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Lars Grüne, Nonlinear Model Predictive Control, p. 2

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Part A: Stabilizing Model Predictive Control



Lars Grüne, Nonlinear Model Predictive Control, p. 3

(1) Introduction

What is Model Predictive Control (MPC)?

Setup

We consider **nonlinear discrete time** control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x_0$$

or, briefly

$$x^+ = f(x, u)$$

with $x \in X, u \in U$

- we consider **discrete time systems** for simplicity of exposition
- **continuous time systems** can be treated by using the discrete time representation of the corresponding **sampled data system** or a **numerical approximation**
- X and U depend on the model. These may be **Euclidean spaces** \mathbb{R}^n and \mathbb{R}^m or more general (e.g., infinite dimensional) spaces. For simplicity of exposition we assume that we have a norm $\|\cdot\|$ on both spaces



Prototype Problem

Assume there exists an equilibrium $x_* \in X$ for $u = 0$, i.e.

$$f(x_*, 0) = x_*$$

Task: **stabilize** the system $x^+ = f(x, u)$ at x_* via static state feedback, i.e., find $\mu : X \rightarrow U$, such that x_* is **asymptotically stable** for the feedback controlled system

$$x_{\mu}(n+1) = f(x_{\mu}(n), \mu(x_{\mu}(n))), \quad x_{\mu}(0) = x_0$$

Additionally, we impose **state constraints** $x_{\mu}(n) \in \mathbb{X}$ and **control constraints** $\mu(x(n)) \in \mathbb{U}$

for all $n \in \mathbb{N}$ and given sets $\mathbb{X} \subseteq X, \mathbb{U} \subseteq U$



Prototype Problem

Asymptotic stability means

Attraction: $x_{\mu}(n) \rightarrow x_*$ as $n \rightarrow \infty$

plus

Stability: Solutions starting close to x_* remain close to x_*

(we will later formalize this property using \mathcal{KL} functions)

Informal interpretation: **control** the system to x_* and **keep it there** while obeying the **state and control constraints**

Idea of MPC: use an optimal control problem which **minimizes the distance** to x_* in order to synthesize a feedback law μ



The idea of MPC

For defining the MPC scheme, we choose a **stage cost** $\ell(x, u)$ penalizing the distance from x_* and the control effort, e.g., $\ell(x, u) = \|x - x_*\|^2 + \lambda \|u\|^2$ for $\lambda \geq 0$

The basic idea of **MPC** is:

- **minimize** the summed stage cost along **trajectories** generated from our model over a **prediction horizon** N
- use the first element of the resulting optimal control sequence as feedback value
- repeat this procedure iteratively for all sampling instants $n = 0, 1, 2, \dots$

Notation in what follows:

- general feedback laws will be denoted by μ
- the **MPC feedback law** will be denoted by μ_N

The basic MPC scheme

Formal description of the basic MPC scheme:

At each time instant n solve for the **current state** $x_{\mu_N}(n)$

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_{\mu_N}(n), \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_{\mu_N}(n)$$

(\mathbf{u} admissible $\Leftrightarrow \mathbf{u} \in \mathbb{U}^N$ and $x_{\mathbf{u}}(k) \in \mathbb{X}$)

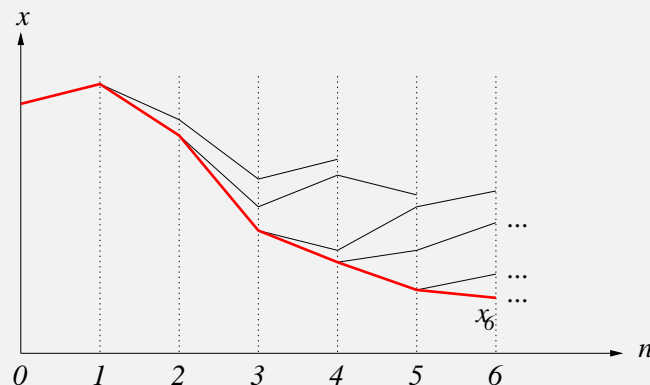
\rightsquigarrow optimal **trajectory** $x^*(0), \dots, x^*(N)$

with optimal **control** $\mathbf{u}^*(0), \dots, \mathbf{u}^*(N-1)$

Define the MPC **feedback law** $\mu(x_{\mu_N}(n)) := \mathbf{u}^*(0)$

$\rightsquigarrow x_{\mu_N}(n+1) = f(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n))) = f(x_{\mu_N}(n), \mathbf{u}^*(0)) = x^*(1)$

MPC from the trajectory point of view



black = predictions (open loop optimization)

red = MPC closed loop, $x_n = x_{\mu_N}(n)$

Model predictive control (aka Receding horizon control)

Idea **first formulated** in [A.I. Propoi, *Use of linear programming methods for synthesizing sampled-data automatic systems*, Automation and Remote Control 1963], often **rediscovered**

used in **industrial applications** since the mid 1970s, mainly for constrained linear systems [Qin & Badgwell, 1997, 2001]

more than 9000 industrial MPC applications in Germany counted in [Dittmar & Pfeifer, 2005]

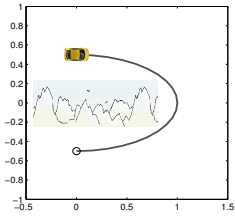
development of theory since ~ 1980 (linear), ~ 1990 (nonlinear)

Central questions:

- When does MPC **stabilize** the system?
- How good is the **performance** of the MPC feedback law?
- How long does the **optimization horizon** N need to be?

and, of course, the development of good algorithms (not topic of this course)

An example



$$\begin{aligned}x_1^+ &= \sin(\varphi + u) \\x_2^+ &= \cos(\varphi + u)/2\end{aligned}$$

$$\text{with } \varphi = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$$

$$\mathbb{X} = \{x \in \mathbb{R}^2 : \|(x_1, 2x_2)^T\| = 1\}, \mathbb{U} = [0, u_{\max}]$$

$$x_* = (0, -1/2)^T, x_0 = (0, 1/2)^T$$

MPC with $\ell(x, u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$ yields asymptotic stability for $N = 11$ but not for $N \leq 10$

Summary of Section (1)

- MPC is an **online optimal control** based method for computing **stabilizing feedback laws**
- MPC computes the feedback law by **iteratively solving finite horizon optimal control problems** using the current state $x_0 = x_{\mu_N}(n)$ as initial value
- the **feedback value** $\mu_N(x_0)$ is the **first element** of the resulting optimal control sequence
- the example shows that MPC does **not always yield an asymptotically stabilizing** feedback law

(2a) Background material: Lyapunov functions

Purpose of this section

We introduce **Lyapunov functions** as a tool to rigorously verify asymptotic stability

In the subsequent sections, this will be used in order to establish asymptotic stability of the **MPC closed loop**

In this section, we consider discrete time systems **without input**, i.e.,

$$x^+ = g(x)$$

with $x \in X$ or, in long form

$$x(n+1) = g(x(n)), \quad x(0) = x_0$$

(later we will apply the results to $g(x) = f(x, \mu_N(x))$)

Note: we do not require g to be **continuous**

Comparison functions

For $\mathbb{R}_0^+ = [0, \infty)$ we use the following classes of **comparison functions**

$$\mathcal{K} := \left\{ \alpha : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \begin{array}{l} \alpha \text{ is continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array} \right\}$$

$$\mathcal{K}_\infty := \left\{ \alpha : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \alpha \in \mathcal{K} \text{ and } \alpha \text{ is unbounded} \right\}$$

$$\mathcal{KL} := \left\{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \begin{array}{l} \beta \text{ is continuous,} \\ \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \in \mathbb{R}_0^+ \\ \text{and } \beta(r, \cdot) \text{ is strictly de-} \\ \text{creasing to 0 for all } r \in \mathbb{R}_0^+ \end{array} \right\}$$

Asymptotic stability revisited

A point x_* is called an **equilibrium** of $x^+ = g(x)$ if $g(x_*) = x_*$

A set $Y \subseteq X$ is called **forward invariant** for $x^+ = g(x)$ if $g(x) \in Y$ holds for each $x \in Y$

We say that x_* is **asymptotically stable** for $x^+ = g(x)$ on a forward invariant set Y if there exists $\beta \in \mathcal{KL}$ such that

$$\|x(n) - x_*\| \leq \beta(\|x(0) - x_*\|, n)$$

holds for all $x \in Y$ and $n \in \mathbb{N}$

How can we **check** whether this property holds?

Lyapunov function

Let $Y \subseteq X$ be a forward invariant set and $x_* \in X$. A function $V : Y \rightarrow \mathbb{R}_0^+$ is called a **Lyapunov function** for $x^+ = g(x)$ if the following two conditions hold for all $x \in Y$:

(i) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$$

(ii) There exists $\alpha_V \in \mathcal{K}$ such that

$$V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$$

Stability theorem

Theorem: If the system $x^+ = g(x)$ admits a **Lyapunov function** V on a forward invariant set Y , then x_* is an **asymptotically stable equilibrium** on Y

Idea of proof: $V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$ implies that V is **strictly decaying** along solutions away from x_*

This allows to **construct** $\tilde{\beta} \in \mathcal{KL}$ with $V(x(n)) \leq \tilde{\beta}(V(x(0)), n)$

The bounds $\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$ imply that **asymptotic stability** holds with $\beta(r, t) = \alpha_1^{-1}(\tilde{\beta}(\alpha_2(r), t))$

Lyapunov functions — discussion

While the convergence $x(n) \rightarrow x_*$ is typically **non-monotone** for an asymptotically stable system, the convergence $V(x(n)) \rightarrow 0$ is **strictly monotone**

It is hence sufficient to check the decay of V **in one time step**

↪ it is typically quite **easy to check** whether a given function is a Lyapunov function

But it is in general **difficult to find** a candidate for a Lyapunov function

For MPC, we will use the **optimal value functions** which we introduce in the next section

(2b) Background material: Dynamic Programming

Purpose of this section

We define the **optimal value functions** V_N for the optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

used within the MPC scheme (with $x_0 = x_{\mu_N}(n)$)

We present the **dynamic programming principle**, which establishes a relation for these functions and will eventually enable us to derive conditions under which V_N is a Lyapunov function

Optimal value functions

We define the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} J_N(x_0, \mathbf{u})$$

setting $V_N(x_0) := \infty$ if x_0 is **not feasible**, i.e., if there is no admissible \mathbf{u} (recall: \mathbf{u} admissible $\Leftrightarrow x_{\mathbf{u}}(k) \in \mathbb{X}, \mathbf{u}(k) \in \mathbb{U}$)

An admissible control sequence \mathbf{u}^* is called **optimal**, if

$$J_N(x_0, \mathbf{u}^*) = V_N(x_0)$$

Note: an optimal \mathbf{u}^* does not need to exist in general. In the sequel **we assume that \mathbf{u}^* exists if x_0 is feasible**

Dynamic Programming Principle

Theorem: (Dynamic Programming Principle) For any feasible $x_0 \in \mathbb{X}$ the optimal value function satisfies

$$V_N(x_0) = \inf_{\substack{u \in \mathbb{U} \\ f(x_0, u) \in \mathbb{X}}} \{ \ell(x_0, u) + V_{N-1}(f(x_0, u)) \}$$

Moreover, if \mathbf{u}^* is an optimal control, then

$$V_N(x_0) = \ell(x_0, \mathbf{u}^*(0)) + V_{N-1}(f(x_0, \mathbf{u}^*(0)))$$

holds.

Idea of Proof: Follows by taking infima in the identity

$$\begin{aligned} J_N(x_0, \mathbf{u}) &= \ell(x_{\mathbf{u}}(0), \mathbf{u}(0)) + \sum_{k=1}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \\ &= \ell(x_0, \mathbf{u}(0)) + J_{N-1}(f(x_0, \mathbf{u}(0)), \mathbf{u}(\cdot + 1)) \end{aligned}$$

Corollaries

Corollary: Let x^* be an optimal trajectory of length N with optimal control u^* and $x^*(0) = x$. Then

(i) The “tail”

$$(x^*(k), x^*(k+1), \dots, x^*(N-1))$$

is an optimal trajectory of length $N - k$.

(ii) The MPC feedback μ_N satisfies

$$\mu_N(x) = \operatorname{argmin}_{u \in \mathbb{U}} \{ \ell(x, u) + V_{N-1}(f(x, u)) \}$$

(i.e., $u = \mu_N(x)$ minimizes this expression),

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

and

$$u^*(k) = \mu_{N-k}(x^*(k)), \quad k = 0, \dots, N-1$$

Dynamic Programming Principle — discussion

We will see later, that under suitable conditions the optimal value function will play the role of a Lyapunov function for the MPC closed loop

The dynamic programming principle and its corollaries will prove to be important tools to establish this fact

In order to see why this can work, in the next section we briefly look at infinite horizon optimal control problems

Moreover, for simple systems the principle can be used for computing V_N and μ_N — we will see an example in the exercises

(2c) Background material:

Relaxed Dynamic Programming

Infinite horizon optimal control

Just like the finite horizon problem we can define the **infinite horizon optimal control problem**

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

and the corresponding **optimal value function**

$$V_\infty(x_0) := \inf_{\mathbf{u} \text{ admissible}} J_\infty(x_0, \mathbf{u})$$

If we could compute an **optimal feedback** μ_∞ for this problem (which is — in contrast to computing μ_N — in general a **very difficult** problem), we would have solved the **stabilization problem**



Infinite horizon dynamic programming principle

Recall the **corollary** from the finite horizon dynamic programming principle

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

The corresponding result which can be proved for the **infinite horizon problem** reads

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x)))$$

↔ if $\ell(x, \mu_\infty(x)) \geq \alpha_V(\|x - x_*\|)$ holds, then we get

$$V_\infty(f(x, \mu_\infty(x))) \leq V_\infty(x) - \alpha_V(\|x - x_*\|)$$

and if in addition $\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$ holds, then V_∞ is a **Lyapunov function** ↔ **asymptotic stability**



Relaxing dynamic programming

Unfortunately, an equation of the type

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x)))$$

cannot be expected if we replace “ ∞ ” by “ N ” everywhere (in fact, it would **imply** $V_N = V_\infty$)

However, we will see that we can establish **relaxed versions** of this inequality in which we

- relax “=” to “ \geq ”
- relax $\ell(x, \mu(x))$ to $\alpha\ell(x, \mu(x))$ for some $\alpha \in (0, 1]$

$$\rightsquigarrow V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

“**relaxed dynamic programming inequality**” [Rantzer et al. '06ff]

What can we conclude from this inequality?



Relaxed dynamic programming

We define the **infinite horizon performance** of the MPC closed loop system $x^+ = f(x, \mu_N(x))$ as

$$J_\infty^{\text{cl}}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \quad x_{\mu_N}(0) = x_0$$

Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a **forward invariant set** for the MPC closed loop and assume that

$$V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

Then for all $x \in Y$ the **infinite horizon performance** satisfies

$$J_\infty^{\text{cl}}(x_0, \mu_N) \leq V_N(x_0)/\alpha$$



Relaxed dynamic programming

Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the **inequalities**

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is **asymptotically stable** on Y with Lyapunov function V_N .

