Nonlinear Model Predictive Control

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Contents

(1) Introduction
What is Model Predictive Control (MPC)?

Setup

We consider nonlinear discrete time control systems

$$x_{\mathbf{u}}(n+1) = f(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x_0$$

or, briefly

$$x^+ = f(x, u)$$

with $x \in X$, $u \in U$

- we consider discrete time systems for simplicity of exposition
- continuous time systems can be treated by using the discrete time representation of the corresponding sampled data system
- X and U depend on the model. These may be Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ or more general (e.g., infinite dimensional) spaces. For simplicity of exposition we assume that we have a norm $\|\cdot\|$ on both spaces

Prototype Problem

Assume there exists an equilibrium $x_* \in X$ for $u = 0$, i.e.

$$f(x_*, 0) = x_*$$

Task: stabilize the system $x^+ = f(x, u)$
at x_* via static state feedback, i.e., find $\mu : X \to U$, such that x_* is asymptotically stable for the feedback controlled system

$$x_\mu(n+1) = f(x_\mu(n), \mu(x_\mu(n))), \;\; x_\mu(0) = x_0$$

Additionally, we impose state constraints $x_\mu(n) \in \mathbb{X}$
and control constraints $\mu(x(n)) \in \mathbb{U}$

for all $n \in \mathbb{N}$ and given sets $\mathbb{X} \subseteq X$, $\mathbb{U} \subseteq U$

Prototype Problem

Asymptotic stability means

Attraction: $x_\mu(n) \to x_*$ as $n \to \infty$

plus

Stability: Solutions starting close to x_* remain close to x_*

(we will later formalize this property using $\mathcal{KL}$ functions)

Informal interpretation: control the system to x_* and keep it there while obeying the state and control constraints

Idea of MPC: use an optimal control problem which minimizes the distance to x_* in order to synthesize a feedback law μ

The idea of MPC

For defining the MPC scheme, we choose a stage cost $\ell(x, u)$ penalizing the distance from x_* and the control effort, e.g., $\ell(x, u) = \|x - x_*\|^2 + \lambda \|u\|^2$ for $\lambda \geq 0$

The basic idea of MPC is:

- minimize the summed stage cost along trajectories generated from our model over a prediction horizon N
- use the first element of the resulting optimal control sequence as feedback value
- repeat this procedure iteratively for all sampling instants $n = 0, 1, 2, \ldots$

Notation in what follows:

- general feedback laws will be denoted by μ
- the MPC feedback law will be denoted by μ_N

The basic MPC scheme

Formal description of the basic MPC scheme:

At each time instant n solve for the current state $x_{\mu_N}(n)$

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \;\; J_N(x_{\mu_N}(n), \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \;\; x_\mathbf{u}(0) = x_{\mu_N}(n)$$

($\mathbf{u}$ admissible $\Leftrightarrow$ $\mathbf{u} \in \mathbb{U}^N$ and $x_\mathbf{u}(k) \in \mathbb{X}$)

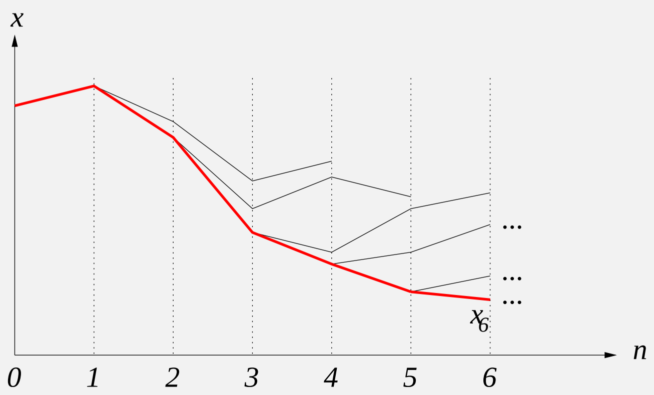
$\rightsquigarrow$ optimal trajectory $x^\star(0), \ldots, x^\star(N)$

with optimal control $\mathbf{u}^\star(0), \ldots, \mathbf{u}^\star(N-1)$

Define the MPC feedback law $\mu(x_\mu(n)) := \mathbf{u}^\star(0)$

$\rightsquigarrow x_{\mu_N}(n+1) = f(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n))) = f(x_{\mu_N}(n), \mathbf{u}^\star(0)) = x^\star(1)$

MPC from the trajectory point of view



black = predictions (open loop optimization)
red = MPC closed loop, $x_n = x_{\mu_N}(n)$

Model predictive control (aka Receding horizon control)

Idea first formulated in [A.I. Propoi, *Use of linear programming methods for synthesizing sampled-data automatic systems*, Automation and Remote Control 1963], often rediscovered

used in industrial applications since the mid 1970s, mainly for constrained linear systems [Qin & Badgwell, 1997, 2001]

more than 9000 industrial MPC applications in Germany counted in [Dittmar & Pfeifer, 2005]

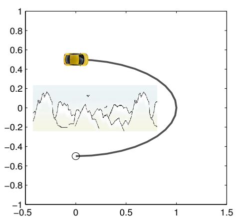
development of theory since ∼1980 (linear), ∼1990 (nonlinear)

Central questions:

- When does MPC stabilize the system?
- How good is the performance of the MPC feedback law?
- How long does the optimization horizon N need to be?

and, of course, the development of good algorithms (not topic of this course)

An example



$$x_1^+ = \sin(\varphi + u)$$
$$x_2^+ = \cos(\varphi + u)/2$$

with $\varphi = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$

$\mathbb{X} = \{x \in \mathbb{R}^2 : \|(x_1, 2x_2)^T\| = 1\}$, $\mathbb{U} = [0, u_{\max}]$

$x_* = (0, -1/2)^T$, $x_0 = (0, 1/2)^T$

MPC with $\ell(x,u) = \|x - x^*\|^2 + |u|^2$ and $u_{\max} = 0.2$ yields asymptotic stability for $N = 11$ but not for $N \leq 10$

Summary of Section (1)

- MPC is an online optimal control based method for computing stabilizing feedback laws
- MPC computes the feedback law by iteratively solving finite horizon optimal control problems using the current state $x_0 = x_{\mu_N}(n)$ as initial value
- the feedback value $\mu_N(x_0)$ is the first element of the resulting optimal control sequence
- the example shows that MPC does not always yield an asymptotically stabilizing feedback law

(2a) Background material:

Lyapunov functions

Purpose of this section

We introduce Lyapunov functions as a tool to rigorously verify asymptotic stability

In the subsequent sections, this will be used in order to establish asymptotic stability of the MPC closed loop

In this section, we consider discrete time systems without input, i.e.,

$$x^+ = g(x)$$

with $x \in X$ or, in long form

$$x(n+1) = g(x(n)), \ \ x(0) = x_0$$

(later we will apply the results to $g(x) = f(x, \mu_N(x))$)

Note: we do not require g to be continuous

Comparison functions

For $\mathbb{R}_0^+ = [0, \infty)$ we use the following classes of comparison functions

$$\mathcal{K} \ := \ \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \ \middle| \ \begin{array}{l} \alpha \text{ is continuous and strictly} \\ \text{increasing with } \alpha(0) = 0 \end{array} \right\}$$

$$\mathcal{K}_\infty \ := \ \left\{ \alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \ \middle| \ \alpha \in \mathcal{K} \text{ and } \alpha \text{ is unbounded} \right\}$$

$$\mathcal{KL} \ := \ \left\{ \beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \ \middle| \ \begin{array}{l} \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \in \mathbb{R}_0^+ \\ \text{and } \beta(r, \cdot) \text{ is strictly de-} \\ \text{creasing to } 0 \text{ for all } r \in \mathbb{R}_0^+ \end{array} \right\}$$

Asymptotic stability revisited

A point x_* is called an equilibrium of $x^+ = g(x)$ if $g(x_*) = x_*$

A set $Y \subseteq X$ is called forward invariant for $x^+ = g(x)$ if $g(x)x \in Y$ holds for each $x \in Y$

We say that x_* is asymptotically stable for $x^+ = g(x)$ on a forward invariant set Y if there exists $\beta \in \mathcal{KL}$ such that

$$\|x(n) - x_*\| \leq \beta(\|x(0) - x_*\|, n) \quad \text{for all } n \in \mathbb{N}$$

How can we check whether this property holds?

