An Introduction to Optimal Control and Recent Solution Strategies

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The latest version of these slides is available from https://t1p.de/optimal_control_padova

The moodle page for this course is https://stem.elearning.unipd.it/enrol/index.php?id=13066

This version of the slides is from 10 September 2025

Outline of the course

Part 1: Optimal Control Problems — An Introduction

Part 2: Solution Concepts

Part 3: Numerical Solution Methods

Part 4: Model Predictive Control

Part 5: Deep Neural Networks for High-Dimensional Problems

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Contents of this part

Part 1: Optimal Control Problems — An Introduction

- Optimal Control Problems
- Continuous-time Optimal Control Problems
- Discrete-time Optimal Control Problems
- Open-loop and Closed-loop Optimal Control

Optimal Control Problems

Setting

We consider nonlinear control systems in continuous time

$$\dot{x}(t) := \frac{d}{dt}x(t) = f(t, x(t), \mathbf{u}(t))$$

or in discrete time

$$x^+(t) := x(t+1) = g(t, x(t), \mathbf{u}(t))$$

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where $f\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g\colon \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are maps with appropriate regularity and $\mathbf{u}(\cdot)$ is either a suitable control function or a control sequence, in both cases with values in $\mathbb{R}^m$

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The bold $\mathbf{u}(\cdot)$ indicates control functions or sequences with $\mathbf{u}(t) = u \in U$

Example: Pendulum on a cart

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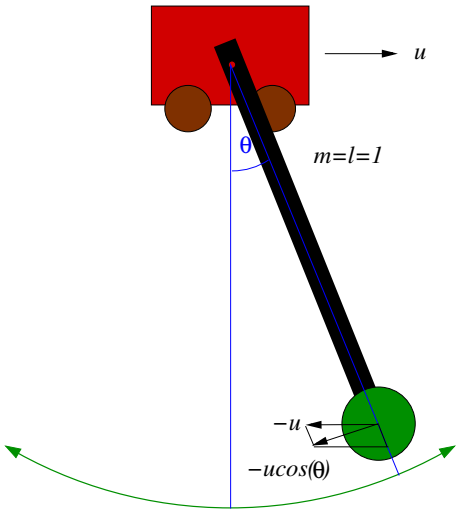

$x_1 = \theta =$ angle
$x_2 =$ angular velocity
$x_3 =$ cart position
$x_4 =$ cart velocity
$u \;\; =$ cart acceleration

$\rightsquigarrow$ control system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
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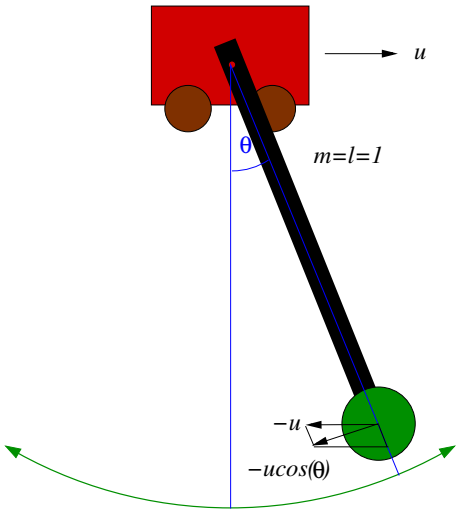

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\dot{x}_5 &= x_6 \\
\dot{x}_6 &= k_3 u(t) - k_4 x_6(t) - k_5 x_5
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Determine a control function that causes a control system to minimize a performance criterion and—at the same time— satisfy physical constraints.

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subject to (OCP$_c$)

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Optimal control in continuous time and discrete time

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$$\forall t \in \mathbb{N}_{[t_0, t_0+N-1]} : \mathbf{u}(t) \in \mathbb{U} \subseteq \mathbb{R}^m$$
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- Class of control functions? Definition of state and input constraints?
- Performance criterion?

Continuous-Time Optimal Control Problems

Continuous-time optimal control problems

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(we may omit t_0, x_0, and/or $\mathbf{u}$ when clear from the context)

Performance criteria

Cost functional $J : \mathbb{R}^n \times L_\infty^{loc} \to \mathbb{R}$ in Lagrange form

$$J(x_0, \mathbf{u}(\cdot)) = \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t)) dt$$

- Stage cost or running cost[1] $\ell : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, continuous and continuously differentiable w.r.t. x
- t_0, t_f can be fixed or free, i.e., we may minimize not only over $\mathbf{u}$ but also over t_0 and/or t_f
- $t_f = \infty$ is possible; we will come back to this later

[1] In older texts also called Lagrange function

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- Cost functional $J : \mathbb{R}^n \times L^{loc}_\infty \to \mathbb{R}$ in Mayer form

$$J(x_0, \mathbf{u}(\cdot)) = L(t_0, x(t_0), t_f, x(t_f))$$

  Initial and terminal cost $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$

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- Cost functional $J : \mathbb{R}^n \times L^{loc}_\infty \to \mathbb{R}$ in Bolza form

$$J(x_0, \mathbf{u}(\cdot)) = L(t_0, x(t_0), t_f, x(t_f)) + \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t)) dt$$

Conversion of criteria

Lagrange form: $$J(x_0, \mathbf{u}(\cdot)) = \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t))dt$$

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Thus, for $\tilde{x} = \begin{pmatrix} c \\ x \end{pmatrix}$, $\tilde{x}_0 = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$ and $L(t_0, \tilde{x}(t_0), t_f, \tilde{x}(t_f)) = c(t_f)$ we obtain

$$L(t_0, \tilde{x}(t_0), t_f, \tilde{x}(t_f)) = \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t))dt$$

Constraints

In this course, we limit ourselves to pointwise state and control constraints

$$x(t) \in \mathbb{X}, \quad \mathbf{u}(t) \in \mathbb{U} \quad \text{for (almost) all } t \in [t_0, t_f]$$

for given sets $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$

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In an implementation, one would describe these sets by inequalities, e.g.,

$$\mathbb{X} = \{x \in \mathbb{R}^n \,|\, g_i(x) \leq 0 \text{ for all } i = 1, \ldots, n_g\}$$

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More general constraints would be, e.g., mixed constraints such as $(x, u) \in \mathbb{Y}$ or integral constraints such as $\int_{t_0}^{t_f} h(x(t), \mathbf{u}(t))dt \leq 0$

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Constraints can also be incorporated in a "soft" way, by choosing ℓ to satisfy

$$\ell(x, u) = \infty \quad \text{if } (x, u) \notin \mathbb{X} \times \mathbb{U}$$

In practice one often chooses ℓ to be very large instead of actually ∞

Example – Formulating an optimal control problem

The simplified dynamics of a car on a 1d road are given by

$$\dot{x} = Ax + Bu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where x_1 is the position, x_2 is the velocity, and the control input u is the acceleration

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Question: How would you specify performance criterion and constraints of a physically meaningful optimal control problem that steers the car from $x_1 = 1$ to $x_1 = 0$ on the interval $[t_0, t_f]$?

Admissible and feasible controls

Definition: A control $\mathbf{u}(\cdot) \in L_\infty^{loc}$ is said to be feasible for initial value x_0 on $[t_0, t_f]$ if

- the solution $x_{\mathbf{u}}(\cdot; t_0, x_0)$ is defined on $[t_0, t_f]$
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Definition: The set

$$\mathcal{U}^{[t_0,t_f]}(x_0) \doteq \left\{\mathbf{u}(\cdot) \in L_\infty^{loc} \mid \mathbf{u}(\cdot) \text{ is feasible for initial value } x_0 \text{ on } [t_0, t_f]\right\}$$

is called the set of feasible controls. Short hand notation: $\mathcal{U}$

Infinite-horizon objectives

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Without discounting, additional conditions on the control system are needed to ensure finiteness. Typically, these are controllability conditions

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For instance, the frequent task of controlling a system to a desired state or reference solution x_{ref} and keeping it there is naturally posed as an infinite-horizon optimal control problem

Example: Pendulum on a cart

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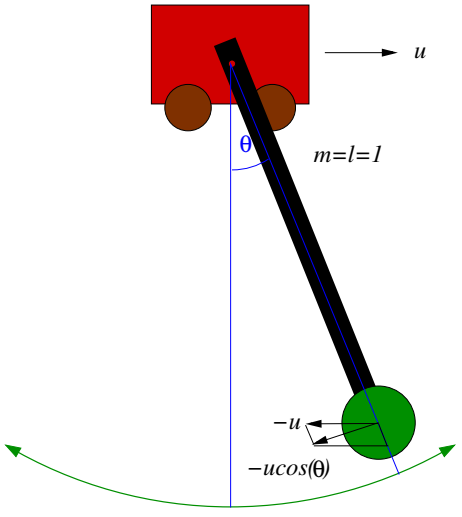