Proof: The assumed inequalities immediately imply that $V = V_N$ is a Lyapunov function for $x^+ = g(x) = f(x, \mu_N(x))$ with

$$\alpha_1(r) = \alpha_3(r), \quad \alpha_V(r) = \alpha \alpha_3(r)$$

\Rightarrow **asymptotic stability**

Relaxed dynamic programming

For proving the **performance estimate** $J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$, the **relaxed dynamic programming inequality** implies

$$\begin{aligned} & \alpha \sum_{n=0}^{K-1} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \\ & \leq \sum_{n=0}^{K-1} (V_N(x_{\mu_N}(n)) - V_N(x_{\mu_N}(n+1))) \\ & = V_N(x_{\mu_N}(0)) - V_N(x_{\mu_N}(K)) \leq V_N(x_{\mu_N}(0)) \end{aligned}$$

Since all summands are ≥ 0 , this implies that the **limit** for $K \rightarrow \infty$ **exists** and we get

$$\alpha J_\infty^{cl}(x_0, \mu_N) = \alpha \sum_{n=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \leq V_N(x_{\mu_N}(0))$$

\Rightarrow **assertion**

Summary of Section (2)

- **Lyapunov functions** are our central tool for verifying asymptotic stability
- **Dynamic programming** provides us with equations which will be heavily used in the subsequent analysis
- **Infinite horizon optimal control** would solve the stabilization problem — if we could **compute** the feedback law μ_∞
- The performance of the MPC controller can be measured by looking at the **infinite horizon value** along the MPC closed loop trajectories
- **Relaxed dynamic programming** gives us conditions under which both asymptotic stability and performance results can be derived

Application of background results

The main task will be to verify the assumptions of the **relaxed dynamic programming theorem**, i.e.,

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

for all x in a **forward invariant set** Y for $x^+ = f(x, \mu_N(x))$

To this end, we present **two different approaches**:

- modify the optimal control problem in the MPC loop by adding **terminal constraints and costs**
- derive assumptions on f and ℓ under which MPC works **without terminal constraints and costs**

(3) Stability with stabilizing constraints

V_N as a Lyapunov Function

Problem: Prove that the MPC feedback law μ_N is stabilizing

Approach: Verify the assumptions

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbf{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

of the relaxed dynamic programming theorem for the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

Why is this difficult?

Let us first consider the inequality

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

The dynamic programming principle for V_N yields

$$V_N(x) \geq \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

↪ we have V_{N-1} where we would like to have V_N

↪ we would get the desired inequality if we could ensure

$$V_{N-1}(f(x, \mu_N(x))) \geq V_N(f(x, \mu_N(x))) + \text{“small error”}$$

(where “small” means that the error can be compensated replacing $\ell(x, \mu_N(x))$ by $\alpha \ell(x, \mu_N(x))$ with $\alpha \in (0, 1)$)

Why is this difficult?

Task: Find conditions under which

$$V_{N-1}(f(x, \mu_N(x))) \geq V_N(f(x, \mu_N(x))) + \text{“small error”}$$

holds

For

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

this appeared to be out of reach until the mid 1990s

Note: $V_{N-1} \leq V_N$ by non-negativity of ℓ ; typically with strict “<”

↪ additional stabilizing constraints were proposed

(3a) Equilibrium terminal constraint

Equilibrium terminal constraint

Optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

Assumption: $f(x_*, 0) = x_*$ and $\ell(x_*, 0) = 0$

Idea: add equilibrium terminal constraint

$$x_{\mathbf{u}}(N) = x_*$$

[Keerthi/Gilbert '88, ...]

↪ we now solve

$$\underset{\mathbf{u} \in \mathbb{U}_{x_*}^N(x_0)}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

with $\mathbb{U}_{x_*}^N(x_0) := \{\mathbf{u} \in \mathbb{U}^N \text{ admissible and } x_{\mathbf{u}}(N) = x_*\}$



Prolongation of control sequences

Let $\tilde{\mathbf{u}} \in \mathbb{U}_{x_*}^{N-1}(x_0) \Rightarrow x_{\tilde{\mathbf{u}}}(N-1) = x_*$

Define $\mathbf{u} \in \mathbb{U}^N$ as $\mathbf{u}(k) := \begin{cases} \tilde{\mathbf{u}}(k), & k = 0, \dots, N-2 \\ 0, & k = N-1 \end{cases}$

$$\Rightarrow x_{\mathbf{u}}(N) = f(x_{\tilde{\mathbf{u}}}(N-1), \mathbf{u}(N-1)) = f(x_*, 0) = x_*$$

$$\Rightarrow \mathbf{u}_N \in \mathbb{U}_{x_*}^N(x_0)$$

↪ every $\tilde{\mathbf{u}} \in \mathbb{U}_{x_*}^{N-1}(x_0)$ can be prolonged to an $\mathbf{u}_N \in \mathbb{U}_{x_*}^N(x_0)$

Moreover, since

$$\ell(x_{\mathbf{u}_N}(N-1), \mathbf{u}_N(N-1)) = \ell(x_*, 0) = 0,$$

the prolongation has zero stage cost

Reversal of $V_{N-1} \leq V_N$

Now, let $\tilde{\mathbf{u}}^* \in \mathbb{U}_{x_*}^{N-1}(x_0)$ be the optimal control for J_{N-1} , i.e.,

$$V_{N-1}(x_0) = J_{N-1}(x_0, \tilde{\mathbf{u}}^*)$$

Denote by $\mathbf{u} \in \mathbb{U}_{x_*}^N(x_0)$ its prolongation

$$\begin{aligned} \Rightarrow V_{N-1}(x_0) &= J_{N-1}(x_0, \tilde{\mathbf{u}}^*) = \sum_{k=0}^{N-2} \ell(x_{\tilde{\mathbf{u}}^*}(k), \tilde{\mathbf{u}}^*(k)) \\ &= \sum_{n=0}^{N-2} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + \underbrace{\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1))}_{=0} \\ &= \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) = J_N(x_0, \mathbf{u}) \geq V_N(x_0) \end{aligned}$$

↪ The inequality $V_{N-1} \leq V_N$ is reversed to $V_{N-1} \geq V_N$

Note: $V_{N-1} \leq V_N$ does no longer hold now

But: the dynamic programming principle remains valid



Relaxed dynamic programming inequality

From the **reversed inequality**

$$V_{N-1}(x) \geq V_N(x)$$

and the **dynamic programming principle**

$$V_N(x) \geq \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

we immediately get

$$V_N(x) \geq \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

This is exactly the desired **relaxed dynamic programming inequality**, even with $\alpha = 1$, since no “small error” occurs

↔ **stability** follows if we can ensure the **additional inequalities**

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$



Feasible sets

The inequality $\inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$ is **easy** to satisfy, e.g., $\ell(x, u) = \|x - x_*\|^2 + \lambda\|u\|^2$ will work (with $\alpha_3(r) = r^2$)

What about $V_N(x) \leq \alpha_2(\|x - x_*\|)$?

Recall: by definition $V_N(x) = \infty$ if x is **not feasible**, i.e., if there is no $\mathbf{u} \in \mathbb{U}_{x_*}^N(x)$

↔ define the **feasible set** $\mathbb{X}_N := \{x \in \mathbb{X} \mid \mathbb{U}_{x_*}^N(x) \neq \emptyset\}$

For $x \notin \mathbb{X}_N$ the inequality $V_N(x) \leq \alpha_2(\|x - x_*\|)$ cannot hold

But: for all $x \in \mathbb{X}_N$ we can **ensure this inequality** under rather mild conditions (details can be given if desired)

↔ the feasible set \mathbb{X}_N is the “natural” **operating region** of MPC with equilibrium terminal constraints



Stability theorem

Theorem: Consider the MPC scheme with **equilibrium terminal constraint** $x_{\mathbf{u}}(N) = x_*$ where x_* satisfies $f(x_*, 0) = x_*$ and $\ell(x_*, 0) = 0$. Assume that

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

holds for all $x \in \mathbb{X}_N$.

Then \mathbb{X}_N is **forward invariant**, the MPC closed loop is **asymptotically stable** on \mathbb{X}_N and the **performance estimate**

$$J_{\infty}^{cl}(x, \mu_N) \leq V_N(x)$$

holds.

Note: The constraint $x_{\mathbf{u}}(N) = x_*$ does **not imply** $x_{\mu_N}(N) = x_*$



Stability theorem — sketch of proof

Sketch of proof: All assertions follow from the **relaxed dynamic programming theorem** if we prove **forward invariance** of \mathbb{X}_N for the MPC closed loop system $x^+ = f(x, \mu_N(x))$

↔ we need to prove $x \in \mathbb{X}_N \Rightarrow x^+ \in \mathbb{X}_N$

(1) The **prolongation property** implies $\mathbb{X}_{N-1} \subseteq \mathbb{X}_N$

(2) For $x \in \mathbb{X}_N$, the **definition** $\mu_N(x) := \mathbf{u}^*(0)$ implies

$$x^+ = f(x, \mu_N(x)) = f(x, \mathbf{u}^*(0)) = x^*(1)$$

and since $x^*(N) = x_*$, the sequence $(x^*(1), \dots, x^*(N))$ is an **admissible trajectory** of length $N - 1$ from $x^*(1) = x^+$ to $x^*(N) = x_*$

(3) This implies $x^+ \in \mathbb{X}_{N-1} \subseteq \mathbb{X}_N$



Equilibrium terminal constraint — Discussion

The additional condition

$$x(N) = x_*$$

ensures asymptotic stability in a **rigorously provable** way, but

- online optimization may become **harder**
- if we want a **large feasible set** \mathbb{X}_N we typically need a **large optimization horizon** N (see the car-and-mountains example)
- system needs to be **controllable to x_* in finite time**
- **not very often used** in industrial practice

(3b) Regional terminal constraint and terminal cost

Regional constraint and terminal cost

Optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

We want V_N to become a **Lyapunov function**

Idea: add local Lyapunov function $F : \mathbb{X}_0 \rightarrow \mathbb{R}_0^+$ as **terminal cost**

$$J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N))$$

F is defined on a region \mathbb{X}_0 around x_* which is imposed as **terminal constraint** $x(N) \in \mathbb{X}_0$

[Chen & Allgöwer '98, Jadbabaie et al. '98 ...]

Regional constraint and terminal cost

We thus **change** the optimal control problem to

$$\underset{\mathbf{u} \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N))$$

with

$$\mathbb{U}_{\mathbb{X}_0}^N(x_0) := \{\mathbf{u} \in \mathbb{U}^N \text{ admissible and } x_{\mathbf{u}}(N) \in \mathbb{X}_0\}$$

Which **properties** do we need for F and \mathbb{X}_0 in order to make this work?

Regional constraint and terminal cost

Assumptions on $F : \mathbb{X}_0 \rightarrow \mathbb{R}_0^+$ and \mathbb{X}_0

There exists a **controller** $\kappa : \mathbb{X}_0 \rightarrow \mathbb{U}$ with the following properties:

- (i) \mathbb{X}_0 is **forward invariant** for $x^+ = f(x, \kappa(x))$:
for each $x \in \mathbb{X}_0$ we have $f(x, \kappa(x)) \in \mathbb{X}_0$
- (ii) F is a **Lyapunov function** for $x^+ = f(x, \kappa(x))$ on \mathbb{X}_0
which is **compatible** with the stage cost ℓ in the following sense:
for each $x \in \mathbb{X}_0$ the inequality

$$F(f(x, \kappa(x))) \leq F(x) - \ell(x, \kappa(x))$$

holds



Prolongation of control sequences

Let $\tilde{\mathbf{u}} \in \mathbb{U}_{\mathbb{X}_0}^{N-1}(x_0) \Rightarrow \tilde{x} := x_{\tilde{\mathbf{u}}}(N-1) \in \mathbb{X}_0$

Define $\mathbf{u} \in \mathbb{U}^N$ as $\mathbf{u}(k) := \begin{cases} \tilde{\mathbf{u}}(k), & k = 0, \dots, N-2 \\ \kappa(\tilde{x}), & k = N-1 \end{cases}$

with κ from (i)

$$\Rightarrow x_{\mathbf{u}}(N) = f(x_{\tilde{\mathbf{u}}}(N-1), \mathbf{u}(N-1)) = f(\tilde{x}, \kappa(\tilde{x})) \in \mathbb{X}_0$$

$$\Rightarrow \mathbf{u} \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)$$

\rightsquigarrow every $\tilde{\mathbf{u}} \in \mathbb{U}_{\mathbb{X}_0}^{N-1}(x_0)$ can be **prolonged** to an $\mathbf{u} \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)$

By (ii) the **stage cost** of the prolongation is **bounded** by

$$\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1)) \leq F(x_{\mathbf{u}}(N-1)) - F(x_{\mathbf{u}}(N))$$



Reversal of $V_{N-1} \leq V_N$

Let $\tilde{\mathbf{u}}^* \in \mathbb{U}_{\mathbb{X}_0}^{N-1}(x_0)$ be the **optimal control** for J_{N-1} , i.e.,

$$V_{N-1}(x_0) = J_{N-1}(x_0, \tilde{\mathbf{u}}^*)$$

Denote by $\mathbf{u} \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)$ its **prolongation**

$$\begin{aligned} \Rightarrow V_{N-1}(x_0) &= J_{N-1}(x_0, \tilde{\mathbf{u}}^*) \\ &= \sum_{k=0}^{N-2} \ell(x_{\tilde{\mathbf{u}}^*}(k), \tilde{\mathbf{u}}^*(k)) + \underbrace{F(x_{\tilde{\mathbf{u}}^*}(N-1))}_{\geq \ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1)) + F(x_{\mathbf{u}}(N))} \\ &\geq \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(n), \mathbf{u}(n)) + F(x_{\mathbf{u}}(N)) \\ &= J_N(x_0, \mathbf{u}) \geq V_N(x_0) \end{aligned}$$

\rightsquigarrow again we get $V_{N-1} \geq V_N$



Feasible sets

Define the **feasible set**

$$\mathbb{X}_N := \{x \in \mathbb{X} \mid \mathbb{U}_{\mathbb{X}_0}^N(x) \neq \emptyset\}$$

Like in the equilibrium constrained case, on \mathbb{X}_N one can **ensure the inequality**

$$V_N(x) \leq \alpha_2(\|x - x_*\|)$$

for some $\alpha_2 \in \mathcal{K}_\infty$ under **mild conditions**, while **outside** \mathbb{X}_N we get $V_N(x) = \infty$



Stability theorem

Theorem: Consider the MPC scheme with **regional terminal constraint** $x_u(N) \in \mathbb{X}_0$ and **Lyapunov function terminal cost** F **compatible** with ℓ . Assume that

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

holds for all $x \in \mathbb{X}_N$.

Then \mathbb{X}_N is **forward invariant**, the MPC closed loop is **asymptotically stable** on \mathbb{X}_N and the **performance estimate**

$$J_\infty^{\text{cl}}(x, \mu_N) \leq V_N(x)$$

holds.