Lyapunov function

Let $Y \subseteq X$ be a forward invariant set and $x_* \in X$. A function $V : Y \to \mathbb{R}_0^+$ is called a Lyapunov function for $x^+ = g(x)$ if the following two conditions hold for all $x \in Y$:

(i) There exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$$

(ii) There exists $\alpha_V \in \mathcal{K}$ such that

$$V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$$

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Stability theorem

Theorem: If the system $x^+ = g(x)$ admits a Lyapunov function V on a forward invariant set Y, then x_* is an asymptotically stable equilibrium on Y

Idea of proof: $V(x^+) \leq V(x) - \alpha_V(\|x - x_*\|)$ implies that V is strictly decaying along solutions away from x_*

This allows to construct $\tilde{\beta} \in \mathcal{KL}$ with $V(x(n)) \leq \tilde{\beta}(V(x(0)), n)$

The bounds $\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$ imply that asymptotic stability holds with $\beta(r,t) = \alpha_1^{-1}(\tilde{\beta}(\alpha_2(r), t))$

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Lyapunov functions — discussion

While the convergence $x(n) \to x_*$ is typically non-monotone for an asymptotically stable system, the convergence $V(x(n)) \to 0$ is strictly monotone

It is hence sufficient to check the decay of V in one time step

$\rightsquigarrow$ it is typically quite easy to check whether a given function is a Lyapunov function

But it is in general difficult to find a candidate for a Lyapunov function

For MPC, we will use the optimal value functions which we introduce in the next section

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(2b) Background material: Dynamic Programming

Purpose of this section

We define the optimal value functions V_N for the optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \quad x_\mathbf{u}(0) = x_0$$

used within the MPC scheme (with $x_0 = x_{\mu_N}(n)$)

We present the dynamic programming principle, which establishes a relation for these functions and will eventually enable us to derive conditions under which V_N is a Lyapunov function

Optimal value functions

We define the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} J_N(x_0, \mathbf{u})$$

setting $V_N(x_0) := \infty$ if x_0 is not feasible, i.e., if there is no admissible $\mathbf{u}$ (recall: $\mathbf{u}$ admissible $\Leftrightarrow$ $x_\mathbf{u}(k) \in \mathbb{X}$, $\mathbf{u}(k) \in \mathbb{U}$)

An admissible control sequence $\mathbf{u}^\star$ is called optimal, if

$$J_N(x_0, \mathbf{u}^\star) = V_N(x_0)$$

Note: an optimal $\mathbf{u}^\star$ does not need to exist in general. In the sequel we assume that $\mathbf{u}^\star$ exists if x_0 is feasible

Dynamic Programming Principle

Theorem: (Dynamic Programming Principle) For any feasible $x_0 \in \mathbb{X}$ the optimal value function satisfies

$$V_N(x_0) = \inf_{u \in \mathbb{U}} \{\ell(x_0, u) + V_{N-1}(f(x_0, u))\}$$

Moreover, if $\mathbf{u}^\star$ is an optimal control, then

$$V_N(x_0) = \ell(x_0, \mathbf{u}^*(0)) + V_{N-1}(f(x_0, \mathbf{u}^\star(0)))$$

holds.

Idea of Proof: Follows by taking infima in the identity

$$\begin{aligned} J_N(x_0, \mathbf{u}) &= \ell(x_\mathbf{u}(0), \mathbf{u}(0)) + \sum_{k=1}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)) \\ &= \ell(x_0, \mathbf{u}(0)) + J_{N-1}(f(x_0, \mathbf{u}(0)), \mathbf{u}(\cdot + 1)) \end{aligned}$$

Corollaries

Corollary: Let $x^\star$ be an optimal trajectory of length N with optimal control $u^\star$ and $x^\star(0) = x$. Then

(i) The "tail"

$$\big(x^\star(k), x^\star(k+1), \ldots, x^\star(N-1)\big)$$

is an optimal trajectory of length $N - k$.

(ii) The MPC feedback μ_N satisfies

$$\mu_N(x) = \operatorname*{argmin}_{u \in \mathbb{U}} \{\ell(x, u) + V_{N-1}(f(x, u))\}$$

(i.e., $u = \mu_N(x)$ minimizes this expression),

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

and

$$u^\star(k) = \mu_{N-k}(x^\star(k)), \quad k = 0, \ldots, N-1$$

Dynamic Programming Principle — discussion

We will see later, that under suitable conditions the optimal value function will play the role of a Lyapunov function for the MPC closed loop

The dynamic programming principle and its corollaries will prove to be important tools to establish this fact

In order to see why this can work, in the next section we briefly look at infinite horizon optimal control problems

Moreover, for simple systems the principle can be used for computing V_N and μ_N — we will see an example in the excercises

(2c) Background material:

Relaxed Dynamic Programming

Infinite horizon optimal control

Just like the finite horizon problem we can define the infinite horizon optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \quad x_\mathbf{u}(0) = x_0$$

and the corresponding optimal value function

$$V_\infty(x_0) := \inf_{\mathbf{u} \text{ admissible}} J_\infty(x_0, \mathbf{u})$$

If we could compute an optimal feedback μ_∞ for this problem (which is — in contrast to computing μ_N — in general a very difficult problem), we would have solved the stabilization problem

Infinite horizon dynamic programming principle

Recall the corollary from the finite horizon dynamic programming principle

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

The corresponding result which can be proved for the infinite horizon problem reads

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x)))$$

$\leadsto$ if $\ell(x, \mu_\infty(x)) \geq \alpha_V(\|x - x_*\|)$ holds, then we get

$$V_\infty(f(x, \mu_\infty(x))) \leq V_\infty(x) - \alpha_V(\|x - x_*\|)$$

and if in addition $\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|)$ holds, then V_∞ is a Lyapunov function $\leadsto$ asymptotic stability

Relaxing dynamic programming

Unfortunately, an equation of the type

$$V_\infty(x) = \ell(x, \mu_\infty(x)) + V_\infty(f(x, \mu_\infty(x)))$$

cannot be expected if we replace "∞" by "N" everywhere (in fact, it would imply $V_N = V_\infty$)

However, we will see that we can establish relaxed versions of this inequality in which we

- relax "$=$" to "$\geq$"
- relax $\ell(x, \mu(x))$ to $\alpha\ell(x, \mu(x))$ for some $\alpha \in (0, 1]$

$$\rightsquigarrow \quad V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

"relaxed dynamic programming inequality" [Rantzer et al. '06ff]

What can we conclude from this inequality?

Relaxed dynamic programming

We define the infinite horizon performance of the MPC closed loop system $x^+ = f(x, \mu_N(x))$ as

$$J_\infty^{cl}(x_0, \mu_N) = \sum_{k=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))), \quad x_{\mu_N}(0) = x_0$$

Theorem: [Gr./Rantzer '08, Gr./Pannek '11] Let $Y \subseteq \mathbb{X}$ be a forward invariant set for the MPC closed loop and assume that

$$V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

holds for all $x \in Y$ and some $N \in \mathbb{N}$ and $\alpha \in (0, 1]$

Then for all $x \in Y$ the infinite horizon performance satisfies

$$J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$$

Relaxed dynamic programming

Theorem (continued): If, moreover, there exists $\alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that the inequalities

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

hold for all $x \in Y$, then the MPC closed loop is asymptotically stable on Y with Lyapunov function V_N.

Proof: The assumed inequalities immediately imply that $V = V_N$ is a Lyapunov function for $x^+ = g(x) = f(x, \mu_N(x))$ with

$$\alpha_1(r) = \alpha_3(r), \quad \alpha_V(r) = \alpha\,\alpha_3(r)$$

$\Rightarrow$ asymptotic stability

Relaxed dynamic programming

For proving the performance estimate $J_\infty^{cl}(x_0, \mu_N) \leq V_N(x_0)/\alpha$, the relaxed dynamic programming inequality implies

$$\begin{aligned} &\alpha \sum_{n=0}^{K-1} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \\ \leq &\sum_{n=0}^{K-1} \Big(V_N(x_{\mu_N}(n)) - V_N(x_{\mu_N}(n+1)) \Big) \\ = &\; V_N(x_{\mu_N}(0)) - V_N(x_{\mu_N}(K)) \;\leq\; V_N(x_{\mu_N}(0)) \end{aligned}$$

Since all summands are ≥ 0, this implies that the limit for $K \to \infty$ exists and we get

$$\alpha J_\infty^{cl}(x_0, \mu_N) = \alpha \sum_{n=0}^{\infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) \leq V_N(x_{\mu_N}(0))$$

$\Rightarrow$ assertion

Summary of Section (2)

- Lyapunov functions are our central tool for verifying asymptotic stability
- Dynamic programming provides us with equations which will be heavily used in the subsequent analysis
- Infinite horizon optimal control would solve the stabilization problem — if we could compute the feedback law μ_∞
- The performance of the MPC controller can be measured by looking at the infinite horizon value along the MPC closed loop trajectories
- Relaxed dynamic programming gives us conditions under which both asymptotic stability and performance results can be derived

Application of background results

The main task will be to verify the assumptions of the relaxed dynamic programming theorem, i.e.,

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

for all x in a forward invariant set Y for $x^+ = f(x, \mu_N(x))$

To this end, we present two different approaches:

- modify the optimal control problem in the MPC loop by adding terminal constraints and costs
- derive assumptions on f and ℓ under which MPC works without terminal constraints and costs

(3) Stability with stabilizing constraints

V_N as a Lyapunov Function

Problem: Prove that the MPC feedback law μ_N is stabilizing

Approach: Verify the assumptions

$$V_N(x) \geq \alpha \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

for some $\alpha \in (0, 1]$, and

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

of the relaxed dynamic programming theorem for the optimal value function

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0$$

Why is this difficult?

Let us first consider the inequality

$$V_N(x) \geq \alpha\ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

The dynamic programming principle for V_N yields

$$V_N(x) \geq \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

$\rightsquigarrow$ we have V_{N-1} where we would like to have V_N

$\rightsquigarrow$ we would get the desired inequality if we could ensure

$$V_{N-1}(f(x, \mu_N(x))) \geq V_N(f(x, \mu_N(x))) + \text{"small error"}$$

(where "small" means that the error can be compensated replacing $\ell(x, \mu_N(x))$ by $\alpha\ell(x, \mu_N(x))$ with $\alpha \in (0, 1)$)

Why is this difficult?