$x_1 = \theta =$ angle
$x_2 =$ angular velocity
$x_3 =$ cart position
$x_4 =$ cart velocity
$u \;\; =$ cart acceleration

$$\ell(x,u) = \|x - x^*\|^2 + \lambda u^2$$

Swing-up to and balancing at the upright position can be achieved by an infinite-horizon optimal control with cost $\ell(x,u) = \|x - x^*\|^2 + \mu u^2$

Discrete-Time Optimal Control Problems

Performance criterion in discrete time

Recall the discrete-time dynamics $x(t+1) = g(t, x(t), \mathbf{u}(t)), \quad x(t_0) = x_0$

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General (Bolza) performance criterion:

$$J(x_0, \mathbf{u}(\cdot)) = \sum_{t=t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t)) + L(t_0, x(t_0), t_f, x(t_f))$$

- Stage cost or running cost $\ell : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$
- Mayer term (initial and/or terminal cost): $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$
- Again, t_0 and t_f can be fixed or free

Infinite-horizon OCP in discrete time

$$\min_{\mathbf{u}(\cdot)} \sum_{t=t_0}^{\infty} \beta^t \ell(t, x(t), \mathbf{u}(t))$$

subject to (OCP$_d$)

$$x(t+1) = g(t, x(t), \mathbf{u}(t)), \quad x(0) = x_0$$
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($\beta = 1$ yields a problem without discounting)

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Note: This does not yet include any numerical computation of $x(h)$ and $\int_{kh}^{(k+1)h} \ell(t, x(t), u)dt$, which may be needed in addition

Open-loop and closed-loop (or feedback) optimal control

Optimal value function and controls

The function

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Note: If $\mathbf{u}^\star(\cdot)$ exists for x_0, then the "inf" in the definition of $V(x_0)$ is a "min"

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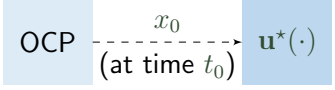

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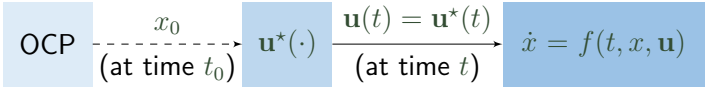

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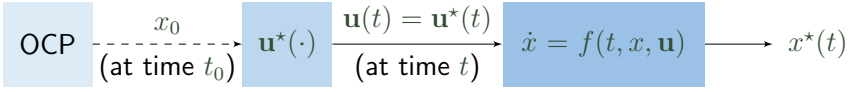

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If there exists a map $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ such that all optimal controls satisfy

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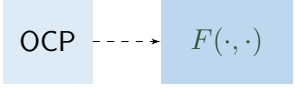

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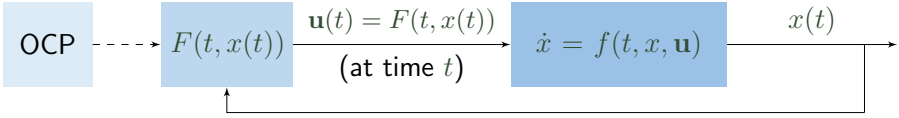

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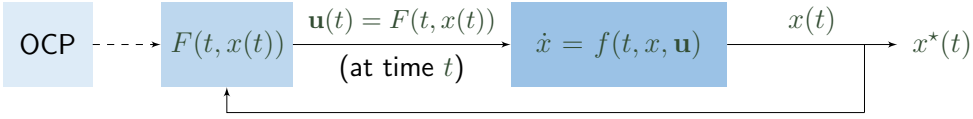

Quiz

Why are (optimal) controls in feedback form preferred?

- They can react to perturbations
- They are easier to implement in practice
- They are easier to compute

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Solution: see the pendulum example

Summary of Part 1: Optimal Control Problems — An Introduction

Outline of the course

Part 1: Optimal Control Problems — An Introduction

Part 2: Solution Concepts

Part 3: Numerical Solution Methods

Part 4: Model Predictive Control

Part 5: Deep Neural Networks for High-Dimensional Problems

Purpose of this part

Introduce classical solution concepts for optimal control problems

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Explain the differences between these concepts

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Contents of this part

Part 2: Solution Concepts

- Dynamic Programming
- Euler-Lagrange Equations
- Pontryagin's Minimum Principle

Dynamic Programming

Optimal value function

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A technical assumption that we make throughout this part:

The cost function ℓ satisfies $\ell \geq 0$ or ℓ is bounded and $\beta < 1$

Dynamic programming principle in discrete time

Theorem: (Dynamic programming principle *or* Bellman equation) Consider the discrete-time infinite-horizon OCP (OCP_d) with time-invariant problem data. The optimal value function satisfies for all $x_0 \in \mathbb{R}^n$

$$V(x_0) = \inf_{u \in \mathbb{U}} \{\ell(x_0, u) + \beta V(g(x_0, u))\} \qquad \text{(DPP)}$$

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"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." [Bellman '57]

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$$\begin{aligned} J(x_0, \mathbf{u}(\cdot)) = \sum_{t=0}^{\infty} \beta^t \ell(x(t), \mathbf{u}(t)) &= \ell(x(0), \mathbf{u}(0)) + \sum_{t=1}^{\infty} \beta^t \ell(x(t), \mathbf{u}(t)) \\ &= \ell(x(0), \mathbf{u}(0)) + \beta \sum_{t=0}^{\infty} \beta^t \ell(x(t+1), \mathbf{u}(t+1)) \\ &= \ell(x(0), \mathbf{u}(0)) + \beta J(x(1), \mathbf{u}(\cdot + 1)) \\ &= \ell(x_0, u) + \beta J(g(x_0, u), \mathbf{u}(\cdot + 1)) \end{aligned}$$

Dynamic programming principle in discrete time

Theorem: (Dynamic programming principle *or* Bellman equation) Consider the discrete-time infinite-horizon OCP (OCP_d) with time-invariant problem data. The optimal value function satisfies for all $x_0 \in \mathbb{R}^n$

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This equality carries over to the infimum over these expressions □

Characterisation of optimal open-loop controls

$$V(x_0) = \inf_{u \in \mathbb{U}} \{\ell(x_0, u) + \beta V(g(x_0, u))\} \qquad \text{(DPP)}$$

Theorem: Consider the discrete-time infinite-horizon OCP (OCP$_d$) with time-invariant problem data and a trajectory $x(\cdot)$ with control $\mathbf{u}(\cdot)$ and assume that

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Dynamic programming principle in continuous time

Consider (OCP_c) with time-invariant problem data on the infinite horizon $[0, \infty)$.

$$V(x_0) = \inf_{\mathbf{u}(\cdot) \in \mathcal{U}} \left\{ \int_0^\tau \beta^t \ell(x(t), \mathbf{u}(t)) dt + \beta^\tau V(x(\tau)) \right\}$$

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(here $\lim_{t\searrow 0}\mathbf{u}(t) = \mathbf{u}(0)$ is assumed; the proof can be modified if this does not hold)

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Task: minimize

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Dynamic programming also works for finite-horizon problems

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and the Hamilton-Jacobi-Bellman equation reads

$$\delta \frac{\partial}{\partial t_0} V(t_0, t_f, x_0) = \inf_{u\in\mathbb{U}} \left\{ \frac{\partial}{\partial x} V(t_0, t_f, x_0) f(x_0, u) + \ell(x, u) \right\} \text{ if } t_0 < t_f, \quad V(t_f, t_f, x) = L(x)$$

A note on optimal feedback laws

Note: It follows from dynamic programming theory, that optimal feedback laws (if they exist)

- do not depend on time if the problem data is time-invariant and $t_f = \infty$
- do in general depend on time in all other cases

Hamilton-Jacobi-Bellman equation

The Hamilton-Jacobi-Bellman equation is a partial differential equation (PDE)

If the optimal value function V is C^1, then it satisfies this equation uniquely (under appropriate boundary conditions)

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$\rightsquigarrow$ It is often much easier to discretize the problem in time and use the discrete-time theory

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- From these equations, optimal feedback laws may be obtained, which are often required in practice
- But, solving the DPP or the HJB equation is in general difficult!
  (we will come back to this later)