Proof: Almost **identical** to the equilibrium constrained case



Regional constraint and terminal cost — Discussion

Compared to the equilibrium constraint, the regional constraint

- yields **easier online optimization problems**
- yields **larger feasible sets**
- does **not need exact controllability** to x_*

But:

- **large feasible set** still needs a **large optimization horizon** N (see again the car-and-mountains example)
- **additional analytical effort** for computing F
- **hardly ever used** in industrial practice

In Section (5) we will see how stability can be proved **without stabilizing terminal constraints**



Summary of Section (3)

- terminal constraints yield that the usual inequality $V_{N-1} \leq V_N$ is **reversed** to $V_{N-1} \geq V_N$
- this enables us to derive the **relaxed dynamic programming inequality** (with $\alpha = 1$) from the dynamic programming principle
- equilibrium constraints demand **more properties** of the system than regional constraints but **do not require a Lyapunov function terminal cost**
- in both cases, the **operating region** is restricted to the feasible set \mathbb{X}_N



(4) Inverse optimality and suboptimality

Performance of μ_N

Once stability can be guaranteed, we can investigate the **performance** of the MPC feedback law μ_N

As already mentioned, we measure the **performance** of the feedback $\mu_N : X \rightarrow U$ via the **infinite horizon functional**

$$J_\infty^{cl}(x_0, \mu_N) := \sum_{n=0}^{\infty} \ell(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n)))$$

Recall: the optimal feedback μ_∞ satisfies $J_\infty^{cl}(x_0, \mu_\infty) = V_\infty(x_0)$

In the literature, two different concepts can be found:

- **Inverse Optimality:** show that μ_N is optimal for an altered running cost $\tilde{\ell} \neq \ell$
- **Suboptimality:** derive upper bounds for $J_\infty^{cl}(x_0, \mu_N)$

Inverse optimality

Theorem: [Poubelle/Bitmead/Gevers '88, Magni/Sepulchre '97]

For both types of terminal constraints, μ_N is **optimal** for

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad \tilde{J}_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \tilde{\ell}(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

with $\tilde{\ell}(x, u) := \ell(x, u) + V_{N-1}(f(x, u)) - V_N(f(x, u))$

Note: $\tilde{\ell} \geq \ell$

Idea of proof: By the **dynamic programming principle**

$$\begin{aligned} V_N(x) &= \inf_{u \in U} \{ \ell(x, u) + V_{N-1}(f(x, u)) \} \\ &= \inf_{u \in U} \{ \tilde{\ell}(x, u) + V_N(f(x, u)) \} \end{aligned}$$

and $V_N(x) = \tilde{\ell}(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$

$\Rightarrow V_N$ and μ_N satisfy the principle for $\tilde{\ell} \Rightarrow$ **optimality**

Inverse optimality

Inverse optimality

- shows that μ_N is an **infinite horizon optimal feedback law**
- thus implies **inherent robustness** against perturbations (sector margin $(1/2, \infty)$)

But

- the running cost

$$\tilde{\ell}(x, u) := \ell(x, u) + V_{N-1}(f(x, u)) - V_N(f(x, u))$$

is **unknown and difficult to compute**

- knowing that μ_N is optimal for $\tilde{J}_\infty(x_0, \mathbf{u})$ doesn't give us a simple way to **estimate** $J_\infty^{cl}(x_0, \mu_N)$

Suboptimality

Recall: For both stabilizing terminal constraints the relaxed dynamic programming theorem yields the **estimate**

$$J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)$$

But: How **large** is V_N ?

Without terminal constraints, the inequality $V_N \leq V_\infty$ is immediate

However, the terminal constraints also **reverse this inequality**, i.e., we have $V_N \geq V_\infty$ and the gap is very difficult to estimate

Suboptimality — example

We consider two examples with $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \mathbb{R}$ for $N = 2$

Example 1: $x^+ = x + u$, $\ell(x, u) = x^2 + u^2$

Terminal constraints $x_{\mathbf{u}}(N) = x_* = 0$

$V_{\infty}(x) \approx 1.618x^2$, $J_{\infty}^{cl}(x, \mu_2) = 1.625x^2$

Example 2: as Example 1, but with $\ell(x, u) = x^2 + u^4$

$V_{\infty}(20) \leq 1726$, $J_{\infty}^{cl}(x, \mu_2) \approx 11240$

General estimates for fixed N appear difficult to obtain. But we can give an asymptotic result for $N \rightarrow \infty$

Asymptotic Suboptimality

Theorem: For both types of terminal constraints the assumptions of the stability theorems ensure

$$V_N(x) \rightarrow V_{\infty}(x)$$

and thus

$$J_{\infty}^{cl}(x, \mu_N) \rightarrow V_{\infty}(x)$$

as $N \rightarrow \infty$ uniformly on compact subsets of the feasible sets, i.e., the MPC performance converges to the optimal one

Idea of proof: uses that any approximately optimal trajectory for J_{∞} converges to x_* and can thus be modified to meet the constraints with only moderately changing its value

Summary of Section (4)

- μ_N is infinite horizon optimal for a suitably altered running cost
- the infinite horizon functional along the μ_N -controlled trajectory is bounded by V_N , i.e.,

$$J_{\infty}^{cl}(x, \mu_N) \leq V_N(x)$$

- $V_N \gg V_{\infty}$ is possible under terminal constraints
- $V_N \rightarrow V_{\infty}$ holds for $N \rightarrow \infty$

(5) Stability and suboptimality without stabilizing constraints

MPC without stabilizing terminal constraints

We return to the basic MPC formulation

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0 = x_{\mu_N}(n)$$

without any stabilizing terminal constraints and costs

In order to **motivate** why we want to avoid terminal constraints and costs, we consider an example of P **double integrators in the plane**



A motivating example for avoiding terminal constraints

Example: [Jahn '10] Consider P 4-dimensional systems

$$\dot{x}_i = f(x_i, u_i) := (x_{i2}, u_{i1}, x_{i4}, u_{i2})^T, \quad i = 1, \dots, P$$

Interpretation: $(x_{i1}, x_{i3})^T = \text{position}$, $(x_{i2}, x_{i4})^T = \text{velocity}$

Stage cost: $\ell(x, u) = \sum_{i=1}^P \|(x_{i1}, x_{i3})^T - x_d\| + \|(x_{i2}, x_{i4})^T\|/50$

with $x_d = (0, 0)^T$ until $t = 20s$ and $x_d = (3, 0)^T$ afterwards

Constraints: no collision, obstacles, limited speed and control

The simulation shows MPC for $P = 128$ (\rightsquigarrow system dimension 512) with sampling time $T = 0.02s$ and horizon $N = 6$



Stabilizing NMPC without terminal constraint

(Some) stability and performance results known in the **literature:**

[Alamir/Bornard '95]

use a **controllability condition** for all $x \in \mathbb{X}$

[Shamma/Xiong '97, Primbs/Nevistić '00]

use **knowledge of optimal value functions**

[Jadbabaie/Hauser '05]

use **controllability of linearization** in x_*

[Grimm/Messina/Tuna/Teel '05, Tuna/Messina/Teel '06,

Gr./Rantzer '08, Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

use **bounds on optimal value functions**

Here we explain the **last approach**



Bounds on the optimal value function

Recall the definition of the **optimal value function**

$$V_N(x) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k, x), \mathbf{u}(k))$$

Boundedness assumption: there exists $\gamma > 0$ with

$$V_N(x) \leq \gamma \ell^*(x) \quad \text{for all } x \in \mathbb{X}, N \in \mathbb{N}$$

where $\ell^*(x) := \min_{u \in \mathbb{U}} \ell(x, u)$

(sufficient conditions for and relaxations of this bound will be discussed later)



Stability and performance index

We choose ℓ , such that

$$\alpha_3(\|x - x_*\|) \leq \ell^*(x) \leq \alpha_4(\|x - x_*\|)$$

holds for $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ (again, $\ell(x, u) = \|x - x_*\|^2 + \lambda\|u\|^2$ works)

Then, the **only inequality left to prove** in order to apply the relaxed dynamic programming theorem is

$$V_N(f(x, \mu_N(x))) \leq V_N(x) - \alpha_N \ell(x, \mu_N(x))$$

for some $\alpha_N \in (0, 1)$ and all $x \in \mathbb{X}$

We can **compute** α_N from the bound $V_N(x) \leq \gamma \ell^*(x)$

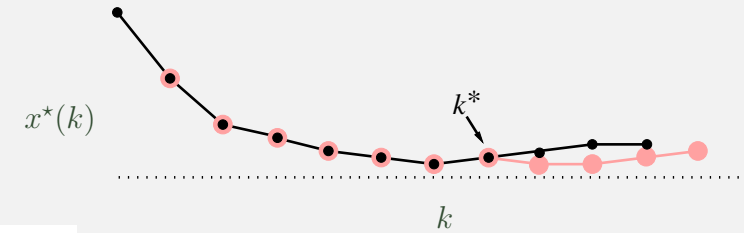
Computing α_N

We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}, N \in \mathbb{N}$ (*)

We want $V_N(x^*(1)) \leq V_N(x^*(0)) - \alpha_N \ell(x^*(0), \mathbf{u}^*(0))$

- use (*) to find $\eta_N > 0, k^* \geq 1$ with $\ell^*(x^*(k^*)) \leq \eta_N \ell^*(x^*(0))$
- concatenate $x^*(1), \dots, x^*(k^*)$ and the optimal trajectory starting in $x^*(k^*) \rightsquigarrow \tilde{x}(\cdot), \tilde{\mathbf{u}}(\cdot)$

$$\Rightarrow V_N(x^*(1)) \leq J_N(x^*(1), \tilde{\mathbf{u}}) \leq V_N(x^*(0)) - \underbrace{(1 - \gamma \eta_N)}_{=\alpha_N} \ell(x^*(0), \mathbf{u}^*(0))$$



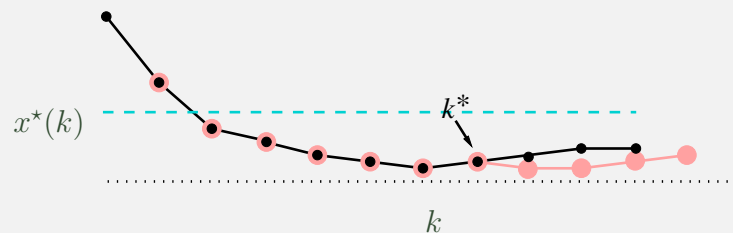
Decay of the optimal trajectory

We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}, N \in \mathbb{N}$

We want $\eta_N > 0, k^* \geq 1$ with $\ell^*(x^*(k^*)) \leq \eta_N \ell^*(x^*(0))$

Variant 1 [Grimm/Messina/Tuna/Teel '05]

$$V_N(x) \leq \gamma \ell^*(x) \Rightarrow \ell(x^*(k), u^*(k)) \leq \gamma \ell^*(x)/N \text{ for at least one } k^* \Rightarrow \alpha_N = 1 - \gamma(\gamma - 1)/N$$



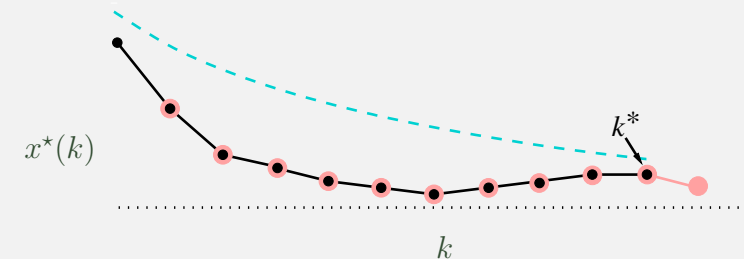
Decay of the optimal trajectory

We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}, N \in \mathbb{N}$

We want $\eta_N > 0, k^* \geq 1$ with $\ell^*(x^*(k^*)) \leq \eta_N \ell^*(x^*(0))$

Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

$$V_N(x) \leq \gamma \ell^*(x) \Rightarrow \ell(x^*(k), u^*(k)) \leq \gamma \left(\frac{\gamma-1}{\gamma}\right)^k \ell^*(x) \\ \Rightarrow k^* = N - 1 \Rightarrow \alpha_N = 1 - (\gamma - 1)^N / \gamma^{N-2}$$



Decay of the optimal trajectory

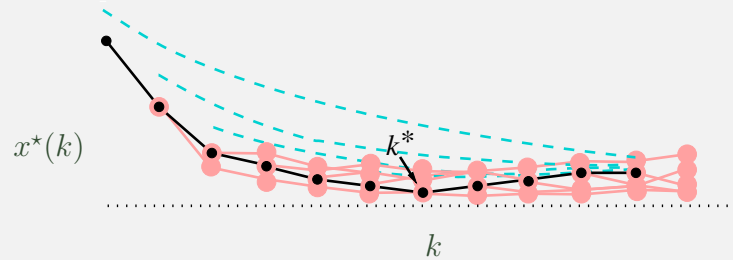
We assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^* \geq 1$ with $\ell^*(x^*(k^*)) \leq \eta_N \ell^*(x^*(0))$

Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

$V_N(x) \leq \gamma \ell^*(x) \Rightarrow$ formulate all constraints and trajectories

\Rightarrow optimize for $\alpha_N \Rightarrow \alpha_N = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-2}}$



Optimization approach to compute α_N

We explain the optimization approach (Variant 3) in **more detail**. We want α_N such that

$$V_N(x^*(1)) \leq V_N(x^*(0)) - \alpha_N \ell(x^*(0), \mathbf{u}^*(0))$$

holds for all **optimal trajectories** $x^*(n), \mathbf{u}^*(n)$ for V_N

The **bound** and the **dynamic programming principle** imply:

$$V_N(x^*(1)) \leq \gamma \ell^*(x^*(1))$$

$$V_N(x^*(1)) \leq \ell(x^*(1), \mathbf{u}^*(1)) + \gamma \ell^*(x^*(2))$$

$$V_N(x^*(1)) \leq \ell(x^*(1), \mathbf{u}^*(1)) + \ell(x^*(2), \mathbf{u}^*(2)) + \gamma \ell^*(x^*(3))$$

$\vdots \quad \vdots \quad \vdots$

Optimization approach to compute α_N

$\rightsquigarrow V_N(x^*(1))$ is **bounded** by sums over $\ell(x^*(n), \mathbf{u}^*(n))$

For sums of these values, in turn, we get bounds from the **dynamic programming principle** and the **bound**:

$$\sum_{n=0}^{N-1} \ell(x^*(n), \mathbf{u}^*(n)) = V_N(x^*(0)) \leq \gamma \ell^*(x^*(0))$$

$$\sum_{n=1}^{N-1} \ell(x^*(n), \mathbf{u}^*(n)) = V_{N-1}(x^*(1)) \leq \gamma \ell^*(x^*(1))$$

$$\sum_{n=2}^{N-1} \ell(x^*(n), \mathbf{u}^*(n)) = V_{N-2}(x^*(2)) \leq \gamma \ell^*(x^*(2))$$