Task: Find conditions under which

$$V_{N-1}(f(x, \mu_N(x))) \geq V_N(f(x, \mu_N(x))) + \text{"small error"}$$

holds

For

$$V_N(x_0) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \quad x_\mathbf{u}(0) = x_0$$

this appeared to be out of reach until the mid 1990s

Note: $V_{N-1} \leq V_N$ by definition; typically with strict "$<$"

$\rightsquigarrow$ additional stabilizing constraints were proposed

(3a) Equilibrium terminal constraint (3a)

Equilibrium terminal constraint

Optimal control problem

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \quad x_\mathbf{u}(0) = x_0$$

Assumption: $f(x_*, 0) = x_*$ and $\ell(x_*, 0) = 0$

Idea: add equilibrium terminal constraint

$$x_\mathbf{u}(N) = x_*$$

[Keerthi/Gilbert '88, ...]

$\rightsquigarrow$ we now solve

$$\underset{\mathbf{u} \in \mathbb{U}^N_{x_*}(x_0)}{\text{minimize}} \quad J_N(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k), \mathbf{u}(k)), \quad x_\mathbf{u}(0) = x_0$$

with $\mathbb{U}^N_{x_*}(x_0) := \{\mathbf{u} \in \mathbb{U}^N \text{ admissible and } x_\mathbf{u}(N) = x_*\}$

Prolongation of control sequences

Let $\tilde{\mathbf{u}} \in \mathbb{U}^{N-1}_{x_*}(x_0) \quad \Rightarrow \quad x_{\tilde{\mathbf{u}}}(N-1) = x_*$

Define $\mathbf{u} \in \mathbb{U}^N$ as $\mathbf{u}(k) := \begin{cases} \tilde{\mathbf{u}}(k), & k = 0, \dots, N-2 \\ 0, & k = N-1 \end{cases}$

$$\Rightarrow \quad x_{\mathbf{u}}(N) = f(x_{\tilde{\mathbf{u}}}(N-1), \mathbf{u}(N-1)) = f(x_*, 0) = x_*$$

$$\Rightarrow \quad \mathbf{u}_N \in \mathbb{U}^N_{x_*}(x_0)$$

$\rightsquigarrow$ every $\tilde{\mathbf{u}} \in \mathbb{U}^{N-1}_{x_*}(x_0)$ can be prolonged to an $\mathbf{u}_N \in \mathbb{U}^N_{x_*}(x_0)$

Moreover, since

$$\ell(x_{\mathbf{u}_N}(N-1), \mathbf{u}_N(N-1)) = \ell(x_*, 0) = 0,$$

the prolongation has zero stage cost

Reversal of $V_{N-1} \le V_N$

Now, let $\tilde{\mathbf{u}}^\star \in \mathbb{U}^{N-1}_{x_*}(x_0)$ be the optimal control for J_{N-1}, i.e.,

$$V_{N-1}(x_0) = J_{N-1}(x_0, \tilde{\mathbf{u}}^\star)$$

Denote by $\mathbf{u} \in \mathbb{U}^N_{x_*}(x_0)$ its prolongation

$$\begin{aligned} \Rightarrow \quad V_{N-1}(x_0) &= J_{N-1}(x_0, \tilde{\mathbf{u}}^\star) = \sum_{k=0}^{N-2} \ell(x_{\tilde{\mathbf{u}}^\star}(k), \tilde{\mathbf{u}}^\star(k)) \\ &= \sum_{n=0}^{N-2} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + \underbrace{\ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1))}_{=0} \\ &= \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) = J_N(x_0, \mathbf{u}) \ge V_N(x_0) \end{aligned}$$

$\rightsquigarrow$ The inequality $V_{N-1} \le V_N$ is reversed to $V_{N-1} \ge V_N$

Note: $V_{N-1} \le V_N$ does no longer hold now

But: the dynamic programming principle remains valid

Relaxed dynamic programming inequality

From the reversed inequality

$$V_{N-1}(x) \ge V_N(x)$$

and the dynamic programming principle

$$V_N(x) \ge \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x)))$$

we immediately get

$$V_N(x) \ge \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x)))$$

This is exactly the desired relaxed dynamic programming inequality, even with $\alpha = 1$, since no "small error" occurs

$\rightsquigarrow$ stability follows if we can ensure the additional inequalities

$$V_N(x) \le \alpha_2(\|x - x_*\|), \qquad \inf_{u \in \mathbb{U}} \ell(x, u) \ge \alpha_3(\|x - x_*\|)$$

Feasible sets

The inequality $\inf_{u \in \mathbb{U}} \ell(x, u) \ge \alpha_3(\|x - x_*\|)$ is easy to satisfy, e.g., $\ell(x, u) = \|x - x_*\|^2 + \lambda\|u\|^2$ will work (with $\alpha_3(r) = r^2$)

What about $V_N(x) \le \alpha_2(\|x - x_*\|)$?

Recall: by definition $V_N(x) = \infty$ if x is not feasible, i.e., if there is no $\mathbf{u} \in \mathbb{U}^N_{x_*}(x)$

$\rightsquigarrow$ define the feasible set $\quad \mathbb{X}_N := \{x \in \mathbb{X} \,|\, \mathbb{U}^N_{x_*}(x) \ne \emptyset\}$

For $x \notin \mathbb{X}_N$ the inequality $V_N(x) \le \alpha_2(\|x - x_*\|)$ cannot hold

But: for all $x \in \mathbb{X}_N$ we can ensure this inequality under rather mild conditions (details can be given if desired)

$\rightsquigarrow$ the feasible set $\mathbb{X}_N$ is the "natural" operating region of MPC with equlibrium terminal constraints

Stability theorem

Theorem: Consider the MPC scheme with equilibrium terminal constraint $x_{\mathbf{u}}(N) = x_*$ where x_* satisfies $f(x_*, u) = x_*$ and $\ell(x_*, 0) = 0$. Assume that

$$V_N(x) \leq \alpha_2(\|x - x_*\|), \quad \inf_{u \in \mathbb{U}} \ell(x, u) \geq \alpha_3(\|x - x_*\|)$$

holds for all $x \in \mathbb{X}_N$.

Then $\mathbb{X}_N$ is forward invariant, the MPC closed loop is asymptotically stable on $\mathbb{X}_N$ and the performance estimate

$$J_\infty^{cl}(x, \mu_N) \leq V_N(x)$$

holds.

Note: The constraint $x_{\mathbf{u}}(N) = x_*$ does not imply $x_{\mu_N}(N) = x_*$

Stability theorem — sketch of proof

Sketch of proof: All assertions follow from the relaxed dynamic programming theorem if we prove forward invariance of $\mathbb{X}_N$ for the MPC closed loop system $x^+ = f(x, \mu_N(x))$

$\leadsto$ we need to prove $x \in \mathbb{X}_N \Rightarrow x^+ \in \mathbb{X}_N$

(1) The prolongation property implies $\mathbb{X}_{N-1} \subseteq \mathbb{X}_N$

(2) For $x \in \mathbb{X}_N$, the definition $\mu_N(x) := \mathbf{u}^\star(0)$ implies

$$x^+ = f(x, \mu_N(x)) = f(x, u^\star(0)) = x^\star(1)$$

and since $x^\star(N) = x_*$, the sequence $(x^\star(1), \ldots, x^\star(N-1))$ is an admissible trajectory of length $N-1$ from $x^\star(1) = x^+$ to $x^\star(N-1) = x_*$

(3) This implies $x^+ \in \mathbb{X}_{N-1} \subseteq \mathbb{X}_N$

Equilibrium terminal constraint — Discussion

The additional condition

$$x(N) = x^*$$

ensures asymptotic stability in a rigorously provable way, but

- online optimization may become harder
- if we want a large feasible set $\mathbb{X}_N$ we typically need a large optimization horizon N (see the car-and-mountains example)
- system needs to be controllable to x^* in finite time
- not very often used in industrial practice

(3b) Regional terminal constraint and terminal cost

Regional constraint and terminal cost

Optimal control problem

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \quad J_N(x_0,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k),\mathbf{u}(k)), \quad x_\mathbf{u}(0)=x_0$$

We want V_N to become a Lyapunov function

Idea: add local Lyapunov function $F:\mathbb{X}_0 \to \mathbb{R}_0^+$ as terminal cost

$$J_N(x_0,u) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k),\mathbf{u}(k)) + F(x_\mathbf{u}(N))$$

F is defined on a region $\mathbb{X}_0$ around x_* which is imposed as terminal constraint $\quad x(N) \in \mathbb{X}_N$

[Chen & Allgöwer '98, Jadbabaie et al. '98 ...]

Regional constraint and terminal cost

We thus change the optimal control problem to

$$\underset{\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)}{\text{minimize}} \quad J_N(x_0,\mathbf{u}) = \sum_{k=0}^{N-1} \ell(x_\mathbf{u}(k),\mathbf{u}(k)) \; + \; F(x_\mathbf{u}(N))$$

with

$$\mathbb{U}^N_{\mathbb{X}_0}(x_0) := \{\mathbf{u}\in\mathbb{U}^N \text{ admissible and } x_\mathbf{u}(N)\in\mathbb{X}_0\}$$

Which properties do we need for F and $\mathbb{X}_0$ in order to make this work?