Euler-Lagrange Equations

Problem Setup

The Euler-Lagrange equations allow to compute open-loop control functions for fixed initial values in a simplified setting:

$$\min_{\mathbf{u}(\cdot)} \quad \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t))dt$$

subject to: (P)

$$\dot{x} = f(t, x, \mathbf{u}), \quad x(t_0) = x_0$$
$$\mathbf{u}(\cdot) \in \mathcal{C}[t_0, t_f]^m$$

$$f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \quad f \in \mathcal{C}^0 \text{ w.r.t. } (t, x, u), \quad f \in \mathcal{C}^1 \text{ w.r.t. } (x, u)$$
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Short hand notation:

$$D_w Z = \frac{\partial}{\partial w} Z, \quad (D_w Z)^\top = Z_w, \quad Z \in \{f, \ell, L, \dots\} \text{ and } w \in \{x, u, t\}$$

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The approach by Euler and Lagrange

$$\min_{\mathbf{u}(\cdot)} \quad \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t))dt$$

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Rewrite equality constraint imposed by the dynamics

$$J(x_0, \mathbf{u}) = \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t)) + \lambda(t)^\top \big(\dot{x}(t) - f(t, x(t), \mathbf{u}(t))\big)\, dt$$

with Lagrange multiplier $\lambda : [t_0, t_f] \to \mathbb{R}^n$

The approach by Euler and Lagrange

$\rightsquigarrow$ Equality constraints included in the objective

$$J(x_0, \mathbf{u}) = \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t)) + \lambda(t)^\top \big(\dot{x}(t) - f(t, x(t), \mathbf{u}(t))\big)\, dt$$

Apply integration by parts to $\lambda^\top \dot{x}$

$$\lambda^\top \dot{x} = -x^\top \dot{\lambda} + \frac{d}{dt}\left(x^\top \lambda\right) \qquad \Leftrightarrow \qquad \int \lambda^\top \dot{x} dt = \int -x^\top \dot{\lambda} dt + \left(x^\top \lambda\right)$$

to obtain

$$J(x_0, \mathbf{u}) = \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t)) - \Big(x(t)^\top \dot{\lambda}(t) + \lambda(t)^\top f(t, x(t), \mathbf{u}(t))\Big) dt + x^\top \lambda \Big|_{t_0}^{t_f}$$

The approach by Euler and Lagrange

Consider a small perturbation of the optimal control $\mathbf{u}^\star(\cdot)$

$$\mathbf{u}^\star(\cdot) + \eta \cdot \mathbf{v}(\cdot), \qquad \eta \in \mathbb{R},\ \eta \approx 0$$

which generates a perturbed state trajectory

$$x_{\mathbf{u}^\star + \eta \cdot \mathbf{v}}(t, x_0)$$

and the limit

$$\lim_{\eta \to 0} \frac{J(x_0, \mathbf{u}^\star + \eta \cdot \mathbf{v}) - J(x_0, \mathbf{u}^\star)}{\eta} = \left.\frac{\partial}{\partial \eta}\right|_{\eta=0} J(x_0, \mathbf{u}^\star + \eta \cdot \mathbf{v})$$

Optimality implies[2]

$$0 = \left.\frac{\partial}{\partial \eta}\right|_{\eta=0} J(x_0, \mathbf{u}^\star + \eta \cdot \mathbf{v}) = \mathfrak{d}J(x_0, \mathbf{u}^\star, \mathbf{v}) = 0$$

[2] $\mathfrak{d}J(x_0, \mathbf{u}^\star, \mathbf{v})$ denotes the Gateaux derivative w.r.t. $\mathbf{u}$.

The approach by Euler and Lagrange

Let

$$\delta x := \frac{\partial}{\partial \eta}\bigg|_{\eta=0} x_{\mathbf{u}^\star + \eta \cdot \mathbf{v}}(t, x_0),$$

called sensitivity or variation of $x(\cdot)$ with respect to the input perturbation $\mathbf{v}$

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We obtain

$$\begin{aligned}0 &= \ell_x(t, x^\star, \mathbf{u}^\star) - f_x^\top(t, x^\star, \mathbf{u}^\star)\lambda - \dot\lambda \\ 0 &= \ell_u(t, x^\star, \mathbf{u}^\star) - f_u^\top(t, x^\star, \mathbf{u}^\star)\lambda \\ 0 &= \lambda(t_f)\end{aligned}$$

First-order necessary conditions of optimality

Theorem (First-order necessary conditions):
Suppose that $u^\star(\cdot) \in \mathcal{C}[t_0, t_f]^m$ is a local minimizer of Problem (P) and $x^\star(\cdot) \in \mathcal{C}^1[t_0, t_f]^n$, $x^\star(t) = x_{\mathbf{u}^\star}(t, t_0, x_0)$ is the corresponding solution.

Then there exists a function $\lambda^\star(\cdot) \in \mathcal{C}^1[t_0, t_f]^n$ such that, for all $t \in [t_0, t_f]$, the triple $(u^\star(\cdot), x^\star(\cdot), \lambda^\star(\cdot))$ satisfies:

$$\begin{aligned} \dot{x^\star} = &\quad f(t, x^\star, \mathbf{u}^\star), && x^\star(t_0) = x_0 \\ \dot{\lambda}^\star = &-\ell_x(t, x^\star, \mathbf{u}^\star) - f_x^\top(t, x^\star, \mathbf{u}^\star)\lambda^\star, && \lambda^\star(t_f) = 0 \qquad \text{(E-L)} \\ 0 = &\quad \ell_u(t, x^\star, \mathbf{u}^\star) + f_{\mathbf{u}}^\top(t, x^\star, \mathbf{u}^\star)\lambda^\star. \end{aligned}$$

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- For general terminal constraints $\psi(t_f, x(t_f)) = 0$, we obtain
$$\lambda(t_f) = L_x(t_f, x(t_f)) + \nu^\top \Psi_x(t_f, x(t_f))$$

First-order necessary conditions – Remarks

The Euler-Lagrange equations can be rewritten more concisely using the Hamiltonian function: $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$

$$H(t, x, u, \lambda) := \ell(t, x, u) + \lambda^\top f(t, x, u) = \ell(t, x, u) + \langle \lambda,\ f(t, x, u) \rangle$$

Notation for scalar product of $w, z, \in \mathbb{R}^n$: $\langle w,\ z \rangle = w^\top z$

[3] H.J. Sussmann and J.C. Willems, "300 years of optimal control: from the Brachystochrone to the Maximum Principle". In: IEEE Control Systems 17.3 (1997), pp. 32–44

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Good overview on the history of optimal control: [Sussmann/Willems '97][3]

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[4] B. Chachuat. Nonlinear and Dynamic Optimization—From Theory to Practice. EPFL, 2009. URL: https://infoscience.epfl.ch/server/api/core/bitstreams/28c307a4-ced8-4df6-bf43-fd02005ed74d/content

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- Change of Hamiltonian along optimal trajectory

$$\frac{d}{dt}H(t, x, \mathbf{u}, \lambda) = H_t + \langle H_x, f(t, x, \mathbf{u})\rangle + \langle H_{\mathbf{u}}, \dot{\mathbf{u}}\rangle + \langle f(t, x, \mathbf{u}), \dot{\lambda}\rangle$$

[4] B. Chachuat. Nonlinear and Dynamic Optimization—From Theory to Practice. EPFL, 2009. URL: https://infoscience.epfl.ch/server/api/core/bitstreams/28c307a4-ced8-4df6-bf43-fd02005ed74d/content

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  $\rightsquigarrow$ Second order necessary conditions (Legendre-Clebsch condition):

$$D^2H = H_{uu} \succeq 0 \text{ for minima,} \qquad D^2H = H_{uu} \preceq 0 \text{ for maxima}$$

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- A readable introduction: [Chachuat '09][4]

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A toy example

$$\min_{\mathbf{u}(\cdot)} \quad \int_0^1 \frac{1}{2}\mathbf{u}^2(t)dt$$

subject to

$$\dot{x}(t) = \mathbf{u}(t) - x(t), \quad x(0) = 1,\ x(1) = 0$$

Task: Write the Euler-Lagrange equations for this problem

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$$\begin{aligned} H(x,u,\lambda) &= \frac{1}{2}u^2 + \lambda^T(u-x) \\ \dot{x}^\star &= \quad H_\lambda(x^\star,\mathbf{u}^\star,\lambda^\star) = u^\star - x^\star \\ \dot{\lambda}^\star &= -H_x(x^\star,\mathbf{u}^\star,\lambda^\star) = \lambda^\star \\ 0 &= \quad H_u(x^\star,\mathbf{u}^\star,\lambda^\star) = u^\star + \lambda^\star \\ x^\star(0) &= 1, \quad x^\star(1) = 0 \end{aligned}$$