$\vdots \quad \vdots$

Verifying the relaxed Lyapunov inequality

Find α_N , such that for all optimal trajectories x^*, \mathbf{u}^* :

$$V_N(x^*(1)) \leq V_N(x^*(0)) - \alpha_N \ell(x^*(0), \mathbf{u}^*(0)) \quad (*)$$

Define $\lambda_n := \ell(x^*(n), \mathbf{u}^*(n))$, $\nu := V_N(x^*(1))$

Then: $(*) \Leftrightarrow \nu \leq \sum_{n=0}^{N-1} \lambda_n - \alpha_N \lambda_0$

The **inequalities from the last slides** translate to

$$\sum_{n=k}^{N-1} \lambda_n \leq \gamma \lambda_k, \quad k = 0, \dots, N-2 \quad (1)$$

$$\nu \leq \sum_{n=1}^j \lambda_n + \gamma \lambda_{j+1}, \quad j = 0, \dots, N-2 \quad (2)$$

We call $\lambda_0, \dots, \lambda_{N-1}, \nu \geq 0$ with (1), (2) **admissible**

Optimization problem

⇒ if α_N is such that the inequality

$$\nu \leq \sum_{n=0}^{N-1} \lambda_n - \alpha_N \lambda_0 \Leftrightarrow \alpha_N \leq \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\lambda_0}$$

holds for all admissible λ_n and ν , then the desired inequality will hold for all optimal trajectories

The largest α_N satisfying this condition is

$$\alpha_N := \min_{\lambda_n, \nu \text{ admissible}} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\lambda_0}$$

This is a linear optimization problem whose solution can be computed explicitly (which is nontrivial) and reads

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}}$$

Stability and performance theorem

Theorem: [Gr./Pannek/Seehafer/Worthmann '10]: Assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$. If

$$\alpha_N > 0 \Leftrightarrow N > 2 + \frac{\ln(\gamma - 1)}{\ln \gamma - \ln(\gamma - 1)} \sim \gamma \ln \gamma$$

then the NMPC closed loop is asymptotically stable with Lyapunov function V_N and we get the performance estimate $J_\infty^{cl}(x, \mu_N) \leq V_\infty(x)/\alpha_N$ with

$$\alpha_N = 1 - \frac{(\gamma - 1)^N}{\gamma^{N-1} - (\gamma - 1)^{N-1}} \rightarrow 1 \text{ as } N \rightarrow \infty$$

Conversely, if $N < 2 + \frac{\ln(\gamma-1)}{\ln \gamma - \ln(\gamma-1)}$, then there exists a system for which $V_N(x) \leq \gamma \ell^*(x)$ holds but the NMPC closed loop is **not** asymptotically stable.

Horizon dependent γ -values

The theorem remains valid if we replace the bound condition

$$V_N(x) \leq \gamma \ell^*(x)$$

by

$$V_N(x) \leq \gamma_N \ell^*(x)$$

for horizon-dependent bounded values $\gamma_N \in \mathbb{R}$, $N \in \mathbb{N}$

$$\rightsquigarrow \alpha_N = 1 - \frac{(\gamma_N - 1) \prod_{i=2}^N (\gamma_i - 1)}{\prod_{i=2}^N \gamma_i - \prod_{i=2}^N (\gamma_i - 1)}$$

This allows for tighter bounds and a refined analysis

Controllability condition

A refined analysis can be performed if we compute γ_N from a controllability condition, e.g., exponential controllability:

Assume that for each $x_0 \in \mathbb{X}$ there exists an admissible control \mathbf{u} such that

$$\ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \leq C \sigma^k \ell^*(x_0), \quad k = 0, 1, 2, \dots$$

for given overshoot constant $C > 0$ and decay rate $\sigma \in (0, 1)$

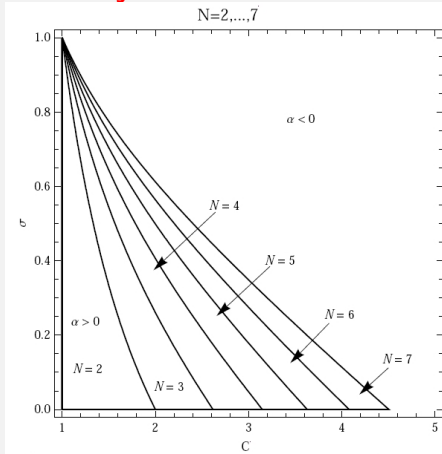
$$\rightsquigarrow V_N(x) \leq \gamma_N \ell^*(x) \text{ for } \gamma_N = \sum_{k=0}^{N-1} C \sigma^k$$

This allows to compute the minimal stabilizing horizon

$$\min\{N \in \mathbb{N} \mid \alpha_N > 0\}$$

depending on C and σ

Stability chart for C and σ



(Figure: Harald Voit)

Conclusion: for short optimization horizon N it is
more important: small C (“small overshoot”)
less important: small σ (“fast decay”)

(we will see in the next section how to use this information)

Comments and extensions

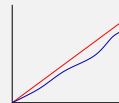
- for **unconstrained linear quadratic** problems:
existence of $\gamma \Leftrightarrow (A, B)$ stabilizable
- additional **weights on the last term** can be incorporated into the analysis [Gr./Pannek/Seehafer/Worthmann '10]
- instead of using γ , α can be **estimated numerically online** along the closed loop [Pannek et al. '10ff]
- positive definiteness of ℓ can be replaced by a **detectability condition** [Grimm/Messina/Tuna/Teel '05]
- under appropriate uniformity assumptions, the results are easily carried over to **tracking time variant references** $x_{\text{ref}}(n)$ instead of an equilibrium x_* [Gr./Pannek '11]

Comments and extensions

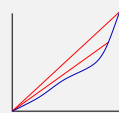
The “linear” inequality $V_N(x) \leq \gamma \ell^*(x)$ may be **too demanding** for nonlinear systems under constraints

Generalization: $V_N(x) \leq \rho(\ell^*(x))$, $\rho \in \mathcal{K}_\infty$

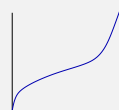
- there is $\gamma > 0$ with $\rho(r) \leq \gamma r$ for all $r \in [0, \infty)$
 \Rightarrow **global asymptotic stability**



- for each $R > 0$
there is $\gamma_R > 0$ with $\rho(r) \leq \gamma_R r$ for all $r \in [0, R]$
 \Rightarrow **semiglobal asymptotic stability**



- $\rho \in \mathcal{K}_\infty$ arbitrary
 \Rightarrow **semiglobal practical asymptotic stability**



[Grimm/Messina/Tuna/Teel '05, Gr./Pannek '11]

Summary of Section (5)

- Stability and performance of MPC without terminal constraints can be ensured by **suitable bounds** on V_N
- An **optimization approach** allows to compute the best possible α_N in the relaxed dynamic programming theorem
- The γ or γ_N can be computed from **controllability properties**, e.g., exponential controllability
- The **overshoot bound** $C > 0$ plays a crucial role or obtaining small stabilizing horizons

(6) Examples for the design of MPC schemes

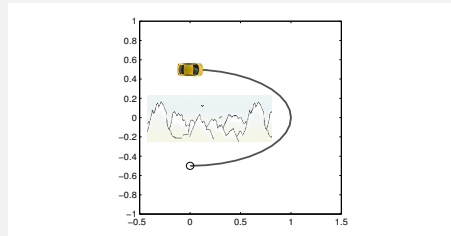
Design of “good” MPC running costs ℓ

We want **small overshoot** C in the estimate

$$\ell(x_{\mathbf{u}}(n), \mathbf{u}(n)) \leq C\sigma^n \ell^*(x_0)$$

The **trajectories** $x_{\mathbf{u}}(n)$ are given, but we can use the **running cost** ℓ as design parameter

The car-and-mountains example reloaded



MPC with $\ell(x, u) = \|x - x_*\|^2 + |u|^2$ and $u_{\max} = 0.2$

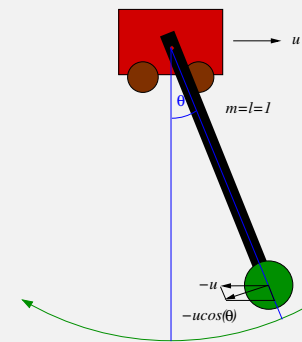
↪ asymptotic stability for $N = 11$ but not for $N \leq 10$

Reason: **detour** around mountains causes large overshoot C

Remedy: put larger weight on x_2 :

$\ell(x, u) = (x_1 - x_{*,1})^2 + 5(x_2 - x_{*,2})^2 + |u|^2$ ↪ as. stab. for $N = 2$

Example: pendulum on a cart



$x_1 = \theta =$ angle

$x_2 =$ angular velocity

$x_3 =$ cart position

$x_4 =$ cart velocity

$u =$ cart acceleration

↪ control system

$$\dot{x}_1 = x_2(t)$$

$$\dot{x}_2 = -g \sin(x_1) - kx_2 - u \cos(x_1)$$

$$\dot{x}_3 = x_4$$

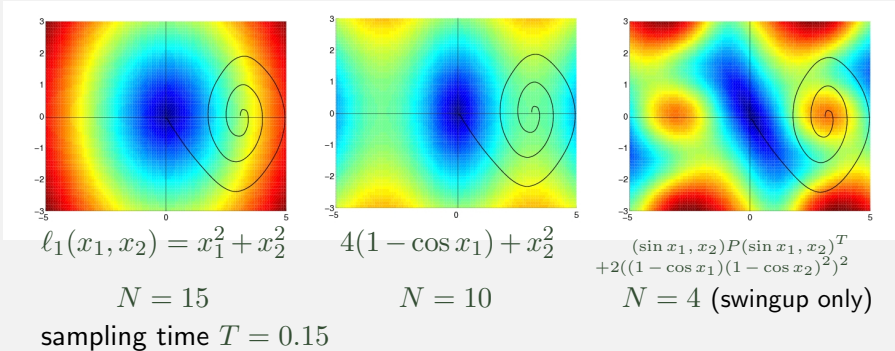
$$\dot{x}_4 = u$$

Example: Inverted Pendulum

Reducing overshoot for **swingup** of the pendulum on a cart:

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= g \sin(x_1) - kx_2 + u \cos(x_1) \\ \dot{x}_3 &= x_4, & \dot{x}_4 &= u \end{aligned}$$

Let $\ell(x) = \sqrt{\ell_1(x_1, x_2) + x_3^2 + x_4^2}$ with



A PDE example

We illustrate this with the **1d controlled PDE**

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0, 1]$

solution $y = y(t, x)$

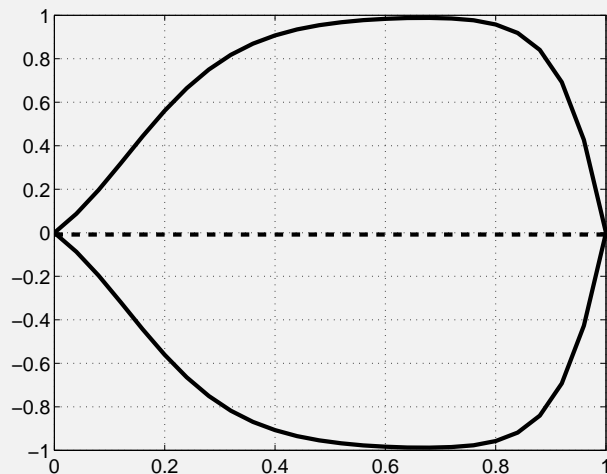
boundary conditions $y(t, 0) = y(t, 1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

and distributed control $u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$

Discrete time system: $y(n) = y(nT, \cdot)$, sampling time $T = 0.025$

The uncontrolled PDE



all equilibrium solutions

MPC for the PDE example

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

Goal: stabilize the **sampled data system** $y(n)$ at $y \equiv 0$

Usual approach: **quadratic L^2 cost**

$$\ell(y(n), u(n)) = \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2$$

For $y \approx 0$ the control u must **compensate** for $y_x \rightsquigarrow u \approx -y_x$

\rightsquigarrow controllability condition

$$\ell(y(n), u(n)) \leq C \sigma^n \ell^*(y(0))$$

$$\Leftrightarrow \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2 \leq C \sigma^n \|y(0)\|_{L^2}^2$$

$$\approx \|y(n)\|_{L^2}^2 + \lambda \|y_x(n)\|_{L^2}^2 \leq C \sigma^n \|y(0)\|_{L^2}^2$$

for $\|y_x\|_{L^2} \gg \|y\|_{L^2}$ this can only hold if $C \gg 0$

MPC for the PDE example

Conclusion: because of

$$\|y(n)\|_{L^2}^2 + \lambda \|y_x(n)\|_{L^2}^2 \leq C\sigma^n \|y(0)\|_{L^2}^2$$

the controllability condition may only hold for very large C

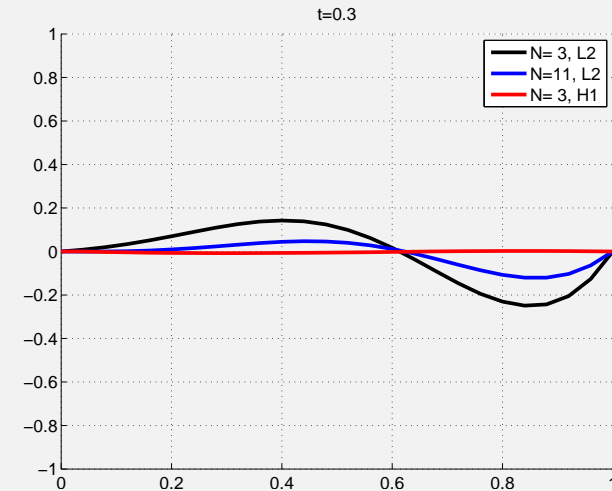
Remedy: use H^1 cost

$$\ell(y(n), u(n)) = \underbrace{\|y(n)\|_{L^2}^2 + \|y_x(n)\|_{L^2}^2}_{=\|y(n)\|_{H^1}^2} + \lambda \|u(n)\|_{L^2}^2.$$

Then an analogous computation yields

$$\|y(n)\|_{L^2}^2 + (1 + \lambda) \|y_x(n)\|_{L^2}^2 \leq C\sigma^n \left(\|y(0)\|_{L^2}^2 + \|y_x(0)\|_{L^2}^2 \right)$$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y + 1)(1 - y)$$

with

domain $\Omega = [0, 1]$

solution $y = y(t, x)$

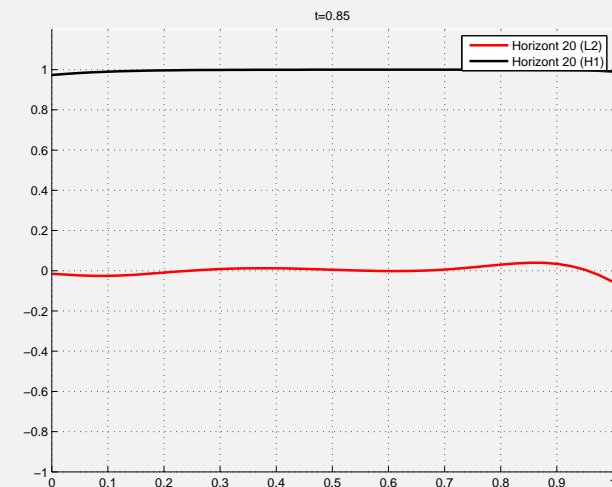
boundary conditions $y(t, 0) = u_0(t)$, $y(t, 1) = u_1(t)$

parameters $\nu = 0.1$ and $\mu = 10$

with boundary control, stability can only be achieved via large gradients in the transient phase

$\rightsquigarrow L^2$ should perform better than H^1

Boundary control, L_2 vs. H_1 , $N = 20$



Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$
Can be made rigorous for many PDEs [Altmüller et al. '10ff]

Summary of Section (6)

- Reducing the overshoot constant C by choosing ℓ appropriately can significantly reduce the horizon N needed to obtain stability
- Computing tight estimates for C is in general a difficult if not impossible task
- But structural knowledge of the system behavior can be sufficient for choosing a “good” ℓ

(7) Feasibility

Feasibility

Consider the feasible sets

$$\mathcal{F}_N := \{x \in \mathbb{X} \mid \text{there exists an admissible } \mathbf{u} \text{ of length } N\}$$

So far we have assumed

$$V_N(x) \leq \gamma \ell^*(x) \text{ for all } x \in \mathbb{X}$$

which implicitly includes the assumption

$$\mathcal{F}_N = \mathbb{X}$$

because $V_N(x) = \infty$ for $x \in \mathbb{X} \setminus \mathcal{F}_N$

What happens if $\mathcal{F}_N \neq \mathbb{X}$ for some $N \in \mathbb{N}$?