Regional constraint and terminal cost

Assumptions on $F:\mathbb{X}_0\to\mathbb{R}_0^+$ and $\mathbb{X}_0$

There exists a controller $\kappa:\mathbb{X}_0\to\mathbb{U}$ with the following properties:

(i) $\mathbb{X}_0$ is forward invariant for $x^+ = f(x,\kappa(x))$:
for each $x\in\mathbb{X}_0$ we have $f(x,\kappa(x))\in\mathbb{X}_0$

(ii) F is a Lyapunov function for $x^+ = f(x,\kappa(x))$ on $\mathbb{X}_0$ which is compatible with the stage cost ℓ in the following sense:
for each $x\in\mathbb{X}_0$ the inequality

$$F(f(x,\kappa(x))) \le F(x) - \ell(x,\kappa(x))$$

holds

Prolongation of control sequences

Let $\tilde{\mathbf{u}}\in\mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0) \quad \Rightarrow \quad \tilde{x} := x_{\tilde{\mathbf{u}}}(N-1)\in\mathbb{X}_0$

Define $\mathbf{u}\in\mathbb{U}^N$ as $\quad \mathbf{u}(k) := \begin{cases} \tilde{\mathbf{u}}(k), & k=0,\dots,N-2 \\ \kappa(\tilde{x}), & k=N-1 \end{cases}$

with κ from (i)

$$\Rightarrow \quad x_\mathbf{u}(N) = f(x_{\tilde{\mathbf{u}}}(N-1),\mathbf{u}(N-1)) = f(\tilde{x},\kappa(\tilde{x}))\in\mathbb{X}_0$$

$$\Rightarrow \quad \mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)$$

$\rightsquigarrow$ every $\tilde{\mathbf{u}}\in\mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ can be prolonged to an $\mathbf{u}\in\mathbb{U}^N_{\mathbb{X}_0}(x_0)$

By (ii) the stage cost of the prolongation is bounded by

$$\ell(x_\mathbf{u}(N-1),\mathbf{u}(N-1)) \le F(x_\mathbf{u}(N-1)) - F(x_\mathbf{u}(N))$$

Reversal of $V_{N-1} \le V_N$

Let $\tilde{\mathbf{u}}^\star \in \mathbb{U}^{N-1}_{\mathbb{X}_0}(x_0)$ be the optimal control for J_{N-1}, i.e.,

$$V_{N-1}(x_0) = J_{N-1}(x_0, \tilde{\mathbf{u}}^\star)$$

Denote by $\mathbf{u} \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$ its prolongation

$$\begin{aligned} \Rightarrow \quad V_{N-1}(x_0) &= J_{N-1}(x_0, \tilde{\mathbf{u}}^\star) \\ &= \sum_{k=0}^{N-2} \ell(x_{\tilde{\mathbf{u}}^\star}(k), \tilde{\mathbf{u}}^\star(k)) + \underbrace{F(x_{\tilde{\mathbf{u}}^\star}(N-1))}_{\ge \ell(x_{\mathbf{u}}(N-1), \mathbf{u}(N-1)) + F(x_{\mathbf{u}}(N))} \\ &\ge \sum_{n=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) + F(x_{\mathbf{u}}(N)) \\ &= J_N(x_0, \mathbf{u}) \ \ge \ V_N(x_0) \end{aligned}$$

$\rightsquigarrow$ again we get $V_{N-1} \ge V_N$

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Feasible sets

Define the feasible set

$$\mathbb{X}_N := \{x \in \mathbb{X} \,|\, \mathbb{U}^N_{\mathbb{X}_0}(x) \ne \emptyset\}$$

Like in the equilibrium constrained case, on $\mathbb{X}_N$ one can ensure the inequality

$$V_N(x) \le \alpha_2(\|x - x_*\|)$$

for some $\alpha_2 \in \mathcal{K}_\infty$ under mild conditions, while outside $\mathbb{X}_N$ we get $V_N(x) = \infty$

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Stability theorem

Theorem: Consider the MPC scheme with regional terminal constraint $x_{\mathbf{u}}(N) \in \mathbb{X}_0$ and Lyapunov function terminal cost F compatible with ℓ. Assume that

$$V_N(x) \le \alpha_2(\|x - x_*\|), \qquad \inf_{u \in \mathbb{U}} \ell(x,u) \ge \alpha_3(\|x - x_*\|)$$

holds for all $x \in \mathbb{X}_N$.

Then $\mathbb{X}_N$ is forward invariant, the MPC closed loop is asymptotically stable on $\mathbb{X}_N$ and the performance estimate

$$J^{cl}_\infty(x, \mu_N) \le V_N(x)$$

holds.

Proof: Almost identical to the equilibrium constrained case

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Regional constraint and terminal cost — Discussion

Compared to the equilibrium constraint, the regional constraint

- yields easier online optimization problems
- yields larger feasible sets
- does not need exact controllability to x_*

But:

- large feasible set still needs a large optimization horizon N (see again the car-and-mountains example)
- additional analytical effort for computing F
- hardly ever used in industrial practice

In Section (5) we will see how stability can be proved without stabilizing terminal constraints

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Summary of Section (3)

- terminal constraints yield that the usual inequality $V_{N-1} \le V_N$ is reversed to $V_{N-1} \ge V_N$
- this enables us to derive the relaxed dynamic programming inequality (with $\alpha = 1$) from the dynamic programming principle
- equilibrium constraints demand more properties of the system than regional constraints but do not require a Lyapunov function terminal cost
- in both cases, the operating region is restricted to the feasible set $\mathbb{X}_N$

(4) Inverse optimality and suboptimality

Performance of μ_N

Once stability can be guaranteed, we can investigate the performance of the MPC feedback law μ_N

As already mentioned, we measure the performance of the feedback $\mu_N : X \to U$ via the infinite horizon functional

$$J_\infty^{cl}(x_0, \mu_N) := \sum_{n=0}^{\infty} \ell(x_{\mu_N}(n), \mu_N(x_{\mu_N}(n)))$$

Recall: the optimal feedback μ_∞ satisfies $J_\infty^{cl}(x_0, \mu_\infty) = V_\infty(x_0)$

In the literature, two different concepts can be found:

- Inverse Optimality: show that μ_N is optimal for an altered running cost $\tilde{\ell} \neq \ell$
- Suboptimality: derive upper bounds for $J_\infty^{cl}(x_0, \mu_N)$

Inverse optimality

Theorem: [Poubelle/Bitmead/Gevers '88, Magni/Sepulchre '97]
For both types of terminal constraints, μ_N is optimal for

$$\underset{\mathbf{u}\text{ admissible}}{\text{minimize}} \ \tilde{J}_\infty(x_0, \mathbf{u}) = \sum_{k=0}^{\infty} \tilde{\ell}(x_{\mathbf{u}}(n), \mathbf{u}(n)), \quad x_{\mathbf{u}}(0) = x_0$$

with $\tilde{\ell}(x,u) := \ell(x,u) + V_{N-1}(f(x,u)) - V_N(f(x,u))$

Note: $\tilde{\ell} \ge \ell$

Idea of proof: By the dynamic programming principle

$$\begin{aligned} V_N(x) &= \inf_{u \in \mathbb{U}} \{\ell(x,u) + V_{N-1}(f(x,u))\} \\ &= \inf_{u \in \mathbb{U}} \{\tilde{\ell}(x,u) + V_N(f(x,u))\} \end{aligned}$$

and $\quad V_N(x) = \tilde{\ell}(x, \mu_N) + V_N(f(x, \mu_N))$

$\Rightarrow$ V_N and μ_N satisfy the principle for $\tilde{\ell}$ $\Rightarrow$ optimality

Inverse optimality

Inverse optimality

- shows that μ_N is an infinite horizon optimal feedback law
- thus implies inherent robustness against perturbations (sector margin $(1/2, \infty)$)

But

- the running cost

$$\tilde{\ell}(x,u) := \ell(x,u) + V_{N-1}(f(x,u)) - V_N(f(x,u))$$

  is unknown and difficult to compute
- knowing that μ_N is optimal for $\widetilde{J}_\infty(x_0,u)$ doesn't give us a simple way to estimate $J_\infty^{cl}(x_0,\mu_N)$

Suboptimality

Recall: For both stabilizing terminal constraints the relaxed dynamic programming theorem yields the estimate

$$J_\infty^{cl}(x_0,\mu_N) \le V_N(x_0)$$

But: How large is V_N ?