Discrete-time counterpart to (E-L)

Continuous-time OCP

$$\min_{\mathbf{u}(\cdot)} \quad \int_0^{t_f} \ell(t, x(t), \mathbf{u}(t))dt$$

subject to

$$\dot{x} = f(t, x, \mathbf{u}), \quad x(0) = x_0$$

Discrete-time OCP (= NLP)

$$\min_{\mathbf{u}(\cdot)} \quad \sum_{t=0}^{N-1} \ell(t, x(t), \mathbf{u}(t))$$

subject to

$$x(t+1) = g(t, x(t), \mathbf{u}(t)), \quad x(0) = x_0$$

With $H = \ell + \lambda^\top f$:

$$\begin{aligned} \dot{x^\star} &= H_\lambda(t, x^\star, \mathbf{u}^\star, \lambda^\star), & x^\star(0) &= x_0 \\ \dot{\lambda}^\star &= -H_x(t, x^\star, \mathbf{u}^\star, \lambda^\star), & \lambda^\star(t_f) &= 0 \\ 0 &= H_u(t, x^\star, \mathbf{u}^\star, \lambda^\star) \end{aligned}$$

?

Discrete-time Euler-Lagrange equations

Define the Lagrangian pointwise in time

$$\widehat{\mathcal{L}}(t) \doteq \ell(x(t), \mathbf{u}(t)) + \lambda(t+1)^\top (g(t, x(t), \mathbf{u}(t)) - x(t+1))$$

and

$$\mathcal{L}(t, x, \mathbf{u}, \lambda) \doteq \lambda(0)^\top (x_0 - x(0)) + \sum_{t=0}^{N-1} \widehat{\mathcal{L}}(t)$$

Stationarity of the Lagrangian

$$\begin{aligned} \mathcal{L}_\lambda = 0 \quad &\rightsquigarrow \quad x^\star(t+1) = g(t, x^\star(t), \mathbf{u}^\star(t)) \\ \mathcal{L}_x = 0 \quad &\rightsquigarrow \quad \lambda^\star(t) = g_x^\top \lambda^\star(t+1) + \ell_x, \quad \lambda^\star(N) = 0 \qquad \text{(E-L}_\text{d}\text{)} \\ \mathcal{L}_u = 0 \quad &\rightsquigarrow \quad 0 = g_u^\top \lambda^\star(t+1) + \ell_u \end{aligned}$$

These are the discrete-time Euler-Lagrange equations

Comparison – Discrete-time and continuous-time Euler-Lagrange

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But: There is no fully equivalent discrete-time counterpart of the Hamiltonian

Pontryagin's Minimum Principle

Preliminaries

We now allow for measurable controls and add constraints to the OCP (P). We start with input constraints

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Problem setup (time invariant, free end time, terminal constraint):

$$\min_{\mathbf{u}(\cdot),t_f} \quad \int_{t_0}^{t_f} \ell(x(t),\mathbf{u}(t))dt$$

subject to

$$\begin{aligned} &\dot{x} = f(x,\mathbf{u}), \quad x(t_0) = x_0 \\ &\mathbf{u}(\cdot) \in L_\infty^{loc}\left([t_0,t_f],\mathbb{U}\right), \quad \mathbb{U} \subseteq \mathbb{R}^m \\ &t_f \in [t_0,T], \quad T < \infty \\ &x(t_f) = x_1 \end{aligned} \tag{$\mathsf{P}_{\mathsf{PMP}}$}$$

$$f:\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, \quad f \in \mathcal{C}^0 \text{ w.r.t. } (x,u), \quad f \in \mathcal{C}^1 \text{ w.r.t. } (x)$$
$$\ell:\ \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}\ , \quad \ell \in \mathcal{C}^0 \text{ w.r.t. } (x,u), \quad \ell \in \mathcal{C}^1 \text{ w.r.t. } (x)$$

Towards Pontryagin's minimum principle

Reformulation in Mayer form

$$c(t) = \int_{t_0}^{t_f} \ell(x(t), \mathbf{u}(t))dt$$

$$\tilde{x}(t) = \begin{pmatrix} c(t) \\ x(t) \end{pmatrix}$$

$$\dot{\tilde{x}}(t) = \tilde{f}(x(t), \mathbf{u}(t)) = \begin{pmatrix} \ell(x(t), u(t)) \\ f(x(t), \mathbf{u}(t)) \end{pmatrix}$$

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$$\min_{\mathbf{u}(\cdot), t_f} \quad c(t_f)$$

subject to

$$\dot{\tilde{x}} = \tilde{f}(x, \mathbf{u}), \quad \tilde{x}(t_0) = [0, x_0]^\top$$

$$\mathbf{u}(\cdot) \in \hat{\mathcal{C}}\left([t_0, t_f], \mathbb{U}\right), \quad \mathbb{U} \subseteq \mathbb{R}^m$$

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Hamiltonian for reformulated problem

$$H(x, \mathbf{u}, \tilde{\lambda}) = \left\langle \tilde{\lambda},\ \tilde{f}(x, \mathbf{u}) \right\rangle = \lambda_0(t)\ell(x(t), \mathbf{u}(t)) + \lambda(t)^\top f(x(t), \mathbf{u}(t)) \qquad \tilde{\lambda} = \begin{pmatrix} \lambda_0 \\ \lambda \end{pmatrix}$$

Pontryagin's minimum principle[5]

Theorem: Suppose that $(u^\star(\cdot), t_f^\star) \in L_\infty^{loc}([t_0, t_f], \mathbb{U}) \times [t_0, T]$ is a global minimizer of Problem ($\mathsf{P}_{\mathsf{PMP}}$) and let $\tilde{x}^\star(\cdot)$ be the corresponding extended solution $\tilde{x}^\star(\cdot) = \tilde{x}_{\mathbf{u}^\star}(\cdot, t_0, \tilde{x}_0)$.

[5] V. G. Boltyanskii et al. "On the theory of optimal processes". In: Doklady Akademii Nauk SSSR 110 (1956), pp. 7–10, L. S. Pontryagin et al. The Mathematical Theory of Optimal Processes. John Wiley & Sons Inc., New York, 1962

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Then there exists an absolutely continuous function

$$\tilde{\lambda}^\star(\cdot) = (\lambda_0^\star(\cdot), \lambda^\star(\cdot))^\top,$$

$\tilde{\lambda}^\star(t) \neq [0, \ldots, 0]^\top$ for all $t \in [t_0, t_f]$, such that $(\mathbf{u}^\star(\cdot),\ \tilde{x}^\star(\cdot),\ \tilde{\lambda}^\star(\cdot))$ satisfy

$$\begin{aligned} \dot{\tilde{x}}^\star(t) &= \quad H_{\tilde{\lambda}}\left(x^\star(t), \mathbf{u}^\star(t), \tilde{\lambda}^\star(t)\right), \qquad \tilde{x}(t_0) = (0, x_0)^\top \\ \dot{\tilde{\lambda}}^\star(t) &= -H_{\tilde{x}}\left(x^\star(t), \mathbf{u}^\star(t), \tilde{\lambda}^\star(t)\right) \end{aligned}$$

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. . . and:

i) The function $H\left(x^\star(t), v, \tilde{\lambda}^\star(t)\right)$ attains its minimum on $\mathbb{U}$ at $v = \mathbf{u}^\star(t)$ for almost all $t \in [t_0, t_f^\star]$:

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ii) For all $t \in [t_0, t_f^\star]$ it holds

$$\lambda_0^\star(t) \equiv const. \geq 0, \qquad H\left(x^\star(t), \mathbf{u}^\star(t), \tilde{\lambda}^\star(t)\right) = const.$$

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iii) If the final time t_f is free, the following transversality condition holds

$$H\left(x^\star(t_f^\star), \mathbf{u}^\star(t_f^\star), \tilde{\lambda}^\star(t_f^\star)\right) = 0.$$

Proofs

D. Liberzon. Calculus of Variations and Optimal Control Theory: A Concise Introduction. Princeton University Press, 2012.
http://liberzon.csl.illinois.edu/teaching/cvoc.pdf

E.R. Pinch. Optimal Control and the Calculus of Variations. Oxford University Press, 1995