The MPC feasibility problem

Even though the open-loop optimal trajectories are forced to satisfy $x^*(k) \in \mathbb{X}$, the closed loop solutions $x_{\mu_N}(n)$ may violate the state constraints, i.e., $x_{\mu_N}(n) \notin \mathbb{X}$ for some n

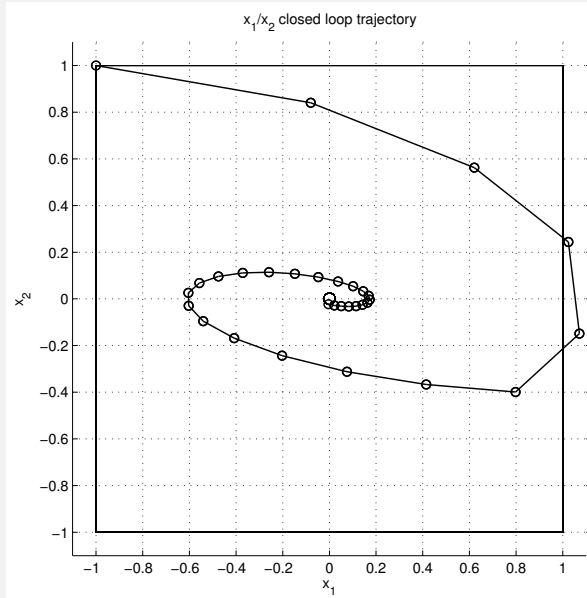
We illustrate this phenomenon by the simple example

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + u/2 \\ x_2 + u \end{pmatrix}$$

with $\mathbb{X} = [-1, 1]^2$ and $\mathbb{U} = [-1/4, 1/4]$. For initial value $x_0 = (-1, 1)^T$, the system can be controlled to 0 without leaving \mathbb{X}

We use MPC with $N = 2$ and $\ell(x, u) = \|x\|^2 + 5u^2$

The MPC feasibility problem: example



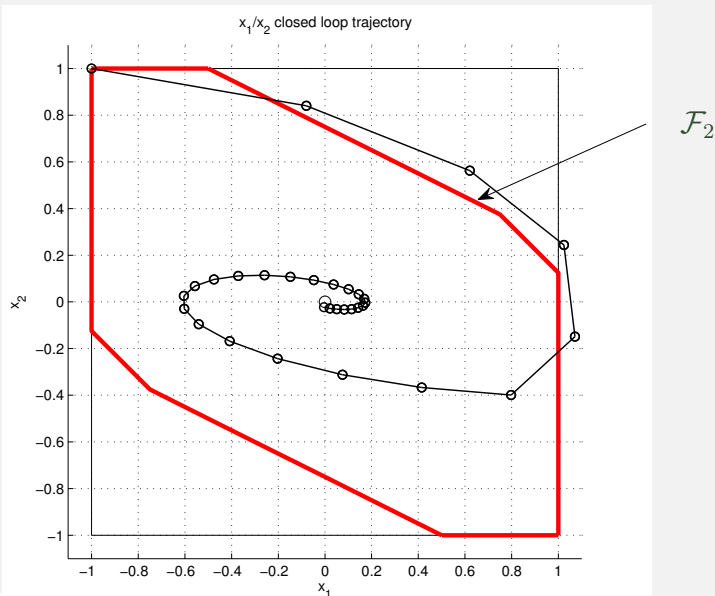
The MPC feasibility problem

How can this happen?

Explanation: In this example $\mathcal{F}_N \subsetneq \mathbb{X}$

- ↪ at time n , the finite horizon state constraints **guarantee** $x^*(1) \in \mathbb{X}$ but in general **not** $x^*(1) \in \mathcal{F}_N$
- ↪ the optimal control problem at time $n + 1$ with initial value $x_{\mu_N}(n + 1) = x^*(1)$ may be **infeasible**
- ↪ $x_{\mu_N}(n + k) \notin \mathbb{X}$ is inevitable for some $k \geq 2$

The MPC feasibility problem: example again



Recursive feasibility

The MPC scheme with horizon N is well defined on a set $A \subseteq \mathcal{F}_N$ if the following **recursive feasibility** condition holds:

$$x \in A \Rightarrow f(x, \mu_N(x)) \in A$$

In terminal constrained MPC, **forward invariance** of the terminal constraint set \mathbb{X}_0 **implies recursive feasibility** of the feasible set

$$\mathbb{X}_N := \{x \in \mathbb{X} \mid \text{there is an admissible } \mathbf{u} \text{ with } x_{\mathbf{u}}(N, x) \in \mathbb{X}_0\}$$

(this was part of the stability theorem in Section 3)

Can we find recursively feasible sets for NMPC **without terminal constraints**?

Recursive feasibility

Theorem: [Kerrigan '00, Gr./Pannek 11] Assume that

$$\mathcal{F}_{N_0} = \mathcal{F}_{N_0-1}$$

holds for some $N_0 \in \mathbb{N}$. Then the set \mathcal{F}_N is **recursively feasible** for all $N \geq N_0$.

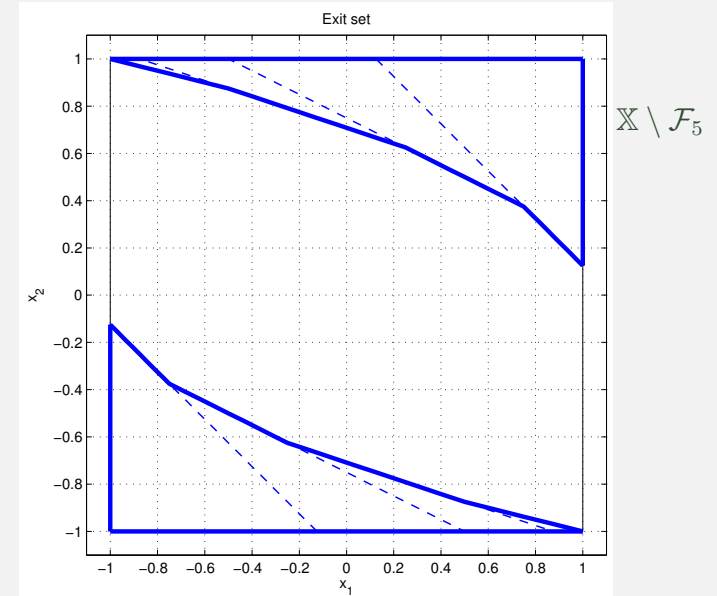
Idea of proof:

- (1) $\mathcal{F}_{N_0} = \mathcal{F}_{N_0-1}$ implies $\mathcal{F}_N = \mathcal{F}_{N_0-1}$ for all $N \geq N_0 - 1$
- (2) $x^*(0) = x \in \mathcal{F}_N$ implies

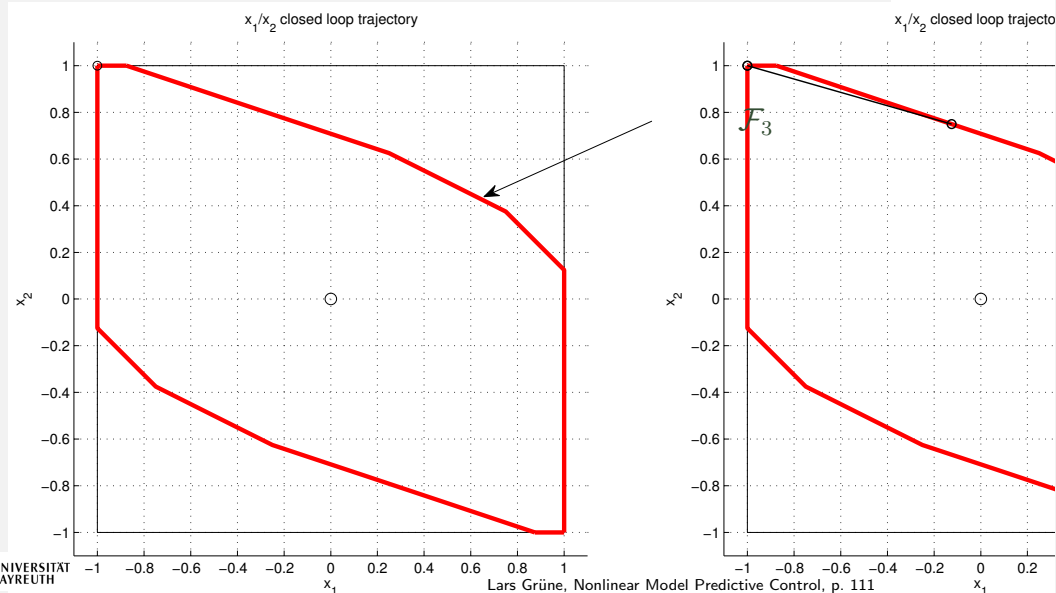
$$f(x, \mu_N(x)) = x^*(1) \in \mathcal{F}_{N-1} = \mathcal{F}_{N_0-1} = \mathcal{F}_N$$

\Rightarrow recursive feasibility of \mathcal{F}_N

Feasible sets for our example



MPC Closed loop solution for $N = 3$



Recursive feasibility

Problem: What if this condition does not hold / cannot be checked?

Theorem: [Gr./Pannek '11, extending Primbs/Nevistić '00]

Assume $V_N(x) \leq \gamma \ell^*(x)$ for all $x \in \mathcal{F}_N$, $N \in \mathbb{N}$

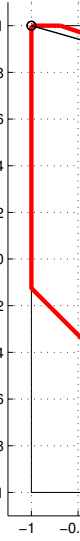
Assume there exists a **forward invariant neighborhood** \mathcal{N} of x_*

Then for each $c > 0$ there exists $N_c > 0$ such that for all $N \geq N_c$ **the level set**

$$A_c := \{x \in \mathcal{F}_N \mid V_N(x) \leq c\}$$

is **recursively feasible** and the MPC closed loop is asymptotically stable with basin of attraction containing A_c

If \mathbb{X} is compact, then $A_c = \mathcal{F}_\infty$ for all sufficiently large N



Idea of proof

$V_N(x) \leq \gamma \ell^*(x)$ implies exponential decay of $\ell^*(x^*(k))$
(as in Variant 2 of the stability proof in Section 5)

$\Rightarrow x^*(N-1) \in \mathcal{N}$ for $x \in A_c$ and $N \geq N_c$

\Rightarrow forward invariance of \mathcal{N} implies that solution can be extended

\Rightarrow recursive feasibility

Discussion

Feasibility properties of MPC without terminal constraints

- Advantage: In contrast to \mathbb{X}_0 in the terminal constrained setting, \mathcal{N} does not need to be known, mere existence is sufficient
- Drawback: In terminal constrained MPC, feasibility at time $n = 0$ implies recursive feasibility. This property is lost without terminal constraints

If this is desired, a forward invariant terminal constraint \mathbb{X}_0 can be used without terminal cost — the stability proof without terminal constraints also works for this setting

Final discussion: comparison of stabilizing MPC with and without terminal constraints

Properties of stabilizing MPC without terminal constraints compared to terminal constrained MPC

- ⊕ needs fewer a priori information to set up the scheme
- ⊖ results are typically less constructive
- ⊕ may exhibit larger operating regions
- ⊖ may need larger N for obtaining stability near x_*

Part B: Economic Model Predictive Control

(8) Economic MPC with terminal constraints

Motivation for economic MPC

Typical approach in practice (e.g., in chemical process control):

- (1) compute an **economically good equilibrium** (x_*, u_*)
("good" = high yield, small energy consumption, etc.)
- (2) design a controller **stabilizing** (x_*, u_*) , e.g., by MPC

This works fine as long as the system state is **close** to x_* **but** on the **way towards** x_* performance in the sense of the chosen criterion may be bad

Idea: Use a stage cost ℓ which does not penalize the distance to some x_* but **directly encodes** the desired economic criterion

Mathematical difference of stabilizing and economic MPC

In **stabilizing MPC**, the stage cost $\ell(x, u)$ **penalizes the distance** to some equilibrium $(x_*, u_*) \in \mathbb{X} \times \mathbb{U}$. In particular, we required

$$\ell(x, u) > \ell(x_*, u_*) \quad \text{for all } (x, u) \in \mathbb{X} \times \mathbb{U}$$

In economic MPC, we remove this requirement. We use the **same algorithm** as in stabilizing MPC, but allow for **more general** ℓ to have more freedom to model economic objectives

We still consider equilibria, but they are now **implicitly defined** via the optimization criterion. In order to distinguish them from (x_*, u_*) in stabilizing MPC, they are denoted by (x^e, u^e)

Example 1: minimum energy control

Example 1: Keep the state of the system **inside an admissible set** \mathbb{X} **minimizing the quadratic control effort**

$$\ell(x, u) = u^2$$

with dynamics

$$x(n+1) = 2x(n) + \mathbf{u}(n)$$

and constraints $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

For this example, it is optimal to **control the system to** $x^e = 0$ **and keep it there with** $u^e = 0 \rightsquigarrow \ell(x^e, u^e) = 0$

Example 2: a macroeconomic problem

Example 2: a (very simple) macroeconomic example
[Brock/Mirman '72]

Minimize the (negative) performance

$$\ell(x, u) = -\ln(Ax^\alpha - u), \quad A = 5, \alpha = 0.34$$

for dynamics $x(n+1) = \mathbf{u}(n)$

and constraints $\mathbb{X} = [0.1, 10], \mathbb{U} = [0.1, 5]$

For this example, the optimal control policy is less obvious



Questions for Economic MPC

Questions:

- In which sense can we expect performance estimates for economic MPC?
- How should terminal constraints be chosen in order to be useful?
- Can we expect asymptotic stability properties?