Without terminal constraints, the inequality $V_N \le V_\infty$ is immediate

However, the terminal constraints also reverse this inequality, i.e., we have $V_N \ge V_\infty$ and the gap is very difficult to estimate

Suboptimality — example

We consider two examples with $\mathbb{X} = \mathbb{R}$, $\mathbb{U} = \mathbb{R}$ for $N = 2$

Example 1: $x^+ = x + u, \quad \ell(x,u) = x^2 + u^2$

Terminal constraints $\quad x_{\mathbf{u}}(N) = x_* = 0$

$V_\infty(x) \approx 1.618x^2, \quad J_\infty^{cl}(x,\mu_2) = 1.625x^2$

Example 2: as Example 1, but with $\quad \ell(x,u) = x^2 + u^4$

$V_\infty(20) \le 1726, \quad J_\infty^{cl}(x,\mu_2) \approx 11240$

General estimates for fixed N appear difficult to obtain. But we can give an asymptotic result for $N \to \infty$

Asymptotic Suboptimality

Theorem: For both types of terminal constraints the assumptions of the stability theorems ensure

$$V_N(x) \to V_\infty(x)$$

and thus

$$J_\infty^{cl}(x,\mu_N) \to V_\infty(x)$$

as $N \to \infty$ uniformly on compact subsets of the feasible sets, i.e., the MPC performance converges to the optimal one

Idea of proof: uses that any approximately optimal trajectory for J_∞ converges to x_* and can thus be modified to meet the constraints with only moderately changing its value

Summary of Section (4)

- μ_N is infinite horizon optimal for a suitably altered running cost
- the infinite horizon functional along the μ_N-controlled trajectory is bounded by V_N, i.e.,

$$J_\infty^{cl}(x, \mu_N) \leq V_N(x)$$

- $V_N \gg V_\infty$ is possible under terminal constraints
- $V_N \to V_\infty$ holds for $N \to \infty$

(5) Stability and suboptimality without stabilizing constraints

MPC without stabilizing terminal constraints

We return to the basic MPC formulation

$$\underset{\mathbf{u} \text{ admissible}}{\text{minimize}} \; J_N(x_0, u) = \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k), \mathbf{u}(k)), \quad x_{\mathbf{u}}(0) = x_0 = x_{\mu_N}(n)$$

without any stabilizing terminal constraints and costs

In order to motivate why we want to avoid terminal constraints and costs, we consider an example of P double integrators in the plane

A motivating example for avoiding terminal constraints

Example: [Jahn '10] Consider P 4-dimensional systems

$$\dot{x}_i = f(x_i, u_i) := (x_{i2}, u_{i1}, x_{i4}, u_{i2})^T, \quad i = 1, \dots, P$$

Interpretation: $(x_{i1}, x_{i3})^T$ = position, $(x_{i2}, x_{i4})^T$ = velocity

Stage cost: $\ell(x, u) = \sum_{i=1}^{P} \|(x_{i1}, x_{i3})^T - x_d\| + \|(x_{i2}, x_{i4})^T\|/50$

with $x_d = (0, 0)^T$ until $t = 20s$ and $x_d = (3, 0)^T$ afterwards

Constraints: no collision, obstacles, limited speed and control

The simulation shows MPC for $P = 128$ ($\rightsquigarrow$ system dimension 512) with sampling time $T = 0.02s$ and horizon $N = 6$

Stabilizing NMPC without terminal constraint

(Some) stability and performance results known in the literature:

[Alamir/Bornard '95]
use a controllability condition for all $x \in \mathbb{X}$

[Shamma/Xiong '97, Primbs/Nevistić '00]
use knowledge of optimal value functions

[Jadbabaie/Hauser '05]
use controllability of linearization in x^e

[Grimm/Messina/Tuna/Teel '05, Tuna/Messina/Teel '06, Gr./Rantzer '08, Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]
use bounds on optimal value functions

Here we explain the last approach

Bounds on the optimal value function

Recall the definition of the optimal value function

$$V_N(x) := \inf_{\mathbf{u} \text{ admissible}} \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k,x), \mathbf{u}(k))$$

Boundedness assumption: there exists $\gamma > 0$ with

$$V_N(x) \le \gamma \ell^\star(x) \quad \text{for all } x \in \mathbb{X}, N \in \mathbb{N}$$

where $\ell^\star(x) := \min_{u \in \mathbb{U}} \ell(x,u)$

(sufficient conditions for and relaxations of this bound will be discussed later)

Stability and performance index

We choose ℓ, such that

$$\alpha_3(\|x - x_*\|) \le \ell^\star(x) \le \alpha_4(\|x - x_*\|)$$

holds for $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ (again, $\ell(x,u) = \|x - x^*\|^2 + \lambda \|u\|^2$ works)

Then, the only inequality left to prove in order to apply the relaxed dynamic programming theorem is

$$V_N(f(x, \mu_N(x))) \le V_N(x) - \alpha_N \ell(x, \mu_N(x))$$

for some $\alpha_N \in (0,1)$ and all $x \in \mathbb{X}$

We can compute α_N from the bound $V_N(x) \le \gamma \ell^\star(x)$

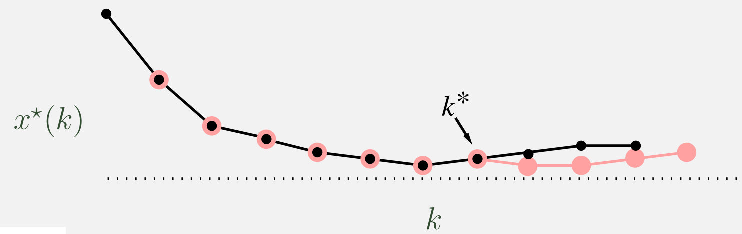
Computing α_N

We assume $\quad V_N(x) \le \gamma \ell^\star(x) \quad$ for all $\quad x \in \mathbb{X}, N \in \mathbb{N} \quad (*)$

We want $\quad V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$

- use $(*)$ to find $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$
- concatenate $x^\star(1), \ldots, x^\star(k^\star)$ and the optimal trajectory starting in $x^\star(k^\star)$ $\rightsquigarrow$ $\tilde{x}(\cdot)$, $\tilde{\mathbf{u}}(\cdot)$

$$\Rightarrow V_N(x^\star(1)) \le J_N(x^\star(1), \tilde{\mathbf{u}}) \le V_N(x^\star(0)) - \underbrace{(1 - \gamma\eta_N)}_{=\alpha_N} \ell(x^\star(0), \mathbf{u}^\star(0))$$



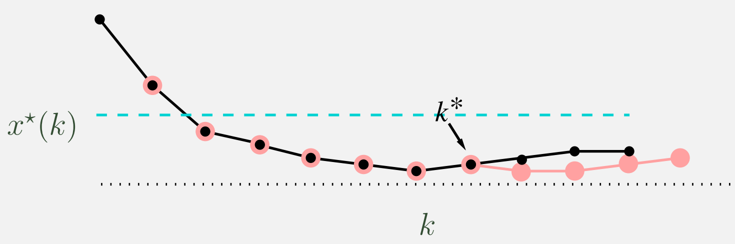
Decay of the optimal trajectory

We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 1 [Grimm/Messina/Tuna/Teel '05]

$V_N(x) \le \gamma \ell^\star(x) \;\Rightarrow\; \ell(x^\star(k), u^\star(k)) \le \gamma \ell^\star(x)/N$ for at least one $k^\star \;\Rightarrow\; \alpha_N = 1 - \gamma(\gamma-1)/N$



Decay of the optimal trajectory

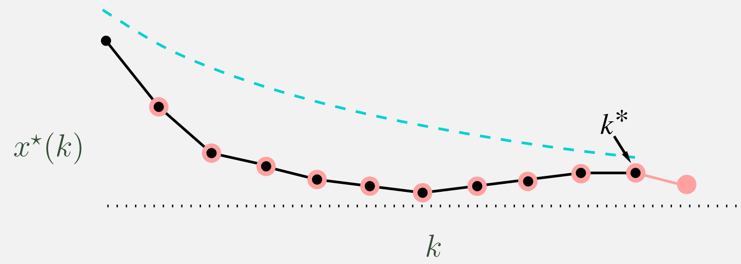
We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 2 [Tuna/Messina/Teel '06, Gr./Rantzer '08]

$V_N(x) \le \gamma \ell^\star(x) \;\Rightarrow\; \ell(x^\star(N-1), u^\star(N-1)) \le \gamma \left(\frac{\gamma-1}{\gamma}\right) \ell^\star(x)$

$\Rightarrow\; k^\star = N-1 \;\Rightarrow\; \alpha_N = 1 - (\gamma-1)^N/\gamma^{N-2}$



Decay of the optimal trajectory

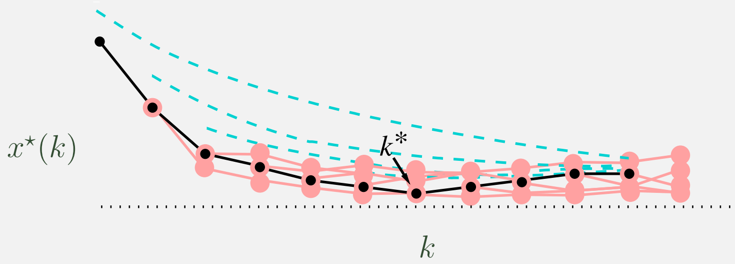
We assume $V_N(x) \le \gamma \ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$

We want $\eta_N > 0$, $k^\star \ge 1$ with $\ell^\star(x^\star(k^\star)) \le \eta_N \ell^\star(x^\star(0))$

Variant 3 [Gr. '09, Gr./Pannek/Seehafer/Worthmann '10]

$V_N(x) \le \gamma \ell^\star(x) \;\Rightarrow$ formulate all constraints and trajectories

$\Rightarrow$ optimize for $\alpha_N \;\Rightarrow\; \alpha_N = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-2}}$