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- For general terminal constraints $\psi(t_f, x(t_f)) = 0$, we obtain

$$\lambda(t_f) = L_x(t_f, x(t_f)) + \nu^\top \Psi_x(t_f, x(t_f))$$

The extra adjoint λ_0

The Hamiltonian in the Euler-Lagrange equations (E-L) is

$$H(x, \mathbf{u}, \lambda) = \quad \ell(x(t), \mathbf{u}(t)) + \lambda^\top f(x(t), \mathbf{u}(t))$$

and in the PMP we have

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In the absence of state constraints[6], we can normalize $\lambda_0 = 1$, such that both Hamiltonians coincide

[6] This condition is only sufficient; if state constraints are present, then this may still be possible. OCPs with $\lambda_0 = 0$ are particularly complicated and called abnormal

The PMP vs. the Euler-Lagrange equations

The Euler-Lagrange equations (E-L) require for $\mathbf{u}^\star$

$$\forall t \in [t_0, t_f^\star]: \quad 0 = \quad \ell_u(x^\star, \mathbf{u}^\star) + f_u^\top(x^\star, \mathbf{u}^\star)\lambda^\star = H_u\left(x^\star, \mathbf{u}^\star, \lambda\right)$$

while the PMP reads

$$\forall t \in [t_0, t_f^\star]: \quad H\left(x^\star, v, \tilde{\lambda}^\star\right) \geq H\left(x^\star, \mathbf{u}^\star, \tilde{\lambda}^\star\right)$$

The PMP vs. the Euler-Lagrange equations

The Euler-Lagrange equations (E-L) require for $\mathbf{u}^\star$

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Note: The principle was originally developed for maximisation problems, hence it was called Pontryagin's Maximum Principle

A toy example

$$\min_{\mathbf{u}(\cdot)} \quad \int_0^1 \frac{1}{2}\mathbf{u}^2(t)dt$$

subject to

$$\dot{x}(t) = \mathbf{u}(t) - x(t), \quad x(0) = 1,\ x(1) = 0$$
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$$H(x, u, \lambda) = \lambda_0 \frac{1}{2}u^2 + \lambda^T(u - x)$$
$$\dot{x}^\star = \quad H_\lambda(x^\star, \mathbf{u}^\star, \lambda^\star) = u^\star - x^\star$$
$$\dot{\lambda}^\star = -H_x(x^\star, \mathbf{u}^\star, \lambda^\star) = \lambda^\star$$
$$\lambda_0 \frac{1}{2}\mathbf{u}^{\star 2} + \lambda^{\star T}(\mathbf{u}^\star - x^\star) \geq \lambda_0 \frac{1}{2}u^2 + \lambda^T(u - x^\star)$$
$$x^\star(0) = 1, \quad x^\star(1) = 0$$

A pitfall example – The PMP for infinite horizon problems

$$\min_{\mathbf{u}(\cdot)} \quad \int_0^{t_f} -(1-x(t))\mathbf{u}(t)dt$$

subject to

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Observe that

$$J(x_0,\mathbf{u}) = -x(t_f) = e^{-\int_0^{t_f} \mathbf{u}(\tau)d\tau} - 1$$

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This suggests that for $t_f = \infty$, the condition might be $\lim_{t\to\infty} \lambda(t) = 0$

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The Hamiltonian reads

$$\begin{aligned} H(x, \mathbf{u}, \lambda_0, \lambda) &= -\lambda_0(1-x)\mathbf{u} + \lambda(1-x)\mathbf{u} \\ &= \quad (\lambda - \lambda_0)(1-x)\mathbf{u} \end{aligned}$$

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$\rightsquigarrow$ regardless of how we choose $\lambda(0)$, we never obtain $\lim_{t\to\infty} \lambda(t) = 0$

The PMP for infinite-horizon problems?

For infinite horizons,

- the general structure of the PMP remains unchanged
- But, if there is no terminal constraint on the state, the PMP does not provide a boundary/transversality condition[7] for the adjoint $\lim_{t\to\infty} \lambda(t)$

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Remedies are known for problems with particular properties[8] (beyond the scope of this course)

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The link between the HJBE and the PMP

Recall the HJBE for undiscounted problems:

$$\frac{\partial}{\partial t}V(t,t_f,x) = \inf_{u\in\mathcal{U}}\left\{\frac{\partial}{\partial x}V(t,t_f,x)f(x,u) + \ell(x,u)\right\} \quad \text{if } t_0 < t_f$$

The point-wise in time minimisation of the Hamiltonian in the PMP gives

$$H\left(x^\star(t),\mathbf{u}^\star,\tilde{\lambda}^\star(t)\right) = \min_{u\in\mathcal{U}} H\left(x^\star(t),u,\tilde{\lambda}^\star(t)\right) = \min_{u\in\mathcal{U}}(\lambda(t)^\star)^\top f(x^\star,u)+\lambda_0\ell(x^\star,u)$$

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and for $\lambda_0 = 1$ we observe the identity

$$\frac{\partial}{\partial x}V(t,t_f,x) = (\lambda^\star)^\top(t)$$

$\rightsquigarrow$ the adjoint $\lambda^\star$ is the derivative of the optimal value function w.r.t. x

Extensions of the presented PMP

- The PMP can be extended to problems with mixed input-state constraints[9]

$$g(x(t), \mathbf{u}(t)) \leq 0$$

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- There exist discrete-time counterparts of the PMP, but they turn out to be more restrictive than KKT conditions

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Quiz

Why is it important that the PMP works for L_∞^{loc}-functions?

- Because there are examples where such functions are needed
- Because this simplifies its proof
- It ain't really important, the authors who wrote it simply did it this way

Why do we need controls from L_∞^{loc}?

This example[10] shows why is it important that the PMP allows for $u \in L_\infty^{loc}$:

$$\min_{\mathbf{u}(\cdot)} \quad \int_0^{10} x_1(t)^2 \, dt$$

subject to

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{u}, \quad x(0) = x_0$$

$$\forall t \in [0, 10]: \quad \mathbf{u}(t) \in [-1, 1]$$

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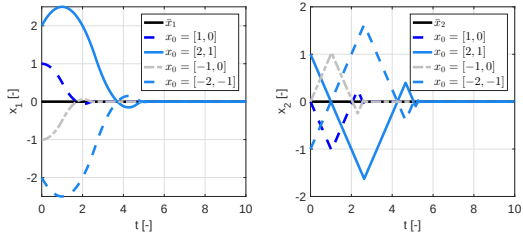

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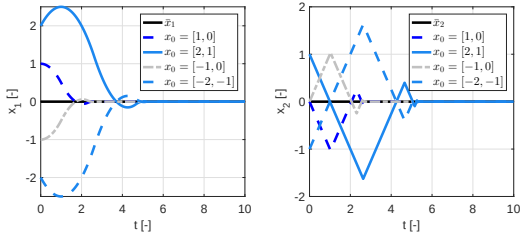

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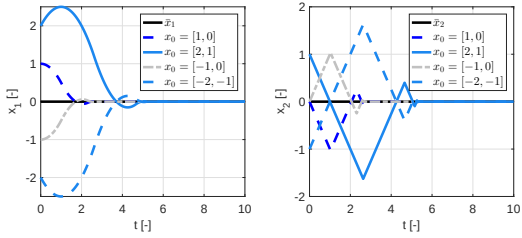

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Easy problems can admit complicated analytic solutions!

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Summary of Part 2: Solution Concepts

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- Dynamic programming gives necessary and sufficient conditions for optimal open-loop and closed-loop controls
- The PMP gives necessary conditions for an optimal open-loop control
- In high dimensions, finding $\lambda^\star$ is usually much easier than finding V

Outline of the course

Part 1: Optimal Control Problems — An Introduction

Part 2: Solution Concepts

Part 3: Numerical Solution Methods

Part 4: Model Predictive Control

Part 5: Deep Neural Networks for High-Dimensional Problems

Purpose of this part

Introduce numerical methods for solving optimal control problems

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Compare methods that compute optimal feedback laws with methods computing open-loop optimal controls and trajectories

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Introduce numerical methods for solving optimal control problems

Compare methods that compute optimal feedback laws with methods computing open-loop optimal controls and trajectories

Discuss the numerical effort

Contents of this part

Part 3: Numerical Solution Methods

- Methods based on Dynamic Programming, computing feedback laws
  - Classical methods for Hamilton-Jacobi-Bellman equations
  - Deep Reinforcement Learning
- Methods computing open-loop optimal controls
  - Methods based on Pontryagin's Maximum Principle
  - Direct Solution Methods

Methods based on Dynamic Programming

Conceptual approach

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in discrete time and

$$DV(x)f(x, F^\star(x)) + \ell(x, F^\star(x)) = \inf_{u \in \mathbb{U}} \{DV(x)f(x, u) + \ell(x, u)\}$$

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We start with the continuous-time case using the Hamilton-Jacobi-Bellman equation

Numerical solution of the HJB equation

Since the Hamilton-Jacobi-Bellman equation

$$\inf_{u \in \mathbb{U}} \{-\delta V(x_0) + DV(x_0) f(x_0, u) + \ell(x_0, u)\} = 0$$

is a partial differential equation, many standard numerical methods for PDEs can be applied. In the literature, one can find finite differences, finite elements, finite volume methods etc.