For answering these questions, we restrict ourselves to an equilibrium analysis (a generalization to periodic orbits is possible)

To this end, recall that $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ is an equilibrium, if

$$f(x^e, u^e) = x^e$$



Economic MPC with terminal constraints

Theorem: [Angeli/Amrit/Rawlings '09] Consider an economic MPC problem with bounded optimal value function V_N which the optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_{\mu_N}(n), \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_{\mu_N}(n)$$

with terminal constraint $x_{\mathbf{u}}(N) = x^e$ is used to generate the MPC feedback law μ_N . Then the inequality

$$\bar{J}_{\infty}^{cl}(x, \mu_N) \leq \ell(x^e, u^e)$$

holds for the averaged closed loop functional

$$\bar{J}_{\infty}^{cl}(x, \mu_N) := \limsup_{K \rightarrow \infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(x_{\mu_N}(k, x), \mu(x_{\mu_N}(k, x)))$$



Sketch of proof

Prolonging an optimal control \mathbf{u}^* with length $N-1$ at the end by the control value u^e yields a control \mathbf{u} satisfying

$$J_N(x, \mathbf{u}) - V_{N-1}(x) \leq \ell(x^e, u^e)$$

This implies

$$V_N(x) - V_{N-1}(x) \leq \ell(x^e, u^e)$$

which together with the dynamic programming principle yields

$$\ell(x, \mu_N(x)) \leq \ell(x^e, u^e) + V_N(x) - V_N(f(x, \mu_N(x)))$$

Summing and averaging then implies

$$\bar{J}_K^{cl}(x, \mu_N) \leq \ell(x^e, u^e) + \frac{1}{K} (V_N(x) - V_N(x_{\mu_N}(K)))$$

which shows the assertion for $K \rightarrow \infty$, since V_N is bounded



Optimality of this estimate

Can we ensure that this estimate is optimal?

Yes, if the system exhibits an infinite horizon averaged optimal equilibrium, i.e., if there exists an equilibrium (x^e, u^e) with

$$\bar{J}_\infty(x, \mathbf{u}) \geq \ell(x^e, u^e)$$

for all $x \in \mathbb{X}$ and all admissible \mathbf{u}

This conclusion is obvious, since

$$\bar{J}_\infty^{cl}(x, \mu_N) \geq \inf_{\mathbf{u} \text{ admissible}} \bar{J}_\infty(x, \mathbf{u})$$

Can we give an easily checkable sufficient condition for the existence of such an equilibrium?



Dissipativity

Given an equilibrium (x^e, u^e) , we use the following

Definition: [Willems '72] The optimal control problem is called strictly dissipative if there exists $\lambda : \mathbb{X} \rightarrow \mathbb{R}$ and $\alpha \in \mathcal{K}_\infty$ such that

$$(D) \quad \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \ell(x^e, u^e) \geq \alpha(\|x - x^e\|)$$

holds for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ and some $\alpha \in \mathcal{K}_\infty$

physical interpretation of (D):

$\lambda(x)$ = energy stored in the system

$\ell(x, u) - \ell(x^e, u^e)$ = energy supplied to the system

strict dissipativity: some amount of energy is dissipated (=lost)



Strict dissipativity

$$(D) \quad \ell(x, u) + \lambda(x) - \lambda(f(x, u)) - \ell(x^e, u^e) \geq \alpha(\|x - x^e\|)$$

Strict dissipativity (D) is

- satisfied for affine linear f and linear quadratic ℓ under mild regularity conditions on f, ℓ, \mathbb{X} and \mathbb{U} [Damm/Gr./Stieler/Worthmann '12]
- more restrictive for nonlinear dynamics, see, e.g., the bilinear example in [Müller/Allgöwer '12]
- sufficient and “close to necessary” for the existence of an infinite horizon averaged optimal equilibrium [Müller/Angeli/Allgöwer '13]



Example 1: minimum energy control

Example 1:

$$x(n+1) = 2x(n) + \mathbf{u}(n), \quad \ell(x, u) = u^2$$

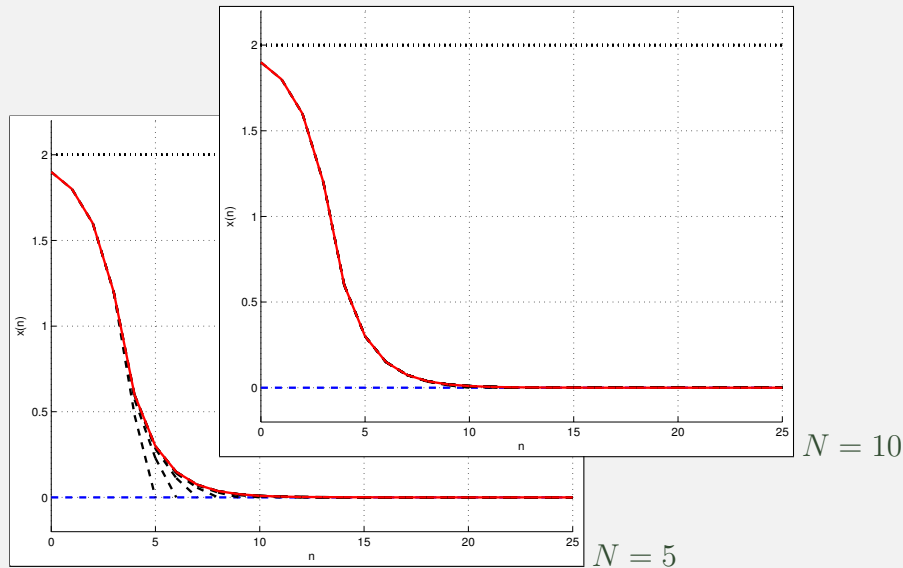
with constraints $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

The system has an optimal equilibrium at $(x^e, u^e) = (0, 0)$ and is strictly dissipative with $\lambda(x) = -x^2/2$

Using the terminal constraint $x_{\mathbf{u}}(N) = 0$, we will see that the closed loop trajectories converge to 0 (and the averaged functional equals 0)



Example 1: trajectories



Example 2: Macroeconomic model

[Brock/Mirman '72]

Minimize the average performance with

$$x(n+1) = \mathbf{u}(n), \quad \ell(x, u) = -\ln(Ax^\alpha - u)$$

with $A = 5, \alpha = 0.34$ and constraints $\mathbb{X} = [0.1, 10], \mathbb{U} = [0.1, 5]$

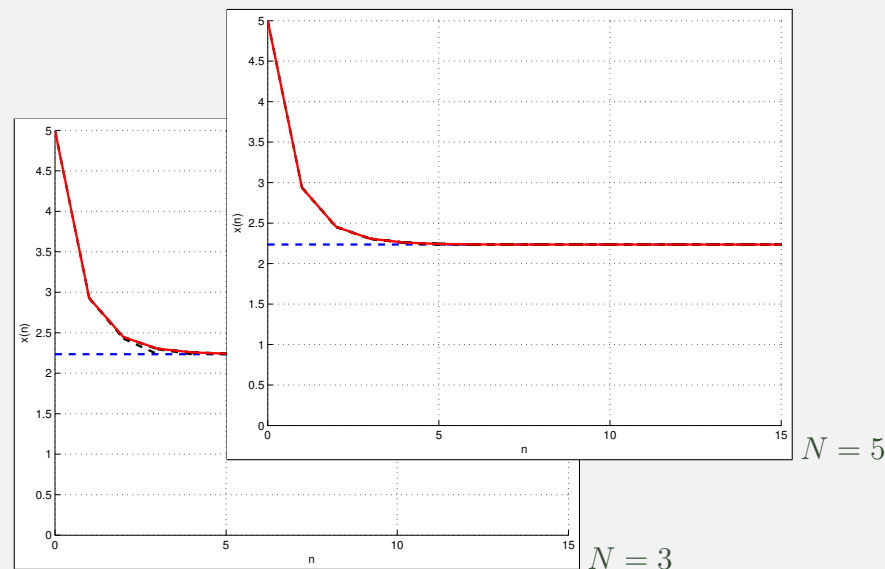
This problem exhibits the **optimal equilibrium**

$$x^e \approx 2.2344 \quad \text{with} \quad \ell(x^e, u^e) \approx 1.4673$$

and is strictly dissipative with $\lambda(x) \approx 0.2306x$

Again, with the terminal constraint $x_{\mathbf{u}}(N) = x^e$ the **closed loop trajectories** converge to x^e (and the **averaged functional** equals $\ell(x^e, u^e)$)

Example 2: trajectories



Discussion

- Averaged optimality is a rather **weak concept**: Trajectories can do **arbitrary detours** as long as in the end $\ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \rightarrow \ell(x^e, u^e)$ holds
- Estimates for their **behavior on finite time intervals** — also called “**transient behaviour**” — have been recently obtained by our group; a paper for CDC2015 is in preparation
- The concept of **good transient behaviour** will be explained in the next section

Extensions: instead of equilibria, the terminal constraints can be formulated for **periodic solutions** [Angeli/Amrit/Rawlings '09]

Regional terminal constraints and **Lyapunov-like terminal costs** are also possible, but their construction is difficult

Asymptotic stability

Assuming an optimal equilibrium exists, what about its **asymptotic stability** for the MPC closed loop? Apparently, this property **holds** for the two **numerical examples**

This is not by chance, since strict dissipativity **(D)** ensures asymptotic stability:

Theorem: [Diehl/Amrit/Rawlings '11, Angeli/Amrit/Rawlings '12] Assume that the optimal control problem is **strictly dissipative** for the equilibrium (x^e, u^e) . Then the MPC closed loop for the scheme with **terminal constraint** $x_u(N) = x^e$ is **asymptotically stable** at x^e .

Sketch of proof

$$(D) \quad \underbrace{\ell(x, u) - \ell(x^e, u^e) + \lambda(x) - \lambda(f(x, u))}_{=: \tilde{\ell}(x, u)} \geq \alpha(\|x - x^e\|)$$

Due to the terminal constraints the functionals J_N (using ℓ) and \tilde{J}_N (using $\tilde{\ell}$) differ only by a constant **independent of u**
 \rightsquigarrow optimal trajectories **coincide**

The optimal control problem with $\tilde{\ell}$ instead of ℓ satisfies all properties for stability of **stabilizing MPC** (with the corresponding optimal value function \tilde{V}_N as Lyapunov function) \rightsquigarrow asymptotic stability for the **modified problem**

Since the optimal trajectories coincide, the **MPC closed loops coincide** \rightsquigarrow asymptotic stability for the **original problem**

Summary of Section (8)

- Economic MPC means that the cost function is **not a-priori related to an equilibrium**
- However, the results become particularly nice if an **optimal equilibrium** (x^e, u^e) exist
- In contrast to stabilizing MPC, this equilibrium **need not be the (unique) minimizer** of ℓ over $\mathbb{X} \times \mathbb{U}$
- The optimal equilibrium can be used as **terminal constraint**
- Optimality can be proven in a (rather weak) **averaged sense**, though simulations suggest **better optimality properties**
- **Strict dissipativity** ensures both the existence of an optimal equilibrium and **asymptotic stability** of the closed loop

(9) Economic MPC without terminal constraints

Economic MPC without terminal constraints

What happens **without terminal constraints**? We investigate this for the examples from the last section:

Example 1: Keep the state of the system **inside an admissible set** \mathbb{X} **minimizing the quadratic control effort**

$$\ell(x, u) = u^2$$

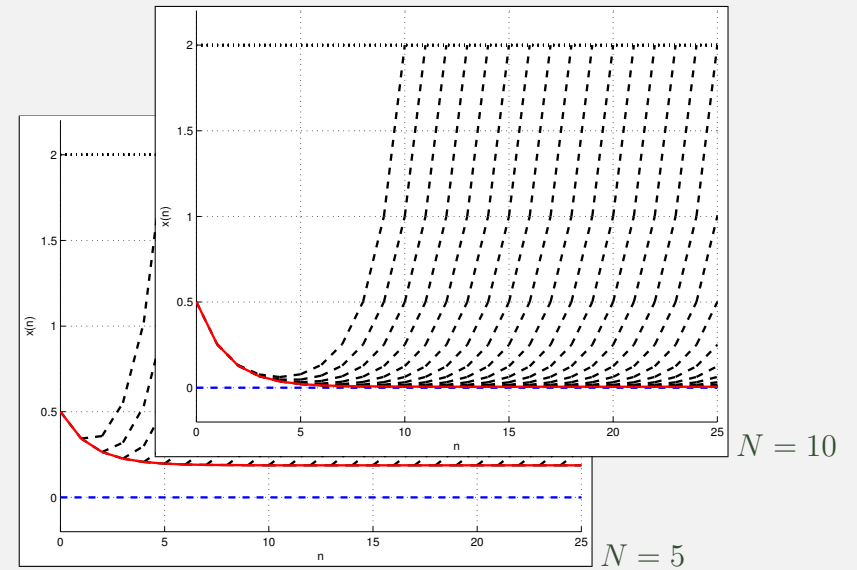
with dynamics

$$x(n+1) = 2x(n) + \mathbf{u}(n)$$

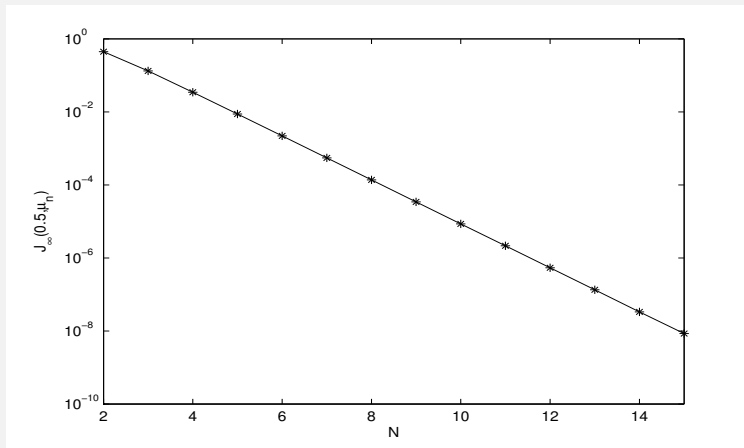
and constraints $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

For this example, it is optimal to **control the system to** $x^e = 0$ **and keep it there with** $u^e = 0 \rightsquigarrow \inf_{\mathbf{u}} J_{\infty}(x, \mathbf{u}) = 0$

Example 1: trajectories



Example: closed loop performance



$J_{\infty}(0.5, \mu_N)$ depending on N , logarithmic scale

Economic MPC without terminal constraints

Next we look once more at the macroeconomic **example** [Brock/Mirman '72]