Optimization approach to compute α_N

We explain the optimization approach (Variant 3) in more detail. We want α_N such that

$$V_N(x^\star(1)) \le V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0))$$

holds for all optimal trajectories $x^\star(n), \mathbf{u}^\star(n)$ for V_N

The bound and the dynamic programming principle imply:

$$\begin{aligned}
V_N(x^\star(1)) &\le \gamma \ell^\star(x^\star(1)) \\
V_N(x^\star(1)) &\le \ell(x^\star(1), \mathbf{u}^\star(1)) + \gamma \ell^\star(x^\star(2)) \\
V_N(x^\star(1)) &\le \ell(x^\star(1), \mathbf{u}^\star(1)) + \ell(x^\star(2), \mathbf{u}^\star(2)) + \gamma \ell^\star(x^\star(3)) \\
\vdots \quad & \quad \vdots \quad \quad \vdots
\end{aligned}$$

Optimization approach to compute α_N

$\rightsquigarrow V_N(x^\star(1))$ is bounded by sums over $\ell(x^\star(n), \mathbf{u}^\star(n))$

For sums of these values, in turn, we get bounds from the dynamic programming principle and the bound:

$$\sum_{n=0}^{N-1} \ell(x^\star(n), \mathbf{u}^\star(n)) = V_N(x^\star(0)) \leq \gamma\ell^\star(x^\star(0))$$

$$\sum_{n=1}^{N-1} \ell(x^\star(n), \mathbf{u}^\star(n)) = V_{N-1}(x^\star(1)) \leq \gamma\ell^\star(x^\star(1))$$

$$\sum_{n=2}^{N-1} \ell(x^\star(n), u^\star(n)) = V_{N-2}(x^\star(2)) \leq \gamma\ell^\star(x^\star(2))$$

$$\vdots \qquad\qquad \vdots$$

Verifying the relaxed Lyapunov inequality

Find α_N, such that for all optimal trajectories $x^\star$, $\mathbf{u}^\star$:

$$V_N(x^\star(1)) \leq V_N(x^\star(0)) - \alpha_N \ell(x^\star(0), \mathbf{u}^\star(0)) \qquad (*)$$

Define $\quad \lambda_n := \ell(x^\star(n), \mathbf{u}^\star(n)), \quad \nu := V_N(x^\star(1))$

Then: $\quad (*) \Leftrightarrow \nu \leq \sum_{n=0}^{N-1} \lambda_n - \alpha_N \lambda_0$

The inequalities from the last slides translate to

$$\sum_{n=k}^{N-1} \lambda_n \leq \gamma\lambda_k, \quad k = 0, \ldots, N-2 \qquad (1)$$

$$\nu \leq \sum_{n=1}^{j} \lambda_n + \gamma\lambda_{j+1}, \quad j = 0, \ldots, N-2 \qquad (2)$$

We call $\lambda_0, \ldots, \lambda_{N-1}, \nu \geq 0$ with (1), (2) admissible

Optimization problem

$\Rightarrow$ if α_N is such that the inequality

$$\nu \leq \sum_{n=0}^{N-1} \lambda_n - \alpha_N \lambda_0 \Leftrightarrow \alpha_N \leq \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\lambda_0}$$

holds for all admissible λ_n and ν, then the desired inequality will hold for all optimal trajectories

The largest α_N satisfying this condition is

$$\alpha_N := \min_{\lambda_n, \nu \text{ admissible}} \frac{\sum_{n=0}^{N-1} \lambda_n - \nu}{\lambda_0}$$

This is a linear optimization problem whose solution can be computed explicitly (which is nontrivial) and reads

$$\alpha_N = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-1}}$$

Stability and performance theorem

Theorem: [Gr./Pannek/Seehafer/Worthmann '10]: Assume $V_N(x) \leq \gamma\ell^\star(x)$ for all $x \in \mathbb{X}$, $N \in \mathbb{N}$. If

$$\alpha_N > 0 \Leftrightarrow N > 2 + \frac{\ln(\gamma-1)}{\ln\gamma - \ln(\gamma-1)} \sim \gamma\ln\gamma$$

then the NMPC closed loop is asymptotically stable with Lyapunov function V_N and we get the performance estimate $J_\infty^{cl}(x, \mu_N) \leq V_\infty(x)/\alpha_N$ with

$$\alpha_N = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-1}} \to 1 \text{ as } N \to \infty$$

Conversely, if $N < 2 + \frac{\ln(\gamma-1)}{\ln\gamma - \ln(\gamma-1)}$, then there exists a system for which $V_N(x) \leq \gamma\ell^\star(x)$ holds but the NMPC closed loop is not asymptotically stable.

Horizon dependent γ-values

The theorem remains valid if we replace the bound condition

$$V_N(x) \le \gamma \ell^\star(x)$$

by

$$V_N(x) \le \gamma_N \ell^\star(x)$$

for horizon-dependent bounded values $\gamma_N \in \mathbb{R}$, $N \in \mathbb{N}$

$$\rightsquigarrow \quad \alpha_N = 1 - \frac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)}$$

This allows for tighter bounds and a refined analysis

Controllability condition

A refined analysis can be performed if we compute γ_N from a controllability condition, e.g., exponential controllability:

Assume that for each $x_0 \in \mathbb{X}$ there exists an admissible control $\mathbf{u}$ such that

$$\ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \le C\sigma^k \ell^\star(x_0), \quad k = 0, 1, 2, \ldots$$

for given overshoot constant $C > 0$ and decay rate $\sigma \in (0, 1)$

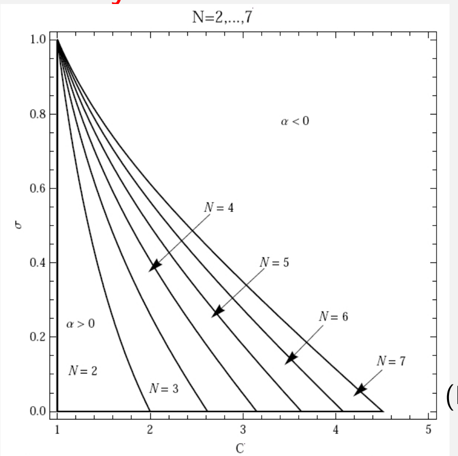
$$\rightsquigarrow \quad V_N(x) \le \gamma_N \ell^\star(x) \quad \text{for} \quad \gamma_N = \sum_{k=0}^{N-1} C\sigma^k$$

This allows to compute the minimal stabilizing horizon

$$\min\{N \in \mathbb{N} \,|\, \alpha_N > 0\}$$

depending on C and σ

Stability chart for C and σ



(Figure: Harald Voit)

Conclusion: for short optimization horizon N it is
more important: small C ("small overshoot")
less important: small σ ("fast decay")

(we will see in the next section how to use this information)

Comments and extensions

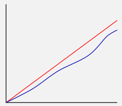
- for unconstrained linear quadratic problems:
  existence of γ $\Leftrightarrow$ (A, B) stabilizable
- additional weights on the last term can be incorporated into the analysis [Gr./Pannek/Seehafer/Worthmann '10]
- instead of using γ, α can be estimated numerically online along the closed loop [Pannek et al. '10ff]
- positive definiteness of ℓ can be replaced by a detectability condition [Grimm/Messina/Tuna/Teel '05]

Comments and extensions

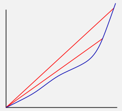
The "linear" inequality $V_N(x) \le \gamma \ell^\star(x)$ may be too demanding for nonlinear systems under constraints

Generalization: $V_N(x) \le \rho(\ell^\star(x)), \quad \rho \in \mathcal{K}_\infty$

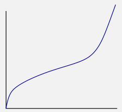
- there is $\gamma > 0$ with $\rho(r) \le \gamma r$ for all $r \in [0, \infty]$
  $\Rightarrow$ global asymptotic stability



- for each $R > 0$
  there is $\gamma_R > 0$ with $\rho(r) \le \gamma_R r$ for all $r \in [0, R]$
  $\Rightarrow$ semiglobal asymptotic stability



- $\rho \in \mathcal{K}_\infty$ arbitrary
  $\Rightarrow$ semiglobal practical asymptotic stability



[Grimm/Messina/Tuna/Teel '05, Gr./Pannek '11]

Summary of Section (5)

- Stability and performance of MPC without terminal constraints can be ensured by suitable bounds on V_N
- An optimization approach allows to compute the best possible α_N in the relaxed dynamic programming theorem
- The γ or γ_N can be computed from controllability properties, e.g., exponential controllability
- The overshoot bound $C > 0$ plays a crucial role or obtaining small stabilizing horizons

(6) Examples for the design of MPC schemes

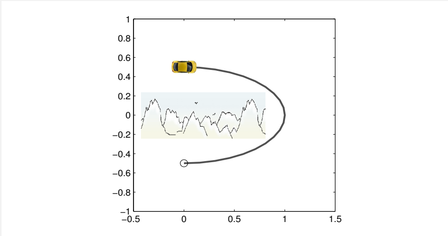
Design of "good" MPC running costs ℓ

We want small overshoot C in the estimate

$$\ell(x_{\mathbf{u}}(n), \mathbf{u}(n)) \le C\sigma^n \ell^\star(x_0)$$

The trajectories $x_{\mathbf{u}}(n)$ are given, but we can use the running cost ℓ as design parameter