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Here we explain the so-called semi-Lagrangian discretisation[11], which is tailored to HJB equations

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Its advantages are unconditional stability and the possibility to obtain error bounds for the feedback law computed from the approximation to V

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Semi-Lagrangian discretisation

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where $\tilde{x}_u(h,x)$ is a numerical approximation of the solution $x_{\mathbf{u}}(h,x)$ with $\mathbf{u} \equiv u$, e.g., the Euler approximation $\tilde{x}_u(h,x) = x + hf(x,u)$

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Rearrange the terms to obtain the semi-discretised equation

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(effectively, this step reverses part of the derivation of the HJB equation)

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Note: Control input constraints are readily implementable, but state constraints lead to boundary conditions that may complicate the computation

2d Example: Optimal Investment in a Firm

[Feichtinger et al. '00, Gr./Semmler '04]

$x_1 =$ invested capital, $x_2 =$ investment, $u =$ change of investment

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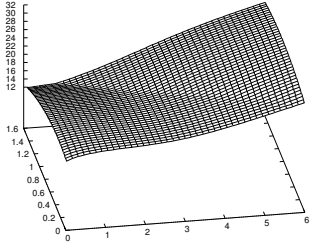

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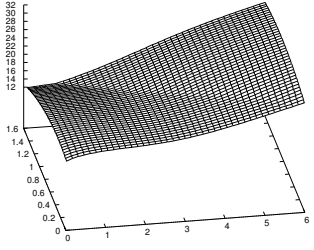

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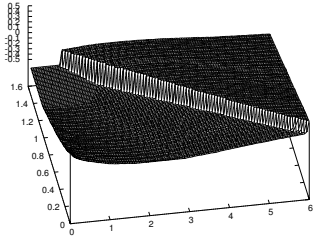

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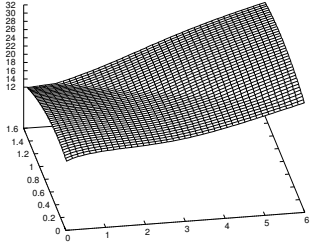

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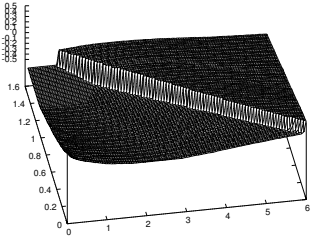

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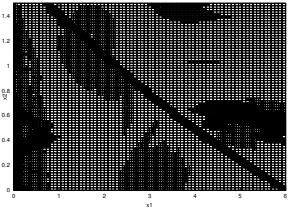

discretization grid

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if we write $g(x,u) = \tilde{x}_u(h,x)$, $\beta = e^{-\delta h}$, ℓ in place of $h\ell$, and V in place of V_h

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- Any method for discrete-time problems can be used in place of Step 2

Solving the Bellman equation

How to find $V \in \mathcal{F}_k$ satisfying

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First, since $\mathcal{F}_k$ is finite dimensional, it suffices to check the equation for x from a finite set $C_k \subset C$

In the classical approaches, V is then obtained iteratively:

- value iteration: $V_{i+1}(x) := \Pi_k \left(\inf_{u \in U} \{\ell(x,u) + \beta V_i(g(x,u))\} \right) \quad \forall\, x \in C_k$

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Value iteration converges linearly, policy iteration quadratically

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An iterative algorithm for obtaining Q is classical Q-learning:

$Q_{i+1}(x_i,u_i) := \ell(x_i,u_i) + \beta \inf_{u\in U} Q_i(x_i^+,u)$ for a sequence of experimental or simulated data x_i, u_i, x_i^+ with $x_i^+ = g(x_i,u_i)$. Often, $x_{i+1} = x_i^+$ is chosen

Deep Reinforcement Learning

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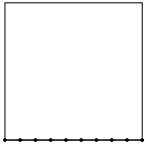

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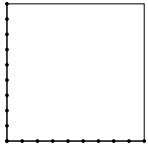

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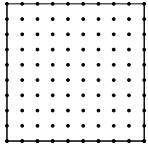

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Besides various technical issues, the main conceptual challenge of numerically approximating V is the notorious curse of dimensionality:

If we solve the Bellman equation with grid- or mesh based spatial approximation techniques on $C \subset \mathbb{R}^n$, then the numerical effort grows exponentially with the space dimension n

Example: Assume 10 discretisation steps are needed in each coordinate direction. Then we obtain

100	discretisation points if $n = 2$
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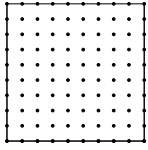

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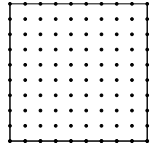

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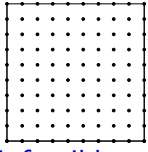


The problem quickly becomes computationally infeasible

Deep Learning approaches for computing V or Q

In deep learning approaches, there is no explicit space discretization via a grid or mesh. Rather the functions V or Q are approximations by deep Neural Networks

For those not familiar with deep neural networks, we provide a brief introduction

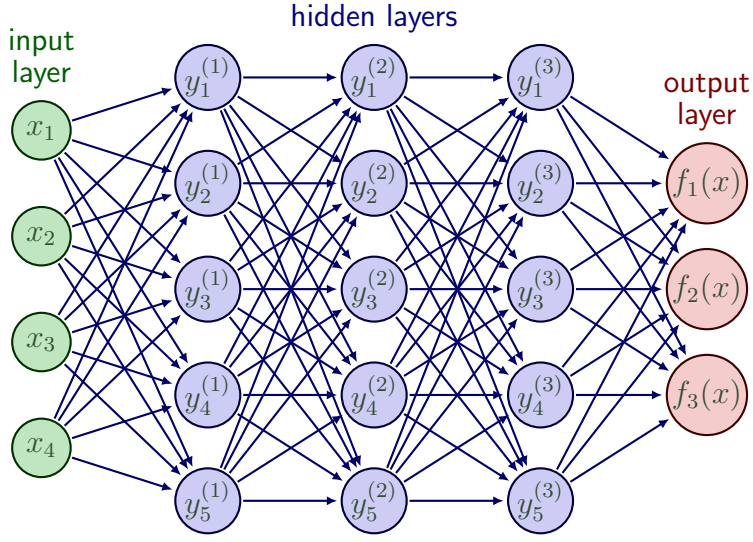
Excursion: Neural Networks

Quiz

Which of the following objects does best describe the nature of a neural network?

- A matrix
- A function
- A sequence
- An equation

Neural Networks



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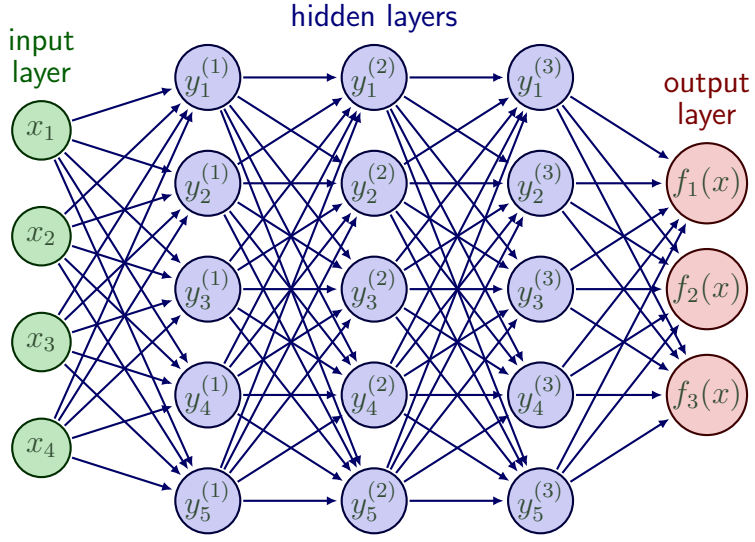