Minimize the average performance with

$$\ell(x, u) = -\ln(Ax^{\alpha} - u), \quad A = 5, \alpha = 0.34$$

with dynamics $x(n+1) = \mathbf{u}(n)$

and constraints $\mathbb{X} = [0.1, 10]$, $\mathbb{U} = [0.1, 5]$

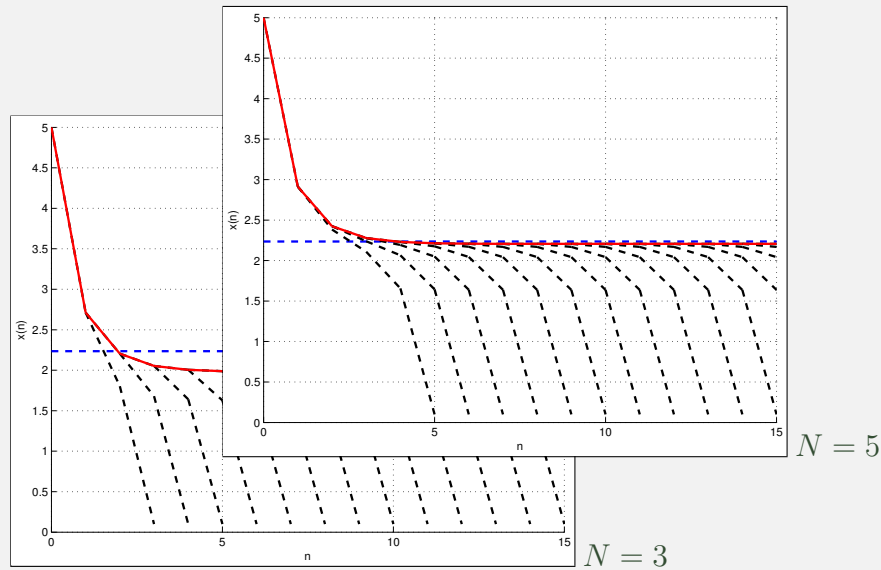
This problem exhibits the **optimal equilibrium**

$$x^e \approx 2.2344 \quad \text{with} \quad \ell(x^e, u^e) \approx 1.4673$$

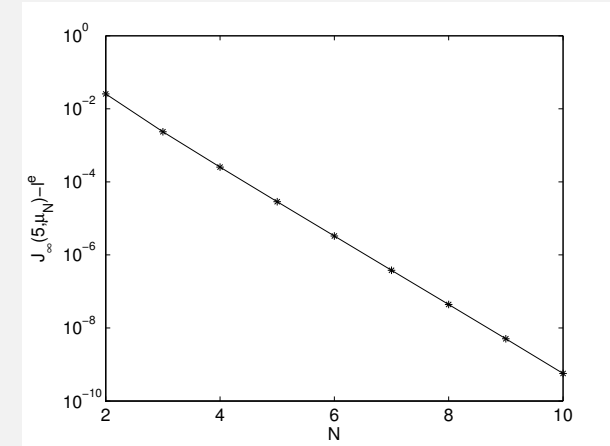
and is strictly dissipative with $\lambda(x) \approx 0.2306x$

Note: now the NMPC algorithm knows neither x^e nor λ

Example: trajectories



Example: averaged closed loop performance



$\bar{J}_\infty^{cl}(5, \mu_N) - \ell(x^e, u^e)$ depending on N , logarithmic scale

Example: a linearized tank reactor

[Diehl/Amrit/Rawlings '11]

Minimize the average performance with

$$\ell(x, u) = \|x\|^2 + 0.05u^2$$

with dynamics

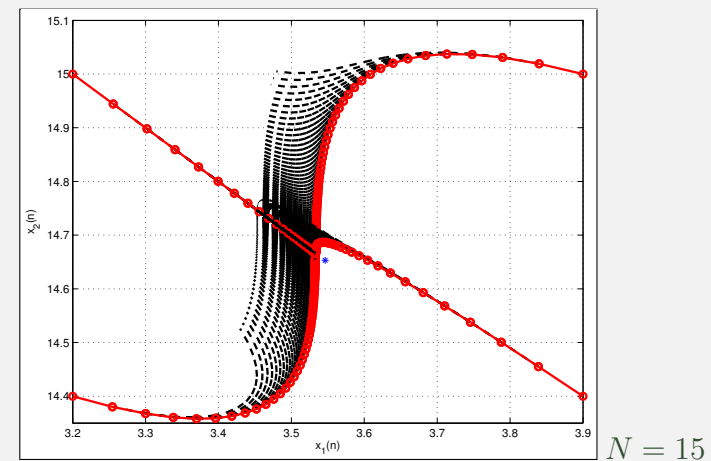
$$x(n+1) = \begin{pmatrix} 0.8353 & 0 \\ 0.1065 & 0.9418 \end{pmatrix} x(n) + \begin{pmatrix} 0.00457 \\ -0.00457 \end{pmatrix} \mathbf{u}(n) + \begin{pmatrix} 0.5559 \\ 0.5033 \end{pmatrix}$$

This problem exhibits the **optimal steady state**

$$x^e \approx \begin{pmatrix} 3.546 \\ 14.653 \end{pmatrix} \quad \text{with} \quad \ell(x^e, u^e) \approx 229.1876$$

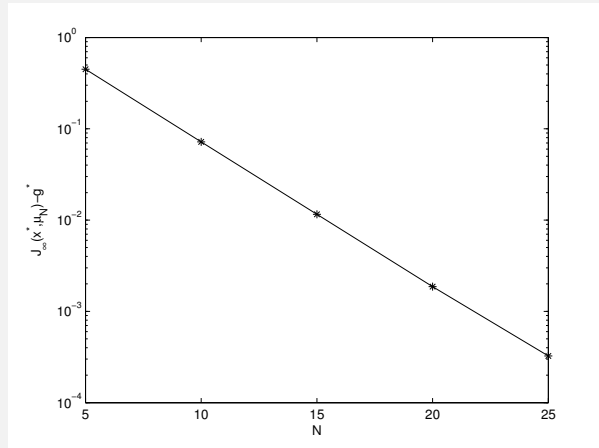
and is **dissipative** with $\lambda(x) = (-368.6684, -503.5415)^T x$

Tank reactor example: trajectories



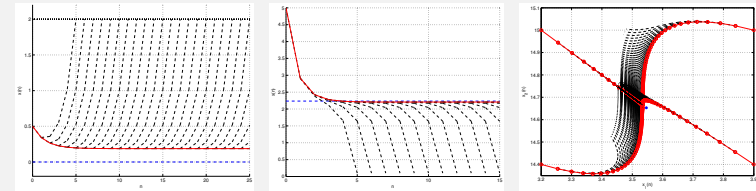
$N = 15$

Tank reactor example: averaged closed loop performance



$\bar{J}_\infty^{cl}(x^e, \mu_N) - \ell(x^e, u^e)$ depending on N , logarithmic scale

Observations



- optimal open loop trajectories first **approach the optimal equilibrium** and then turn away – “turnpike property”
- closed loop trajectories **converge to a neighborhood** of the optimal equilibrium whose size tends to 0 as $N \rightarrow \infty$
- the closed loop performance satisfies $\bar{J}_\infty^{cl}(x, \mu_N) \rightarrow \ell(x^e, u^e)$ as $N \rightarrow \infty$, exponentially fast

Can we **prove** this behavior?

Idea of proof

The following **inequality** plays the role of the “ α_N -inequality” from stabilizing NMPC:

$$V_{N+1}(x) - V_N(x) \leq \ell(x^e, u^e) + \text{“error”}$$

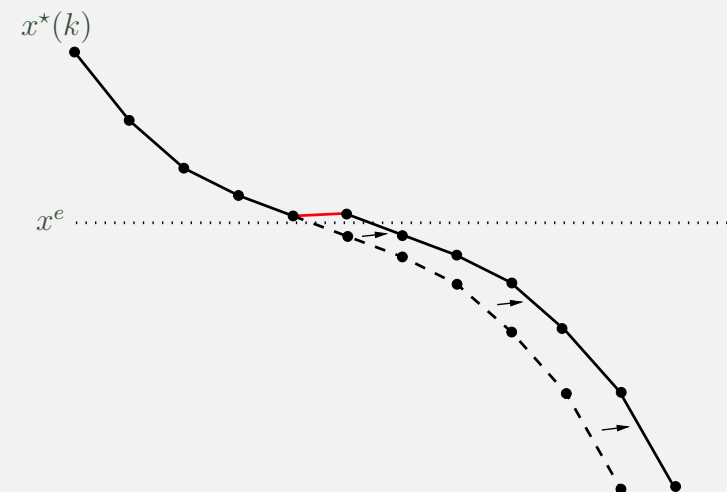
In **stabilizing MPC** or under **terminal constraints**, we have seen that this inequality can be established by “prolonging” the finite horizon optimal trajectory **at the end**

But: this method does not work here, since at the end the finite horizon optimal trajectories are **far away from** x^e

Remedy: prolong the optimal trajectory **in the middle**

Prolonging in the middle

Sketch of the idea:



Assumptions needed for this construction

What do we need to make this construction work? [Gr. '13]

- (1) **Continuity** of V_N near x^e (uniform in x and N)
 - ▶ ensures that we can prolong the trajectory in the middle without changing the value of the tail too much
- (2) **Turnpike property**
 - ▶ ensures that the finite horizon optimal trajectory satisfies

$$\min_{k \in \{0, \dots, N\}} \|x^*(k) - x^e\| \leq \sigma(N)$$

with $\sigma(N) \rightarrow 0$ as $N \rightarrow \infty$

- ▶ note: in numerical examples we often observe **exponential turnpike**, i.e., $\sigma(N) = \theta^N$

The next theorem provides **checkable sufficient conditions** for these properties



Economic MPC theorem

Theorem: [Gr./Stieler '14]

Let f and ℓ be Lipschitz, \mathbb{X} and \mathbb{U} be compact and assume

- (i) **local controllability** near x^e
 - (ii) **strict dissipativity**
 - (iii) **reachability of x^e from all $x \in \mathbb{X}$**
 - (iv) **polynomial growth conditions for $\tilde{\ell}$**
- $\left. \begin{array}{l} \text{(i)} \\ \text{(ii)} \end{array} \right\} \Rightarrow$ uniform continuity of V_N
 $\left. \begin{array}{l} \text{(ii)} \\ \text{(iii)} \end{array} \right\} \Rightarrow$ turnpike property

(i)–(iv) \Rightarrow **exponential turnpike**

[Damm/Gr./Stieler/Worthmann '14]

(for alternative conditions see also [Porretta/Zuazua '13])



Economic MPC theorem

Under assumptions (i)–(iii), there exist $\varepsilon_1(N), \varepsilon_2(K) \rightarrow 0$ as $N \rightarrow \infty$ and $K \rightarrow \infty$, exponentially fast if additionally (iv) holds, such that the following properties hold

- (1) Approximate **average optimality**:

$$\bar{J}_\infty^{cl}(x, \mu_N) \leq \ell(x^e, u^e) + \varepsilon_1(N)$$

- (2) Practical **asymptotic stability**: there is $\beta \in \mathcal{KL}$:

$$\|x_{\mu_N}(k, x) - x^e\| \leq \beta(\|x - x^e\|, k) + \varepsilon_1(N) \text{ for all } k \in \mathbb{N}$$

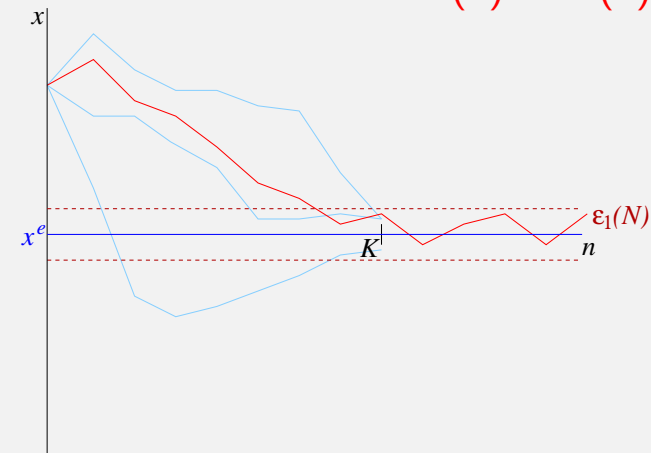
- (3) Approximate **transient optimality**: for all $K \in \mathbb{N}$:

$$J_K^{cl}(x, \mu_N(x)) \leq J_K(x, \mathbf{u}) + K\varepsilon_1(N) + \varepsilon_2(K)$$

for all admissible \mathbf{u} with $\|x_{\mathbf{u}}(K, x) - x^e\| \leq \beta(\|x - x^e\|, K) + \varepsilon_1(N)$



Illustration of (2) and (3)



- (2): $x_{\mu_N}(n)$ **converges** to the $\varepsilon_1(N)$ -ball around x^e

- (3): cost of **light blue** trajectories is **higher** than that of

- (3): $x_{\mu_N}(n)$ up to error terms $K\varepsilon_1(N) + \varepsilon_2(K)$



Linear quadratic convex problems

Theorem: [Gr./Stieler '14] For $\mathbb{X} = \mathbb{R}^n$, $U = \mathbb{R}^m$ and

$$f(x, u) = Ax + Bu + c$$

$$\ell(x, u) = x^T R x + u^T Q u + d^T x + e^T u, \quad R, Q > 0$$

the condition

(A, B) is stabilizable

is **necessary and sufficient** for practical asymptotic stability and approximate optimality of the MPC closed loop.