The car-and-mountains example reloaded



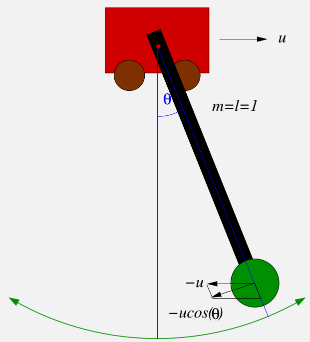
MPC with $\ell(x,u) = \|x - x^*\|^2 + |u|^2$ and $u_{\max} = 0.2$
$\rightsquigarrow$ asymptotic stability for $N = 11$ but not for $N \leq 10$

Reason: detour around mountains causes large overshoot C

Remedy: put larger weight on x_2:

$\ell(x,u) = (x_1 - x_1^*)^2 + 5(x_2 - x_2^*)^2 + |u|^2$ $\rightsquigarrow$ as. stab. for $N = 2$

Example: pendulum on a cart



$x_1 = \theta =$ angle
$x_2 =$ angular velocity
$x_3 =$ cart position
$x_4 =$ cart velocity
$u =$ cart acceleration

$\rightsquigarrow$ control system

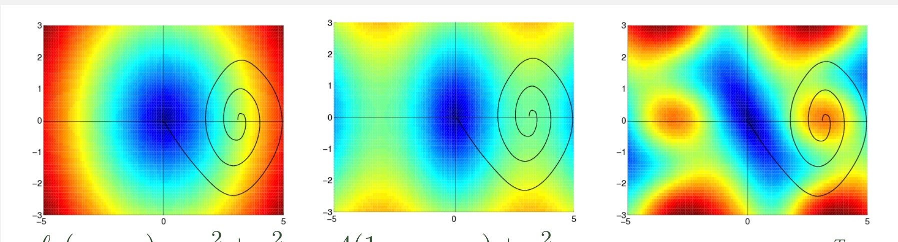
$$\begin{aligned} \dot x_1 &= x_2(t) \\ \dot x_2 &= -g\sin(x_1) - kx_2 \\ &\quad -u\cos(x_1) \\ \dot x_3 &= x_4 \\ \dot x_4 &= u \end{aligned}$$

Example: Inverted Pendulum

Reducing overshoot for swingup of the pendulum on a cart:

$$\begin{aligned} \dot x_1 &= x_2, & \dot x_2 &= g\sin(x_1) - kx_2 + u\cos(x_1) \\ \dot x_3 &= x_4, & \dot x_4 &= u \end{aligned}$$

Let $\ell(x) = \sqrt{\ell_1(x_1,x_2) + x_3^2 + x_4^2}$ with



$\ell_1(x_1,x_2) = x_1^2 + x_2^2$	$4(1-\cos x_1) + x_2^2$	$(\sin x_1, x_2)P(\sin x_1, x_2)^T$ $+2((1-\cos x_1)(1-\cos x_2)^2)^2$
$N = 15$	$N = 10$	$N = 4$ (swingup only)

sampling time $T = 0.15$

A PDE example

We illustrate this with the 1d controlled PDE

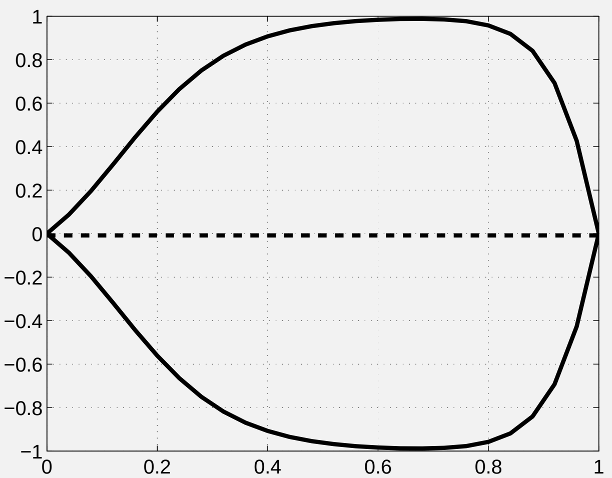
$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0,1]$
solution $y = y(t,x)$
boundary conditions $y(t,0) = y(t,1) = 0$
parameters $\nu = 0.1$ and $\mu = 10$
and distributed control $u : \mathbb{R} \times \Omega \to \mathbb{R}$

Discrete time system: $y(n) = y(nT, \cdot)$, sampling time $T = 0.025$

The uncontrolled PDE



all equilibrium solutions

MPC for the PDE example

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

Goal: stabilize the sampled data system $y(n)$ at $y \equiv 0$

Usual approach: quadratic L^2 cost

$$\ell(y(n), u(n)) = \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2$$

For $y \approx 0$ the control u must compensate for $y_x \rightsquigarrow u \approx -y_x$

$\rightsquigarrow$ controllability condition

$$\begin{aligned} \ell(y(n), u(n)) &\leq C\sigma^n \ell^*(y(0)) \\ \Leftrightarrow \quad \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2 &\leq C\sigma^n \|y(0)\|_{L^2}^2 \\ \approx \quad \|y(n)\|_{L^2}^2 + \lambda \|y_x(n)\|_{L^2}^2 &\leq C\sigma^n \|y(0)\|_{L^2}^2 \end{aligned}$$

for $\|y_x\|_{L^2} \gg \|y\|_{L^2}$ this can only hold if $C \gg 0$

MPC for the PDE example

Conclusion: because of

$$\|y(n)\|_{L^2}^2 + \lambda \|y_x(n)\|_{L^2}^2 \leq C\sigma^n \|y(0)\|_{L^2}^2$$

the controllability condition may only hold for very large C

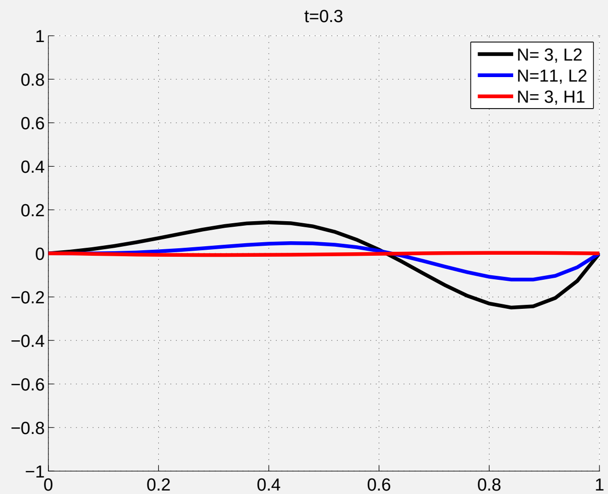
Remedy: use H^1 cost

$$\ell(y(n), u(n)) = \underbrace{\|y(n)\|_{L^2}^2 + \|y_x(n)\|_{L^2}^2}_{=\|y(n)\|_{H^1}^2} + \lambda \|u(n)\|_{L^2}^2.$$

Then an analogous computation yields

$$\|y(n)\|_{L^2}^2 + (1+\lambda)\|y_x(n)\|_{L^2}^2 \leq C\sigma^n \Big(\|y(0)\|_{L^2}^2 + \|y_x(0)\|_{L^2}^2 \Big)$$

MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

domain $\Omega = [0,1]$

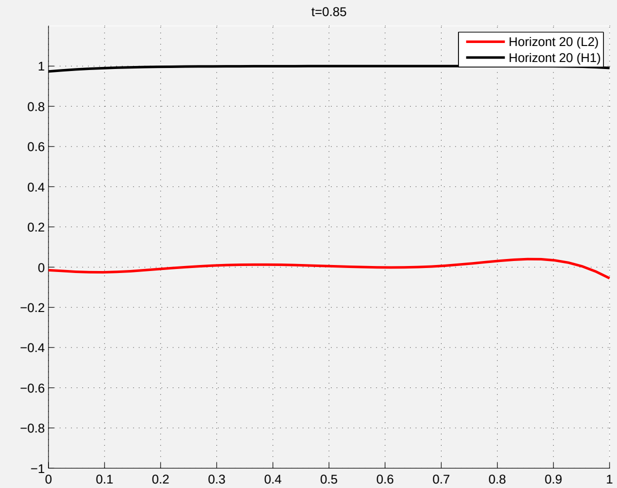
solution $y = y(t,x)$

boundary conditions $y(t,0) = u_0(t)$, $y(t,1) = u_1(t)$

parameters $\nu = 0.1$ and $\mu = 10$

with boundary control, stability can only be achieved via large gradients in the transient phase
$\rightsquigarrow$ L^2 should perform better that H^1

Boundary control, L_2 vs. H_1, $N = 20$



Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$
Can be made rigorous for many PDEs [Altmüller et al. '10ff]

Summary of Section (6)

- Reducing the overshoot constant C by choosing ℓ appropriately can significantly reduce the horizon N needed to obtain stability
- Computing tight estimates for C is in general a difficult if not impossible task
- But structural knowledge of the system behavior can be sufficient for choosing a "good" ℓ

(7) Feasibility

Feasibility

Consider the feasible sets

$$\mathcal{F}_N := \{x \in \mathbb{X} \,|\, \text{there exists an admissible } \mathbf{u} \text{ of length } N\}$$

So far we have assumed

$$V_N(x) \le \gamma \ell^\star(x) \text{ for all } x \in \mathbb{X}$$

which implicitly includes the assumption

$$\mathcal{F}_N = \mathbb{X}$$

because $V_N(x) = \infty$ for $x \in \mathbb{X} \setminus \mathcal{F}_N$

What happens if $\mathcal{F}_N \neq \mathbb{X}$ for some $N \in \mathbb{N}$?