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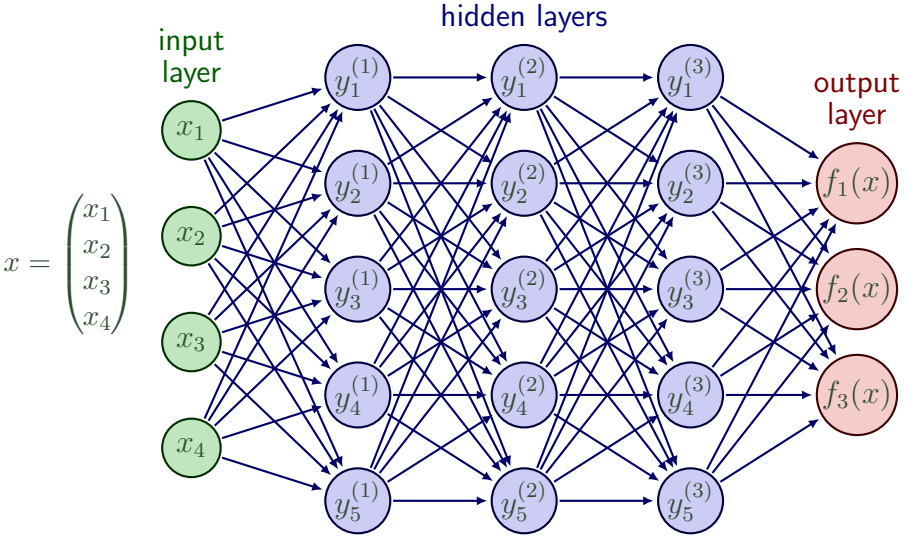

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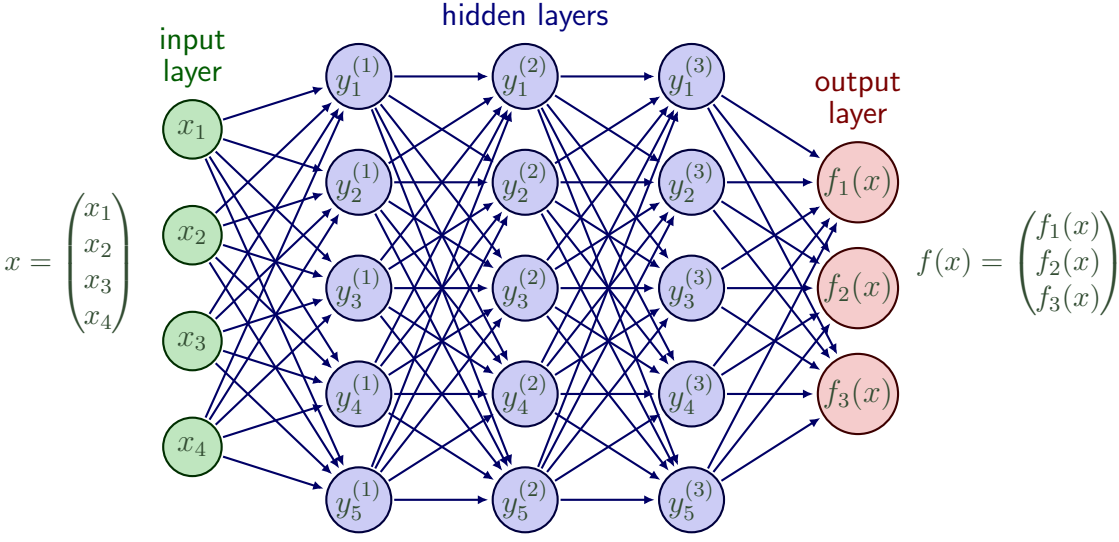

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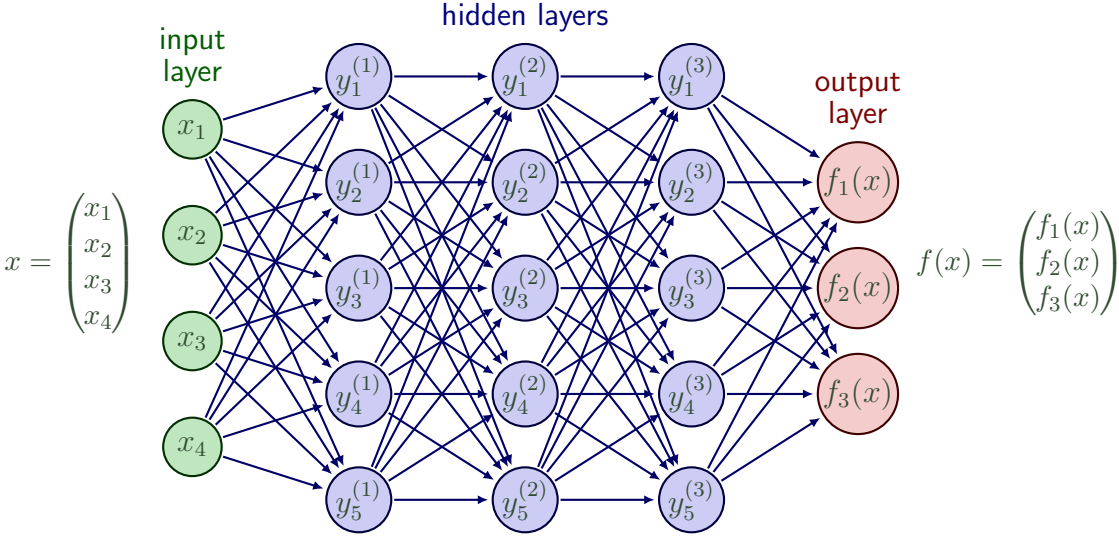

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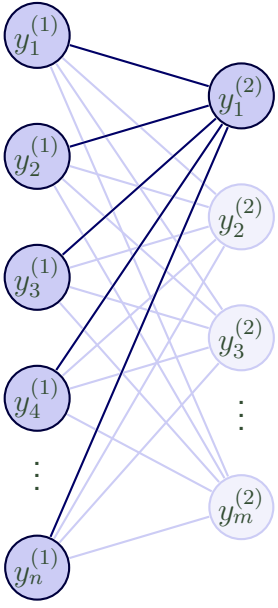

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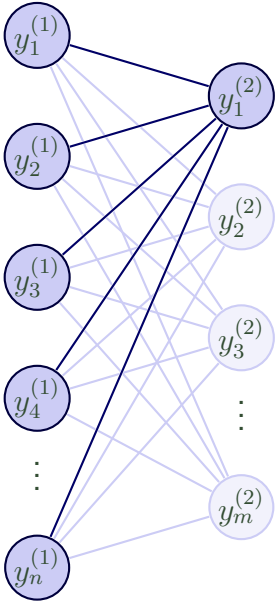

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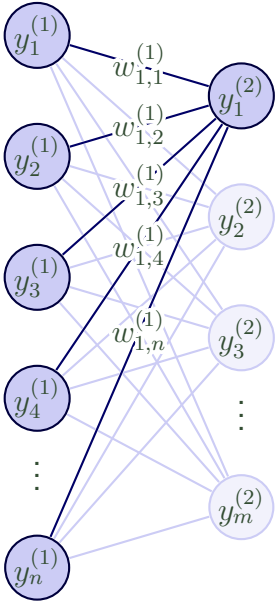

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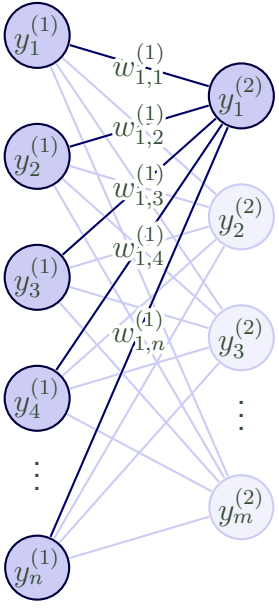

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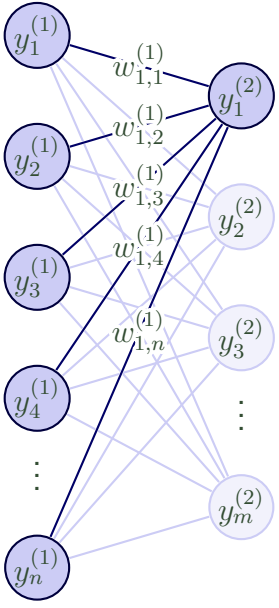