Moreover, all error terms converge to 0 exponentially fast

Summary of Section (9)

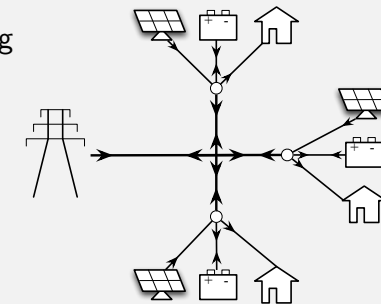
- Without terminal constraints, average performance is **only** achieved **approximately** — the larger N , the better
- Likewise, asymptotic stability is **only** achieved up to a small **neighborhood** of x^e , i.e., “**practically**”
- On the other hand, **no a priori knowledge** of the optimal equilibrium is needed
- In addition, **transient optimality** is achieved (recently also established for terminal constrained variant)
- **Exponential turnpike** plus polynomial bounds in addition ensure **exponential decay** of the error terms
- As in the case with terminal constraints **dissipativity** plus **controllability** (or **stabilizability**) are the important structural conditions

(10) Application to a smart grid control problem

with Philipp Braun (Bayreuth), Chris Kellett (Newcastle), Steve Weller (Newcastle) and Karl Worthmann (Ilmenau)

An application to a smart grid control problem

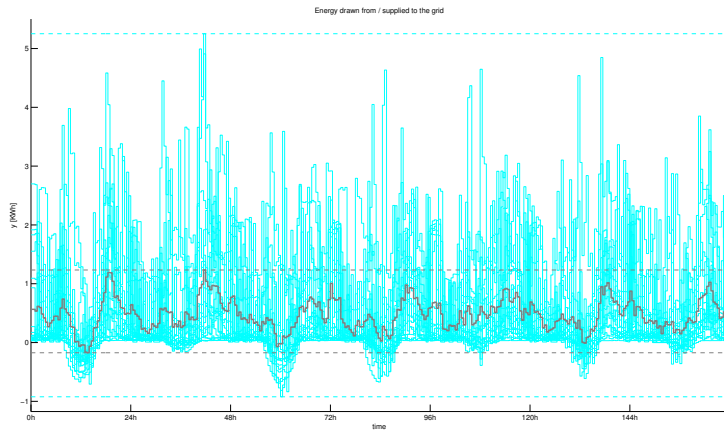
Consider the following setting in a future **smart grid**:



(batteries could be replaced by other storage devices)

Control goal: Use the batteries as buffer in order to avoid
Control goal: **large variations** in demand and supply

Data: net energy demand



Ausgrid Data: individual units, averaged

In practice, forecasted data will be used

Model

For each unit $i = 1, \dots, P$ we define

x_i = state of battery of i th unit

$$0 \leq x_i \leq C_i$$

u_i = battery charge/discharge

$$\underline{u}_i \leq u_i \leq \bar{u}_i$$

w_i = energy load minus production in i th unit

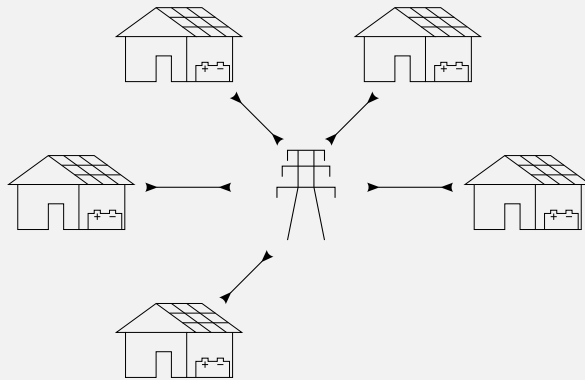
y_i = power drawn from/supplied to the outside

$$x_i(k+1) = x_i(k) + T u_i(k)$$

$$y_i(k) = w_i(k) + u_i(k)$$

sampling time $T = 30$ min

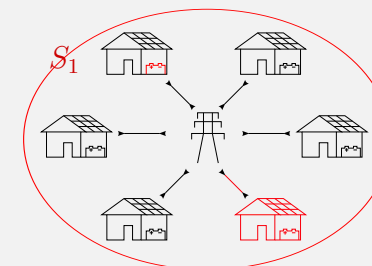
MPC approach



Objective: keep y_i close to average (in time) consumption using MPC with ℓ penalizing the deviation from the average

Control Schemes

Centralized Control



Compute at sampling instant n

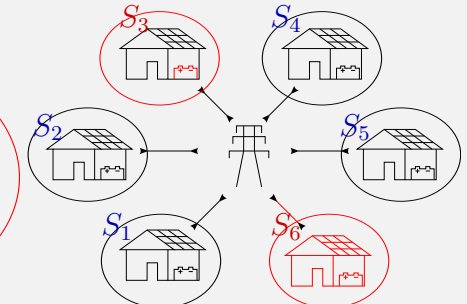
$$\bar{\zeta}(n) = \frac{1}{NP} \sum_{i=1}^P \sum_{j=0}^{N-1} w_i(n+j)$$

and minimize over (u_1, \dots, u_P)

$$\sum_{j=0}^{N-1} \left(\bar{\zeta}(n) - \frac{1}{P} \sum_{i=1}^P y_i(n+j) \right)^2$$

w.r.t. global constraints

Decentralized Control



For each unit i compute

$$\bar{\zeta}_i(n) = \frac{1}{N} \sum_{j=0}^{N-1} w_i(n+j)$$

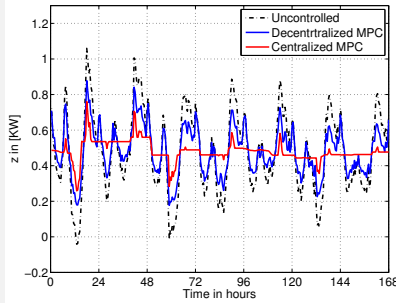
and minimize over u_i

$$\sum_{j=0}^{N-1} \left(\bar{\zeta}_i(n) - y_i(n+j) \right)^2$$

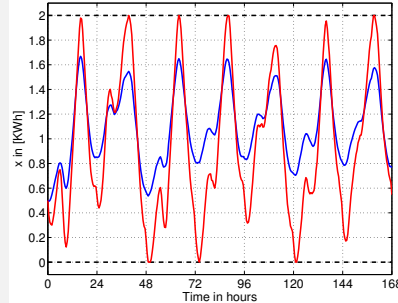
w.r.t. local constraints

Numerical Results

Performance



Average Battery Profiles

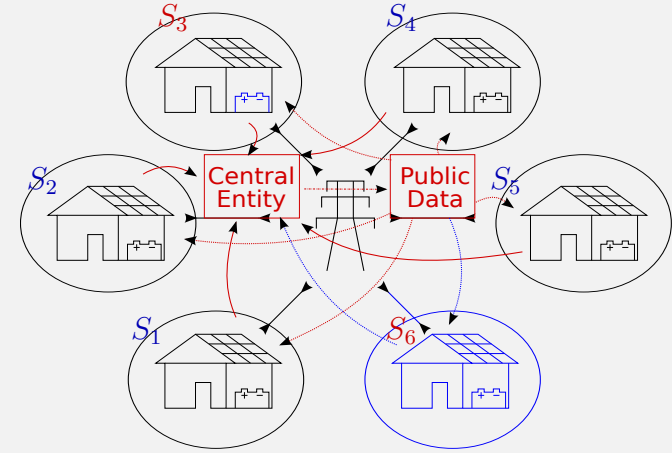


Setting:

- 100 units; 1 week simulation length
- prediction horizon 24[h]; sampling time 0.5[h]
- maximal charging/discharging rates per hour: 0.3[kWh]

Control Schemes

Alternative: Distributed Control (Optimization in units with communication via Central Entity)



The Centralized Optimization Algorithm

At each sampling instant n :

1. Set $x_0 = [x_1(n), \dots, x_P(n)]^T$
2. Compute $\bar{\zeta}(n) = \frac{1}{NP} \sum_{i=1}^P \sum_{j=0}^{N-1} w_i(n+k)$
3. Minimize $J_N(x_0, y(n+\cdot)) = \sum_{k=0}^{N-1} \left(\bar{\zeta}(n) - \frac{1}{P} \sum_{i=1}^P y_i(n+k) \right)^2$
s.t.
 - ▶ $x_i(0) = x_{\mu_N, i}(n)$ and $x_i(k+1) = x_i(k) + Tu_i(k)$
 - ▶ $y_i(n+k) = w_i(n+k) + u_i(k)$
 - ▶ $0 \leq x_i(k+1) \leq C_i$ and $\underline{u}_i \leq u_i(k) \leq \bar{u}_i$

for $k = 0, \dots, N-1$ and $i = 1, \dots, P$

$$\rightsquigarrow \begin{cases} \text{optimal control sequence} & u^*(0), \dots, u^*(N-1) \\ \text{performance output} & y^*(0), \dots, y^*(N-1) \end{cases}$$

The Distributed Optimization Algorithm

At each sampling instant n :

1. Initialize $y_i^0(j) := w_i(j)$, $j = n, \dots, n+N-1$ (i.e., $u_i \equiv 0$)
2. Perform iteratively for $\ell = 0, 1, \dots$
 - a. **Units:** send y_i^ℓ to the Central Entity
 - b. **Central Entity:** Compute and broadcast $\bar{\zeta}(n)$ and $Y^\ell(j) := \sum_{i=1}^P y_i^\ell(j)$, $j = n, 1, \dots, n+N-1$
 - c. **Units:** For each $i \in \{1, \dots, P\}$ minimize (in parallel) $J_{N,i}(x_i, y_i(\cdot)) = \sum_{j=n}^{n+N-1} (P\bar{\zeta}(n) - Y^\ell(j) + y_i^\ell(j) - y_i(j))^2$
send the (unique) minimizer $y_i^{\ell,*}(\cdot)$ to the Central Entity
 - d. **Central Entity:** Compute and broadcast $\theta = \operatorname{argmin}_{\theta \in [0,1]} \sum_{j=n}^{n+N-1} \left(\bar{\zeta}(n) - \frac{1}{P} \sum_{i=1}^P [(1-\theta)y_i^\ell(j) + \theta y_i^{\ell,*}(j)] \right)^2$
 - e. **Units:** Set $y_i^{\ell+1}(\cdot) = (1-\theta)y_i^\ell(\cdot) + \theta y_i^{\ell,*}(\cdot)$

Convergence of Distributed Optimization (1)

Lemma: If $y^{\ell,*}(\cdot) \neq y^\ell(\cdot)$, then $V^{\ell+1} < V^\ell$ holds for

$$V^\ell := \sum_{j=n}^{n+N-1} \left(\bar{\zeta}(n) - \frac{1}{P} \sum_{i=1}^P Y^\ell(j) \right)^2$$

Proof:

$$\begin{aligned} V^{\ell+1} &= \sum_{j=n}^{n+N-1} \left(\bar{\zeta}(n) - \frac{1}{P} \sum_{i=1}^P Y^{\ell+1}(j) \right)^2 \\ &= \sum_j \left(\bar{\zeta}(n) - Y^\ell(j) + \frac{1}{P} \sum_i \theta \left(y_i^{\ell,*}(j) - y_i^\ell(j) \right) \right)^2 \\ &\leq \sum_j \left(\sum_i \frac{1}{P} \left(\bar{\zeta} - Y^\ell(j) + \frac{1}{P} \left(y_i^{\ell,*}(j) - y_i^\ell(j) \right) \right) \right)^2 \quad \theta = \frac{1}{P} \\ &\leq \sum_j \sum_i \frac{1}{P} \left(\bar{\zeta} - Y^\ell(j) + \frac{1}{P} \left(y_i^{\ell,*}(j) - y_i^\ell(j) \right) \right)^2 \quad \text{Jen.'s Ineq.} \\ &< \frac{1}{P} \sum_i \sum_j \left(\bar{\zeta}(n) - Y^\ell(j) \right)^2 = V^\ell. \quad \text{Local Minimization} \end{aligned}$$

Convergence of Distributed Optimization (2)

Lemma: If $y^{\ell,*}(\cdot) \neq y^\ell(\cdot)$, then $V^{\ell+1} < V^\ell$ holds for

$$V^\ell := \sum_{j=n}^{n+N-1} \left(\bar{\zeta}(n) - \frac{1}{P} \sum_{i=1}^P Y^\ell(j) \right)^2$$

Corollary: $\lim_{\ell \rightarrow \infty} V^\ell$ exists

Proof: $V^\ell \geq 0$ is bounded from below and decreasing.

Theorem: The limit $V^* = \lim_{\ell \rightarrow \infty} V^\ell$ generated by distributed optimization coincides with the optimal value $V^\#$ of the centralized optimization

Proof (by contradiction): Assume $V^\# < V^*$

Local minimization leads to $y^{\ell,*}(\cdot) \neq y^\ell(\cdot)$ in the limit which by the lemma above implies an improvement of V^* . \downarrow

Convergence of Distributed Optimization (3)

Theorem: The limit $V^* = \lim_{\ell \rightarrow \infty} V^\ell$ generated by distributed optimization coincides with the optimal value $V^\#$ of the centralized optimization.

Question: Do the minimizers also converge?

Answer: Not necessarily, because the centralized minimizer is not unique. But we obtain something slightly weaker:

Lemma Let $(y^\ell)_{\ell \in \mathbb{N}_0}$ be the sequence of minimizers generated by distributed optimization. Then

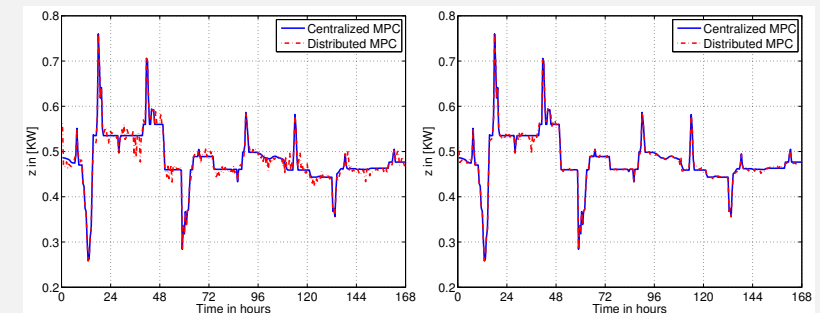
$$\|y^\ell - y^{\ell-1}\| \rightarrow 0 \quad \text{for } \ell \rightarrow \infty.$$

The proof is based on sensitivity analysis [Fiacco]

Question: When should the iterative distributed optimization be **terminated**? \rightarrow numerical simulation studies

Numerical Results

Closed loop (MPC) performance with incomplete optimization



- iteration until $\ell = 3$ (left) and $\ell = 10$ (right) at every sampling instant
- Simulation for 100 units, simulation length one week

Summary of Section (10)

For the particular networked situation, we were able to derive a **distributed optimization routine** (to be carried out in each step of the MPC scheme) providing

- **Flexibility** due to local optimization
- **Rather fast convergence to the centralized optimum**
- **Price to pay:** existence of a Central Entity and communication during the iteration

Open Questions for MPC

Independent from the optimization algorithm developed for this application, there are several **open questions for MPC**:

- Can we derive a performance bound for the **time varying situation** of this example?
- What replaces the **optimal equilibrium** for this time-varying problem? Is there a suitable **dissipativity notion**?
- What can we say about the MPC closed loop if the units cannot reach an optimum but, e.g., only a **Nash equilibrium**?

Selected literature

D.Q. Mayne, J.B. Rawlings, C.V. Rao, P.O.M. Scokaert, Constrained model predictive control: stability and optimality, *Automatica*, 36(2000), 789–814 (**“The” classical reference**)

L. Grüne and J. Pannek, *Nonlinear Model Predictive Control*, Springer, 2011 (**contains most of the material from Part A**)

L. Grüne, NMPC without terminal constraints, *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control, 2012*, 1–13 (**survey of some results from Part A and B**)

D. Angeli, R. Amrit, J.B. Rawlings, On average performance and stability of economic model predictive control, *IEEE Trans. Autom. Control*, 57 (2012), 1615–1626 (**results from Section (8)**)

L. Grüne and M. Stieler, Asymptotic stability and transient optimality of economic MPC without terminal conditions, *Journal of Process Control*, 24 (2014), 1187–1196 (**results from Section (9)**)

K. Worthmann, C.M. Kellett, P. Braun, L. Grüne, S.R. Weller, Distributed and decentralized control of residential energy systems incorporating battery storage, *IEEE Transactions on Smart Grid*, to appear (**results from Section (10)**)