The MPC feasibility problem

Even though the open-loop optimal trajectories are forced to satisfy $x^\star(k) \in \mathbb{X}$, the closed loop solutions $x_{\mu_N}(n)$ may violate the state constraints, i.e., $x_{\mu_N}(n) \notin \mathbb{X}$ for some n

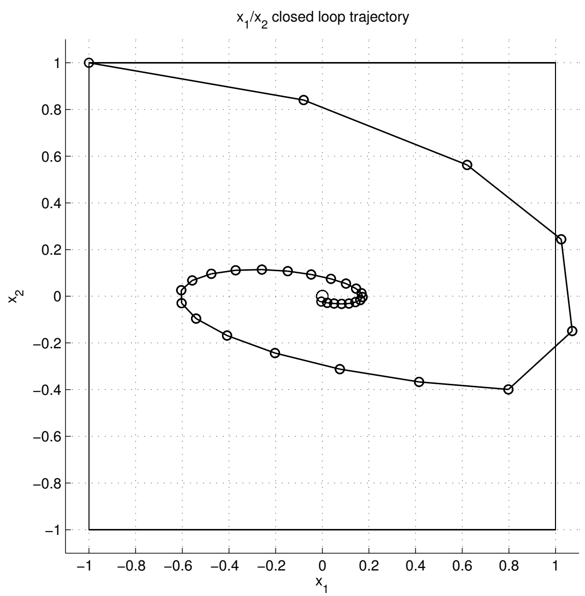
We illustrate this phenomenon by the simple example

$$\begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + u/2 \\ x_2 + u \end{pmatrix}$$

with $\mathbb{X} = [-1,1]^2$ and $\mathbb{U} = [-1/4, 1/4]$. For initial value $x_0 = (-1,1)^T$, the system can be controlled to 0 without leaving $\mathbb{X}$

We use MPC with $N = 2$ and $\ell(x,u) = \|x\|^2 + 5u^2$

The MPC feasibility problem: example



The MPC feasibility problem

How can this happen?

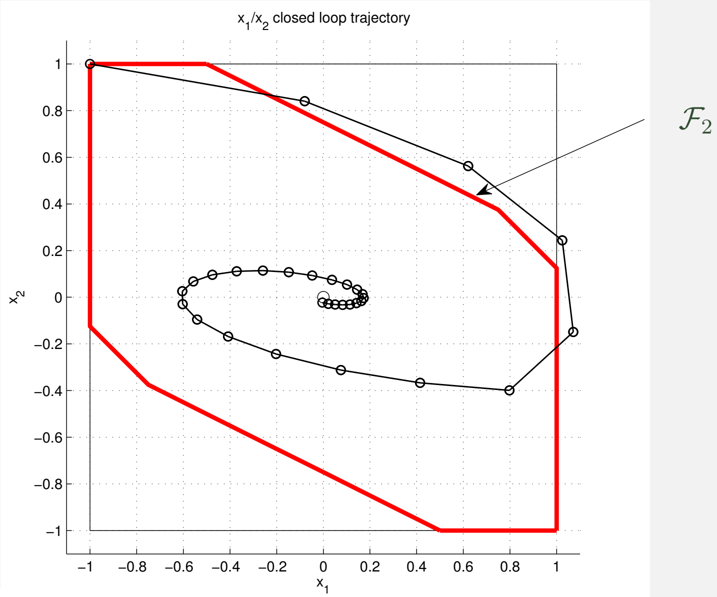
Explanation: In this example $\mathcal{F}_N \subsetneq \mathbb{X}$

⇝ at time n, the finite horizon state constraints guarantee $x^\star(1) \in \mathbb{X}$ but in general not $x^\star(1) \in \mathcal{F}_N$

⇝ the optimal control problem at time $n+1$ with initial value $x_{\mu_N}(n+1) = x^\star(1)$ may be infeasible

⇝ $x_{\mu_N}(n+k)$ is inevitable for some $k \ge 2$

The MPC feasibility problem: example again



Recursive feasibility

The MPC scheme with horizon N is well defined on a set $A \subseteq \mathcal{F}_N$ if the following recursive feasibility condition holds:

$$x \in A \quad \Rightarrow \quad f(x, \mu_N(x)) \in A$$

In terminal constrained MPC, forward invariance of the terminal constraint set $\mathbb{X}_0$ implies recursive feasibility of the feasible set

$$\mathbb{X}_N := \{x \in \mathbb{X} \,|\, \text{there is an admissible } \mathbf{u} \text{ with } x_{\mathbf{u}}(N, x) \in \mathbb{X}_0\}$$

(this was part of the stability theorem in Section 3)

Can we find recursively feasible sets for NMPC without terminal constraints?

Recursive feasibility

Theorem: [Kerrigan '00, Gr./Pannek 11] Assume that

$$\mathcal{F}_{N_0} = \mathcal{F}_{N_0-1}$$

holds for some $N_0 \in \mathbb{N}$. Then the set $\mathcal{F}_N$ is recursively feasible for all $N \geq N_0$.

Idea of proof:

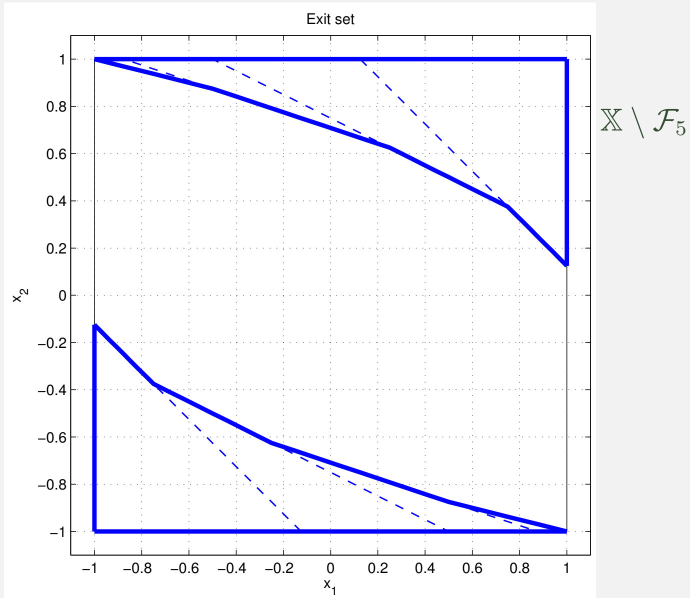
(1) $\mathcal{F}_{N_0} = \mathcal{F}_{N_0-1}$ implies $\mathcal{F}_N = \mathcal{F}_{N_0-1}$ for all $N \geq N_0 - 1$

(2) $x^\star(0) = x \in \mathcal{F}_N$ implies

$$f(x, \mu_N(x)) = x^\star(1) \in \mathcal{F}_{N-1} = \mathcal{F}_{N_0-1} = \mathcal{F}_N$$

$\Rightarrow$ recursive feasibility of $\mathcal{F}_N$

Feasible sets for our example



Recursive feasibility

Problem: What if this condition does not hold / cannot be checked?

Theorem: [Gr./Pannek '11, extending Primbs/Nevistić '00]

Assume $V_N(x) \leq \gamma \ell^\star(x)$ for all $x \in \mathcal{F}_N$, $N \in \mathbb{N}$

Assume there exists a forward invariant neighborhood $\mathcal{N}$ of x_*

Then for each $c > 0$ there exists $N_c > 0$ such that for all $N \geq N_c$ the level set

$$A_c := \{x \in \mathcal{F}_N \,|\, V_N(x) \leq c\}$$

is recursively feasible and the MPC closed loop is asymptotically stable with basin of attraction containing A_c

If $\mathbb{X}$ is compact, then $A_c = \mathcal{F}_\infty$ for all sufficiently large N

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Idea of proof

$V_N(x) \leq \gamma \ell^\star(x)$ implies exponential decay of $\ell^\star(x^\star(k))$ (as in Variant 2 of the stability proof in Section 5)

$\Rightarrow$ $x^\star(N-1) \in \mathcal{N}$ for $x \in A_c$ and $N \geq N_c$

$\Rightarrow$ forward invariance of $\mathcal{N}$ implies that solution can be extended

$\Rightarrow$ recursive feasibility

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Discussion

Feasibility properties of MPC without terminal constraints

- Advantage: In contrast to $\mathbb{X}_0$ in the terminal constrained setting, $\mathcal{N}$ does not need to be known, mere existence is sufficient
- Drawback: In terminal constrained MPC, feasibility at time $n = 0$ implies recursive feasibility. This property is lost without terminal constraints

  If this is desired, a forward invariant terminal constraint $\mathbb{X}_0$ can be used without terminal cost — the stability proof without terminal constraints also works for this setting

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Final discussion: comparison of MPC with and without terminal constraints

Properties of MPC without terminal constraints compared to terminal constrained MPC

- ⊕ needs fewer a priori information to set up the scheme
- ⊖ results are typically less constructive
- ⊕ may exhibit larger operating regions
- ⊖ may need larger N for obtaining stability near x_*

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