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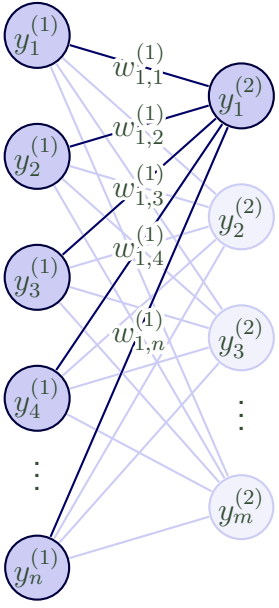

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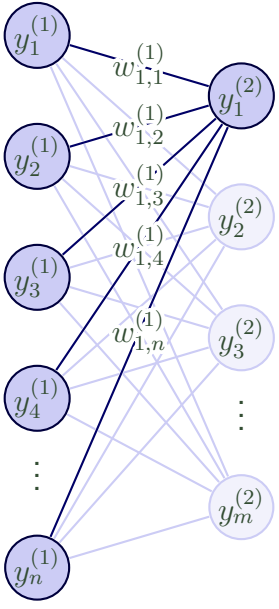

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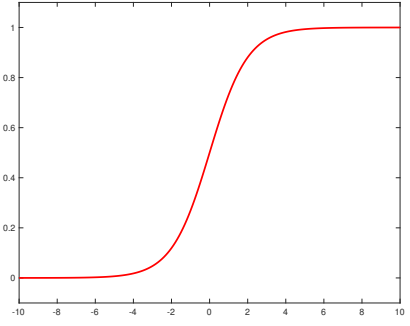

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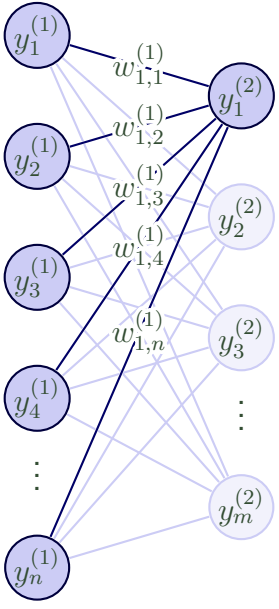

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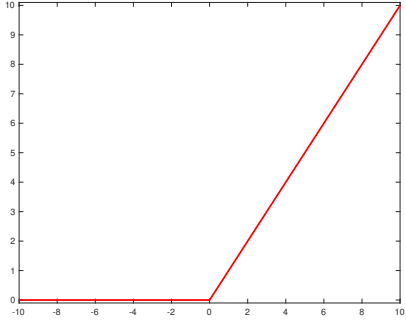

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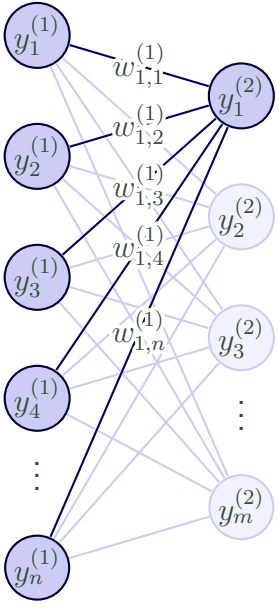

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softplus:
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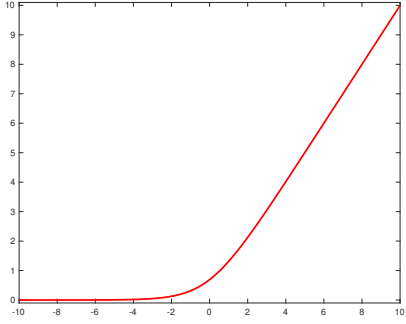

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This is the standard loss function for regression ("supervised learning"), i.e., for learning a function with prescribed values at the data points

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$\rightarrow$ Example: Learning a sine function

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for data x_j or, respectively,

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$$\frac{1}{N}\sum_{j=1}^{N} |\delta V_{NN}(x_j) + \inf_{u\in U}\{\ell(x_j,u) + DV_{NN}(x_j)f(x_j,u))\}|^2$$

Deep Learning approaches, implementation details

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Minimising these loss functions is a hard problem

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- Instead of computing the "inf" in L, an approximate minimizer $F_{NN,i}$ is stored in a second Neural Network $\rightsquigarrow$ "actor-critic method"
- Once an approximation to F is stored, the fact that F optimizes the cost can be used in the design of the loss function $\rightsquigarrow$ "policy gradient method"

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- Data points are often generated along trajectories, as long as constraints are violated or a pre-defined goal is reached $\rightsquigarrow$ "episodes"

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We will see in the Part 5 of this course whether Deep Learning can really do better than grid- or meshed-based approaches

Methods computing open-loop optimal controls

How to solve OCPs for open-loop controls?

Core challenge: How to compute the function $\mathbf{u}^\star(\cdot)$?

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Methods based on Pontryagin's Maximum Principle
"Indirect Solution Methods"

Basic idea of indirect shooting

"First optimise, then discretise"

$$\min_{\mathbf{u}(\cdot)} \quad \int_{t_0}^{t_f} \ell(t, x(t), \mathbf{u}(t))dt + L(t_f, x(t_f))$$

subject to: (P)

$$\begin{aligned} \dot{x} &= f(t, x, \mathbf{u}), \quad x(t_0) = x_0 \\ \mathbf{u}(\cdot) &\in L_\infty([t_0, t_f], \mathbb{U}) \\ 0 &= \Psi(t_f, x(t_f)) \end{aligned}$$

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NCOs:

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A basic indirect shooting algorithm

Choose $\varepsilon > 0$. Guess λ_0^0, ν^0. Set $k = 0$

1. Solve numerically from t_0 to t_f

$$\begin{aligned} \dot{x}^k &= \quad H_\lambda(t, x^k, \mathbf{u}^k, \lambda^k), \quad x(t_0)^k = x_0 \\ \dot{\lambda}^k &= -H_x(t, x^k, \mathbf{u}^k, \lambda^k), \quad \lambda(t_0)^k = \lambda_0^k \\ 0 &= \quad H_u(t, x^k, \mathbf{u}^k, \lambda^k) \end{aligned}$$

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2. Compute defect of transversality and terminal conditions:

$$\mathcal{F}(\lambda_0^k, \nu^k) \doteq \begin{pmatrix} \lambda(t_f)^k - \quad L_x(t_f, x(t_f)^k) - (\nu^k)^\top \Psi_x(t_f, x(t_f)^k) \\ \Psi(t_f, x(t_f)^k) \end{pmatrix}$$

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3. If $\|\mathcal{F}(\lambda_0^k, \nu^k)\| \leq \varepsilon \quad \rightarrow \quad$ STOP
4. Update λ_0^k, ν^k to enforce $\mathcal{F}(\lambda_0^k, \nu^k) \to 0$. $k \leftarrow k + 1$. GOTO Step 1

How to do Step 4?

- We want

$$\lim_{k\to\infty} \mathcal{F}(\lambda_0^k, \nu^k) = 0.$$

This is a root finding problem, e.g., we can apply Newton's method[12]

[12] Or any quasi Newton method, also stepsize parameters could be introduced.

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- Update

$$\begin{pmatrix} \lambda_0^{k+1} \\ \nu^{k+1} \end{pmatrix} = \begin{pmatrix} \lambda_0^k \\ \nu^k \end{pmatrix} + \begin{pmatrix} \Delta\lambda^k \\ \Delta\nu^k \end{pmatrix}.$$

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Example

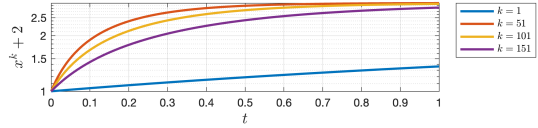
$$\min_{\mathbf{u}(\cdot)} \quad \int_0^1 \frac{1}{2}\mathbf{u}^2(t)dt$$

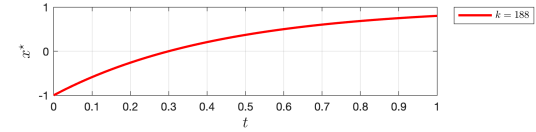
subject to

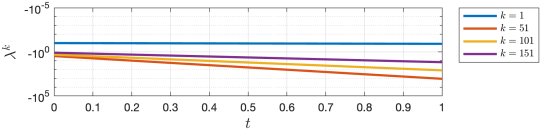
$$\dot{x}(t) = \mathbf{u}(t)(1 - x(t))$$
$$x(0) = x_0,\ x(1) = x_1$$

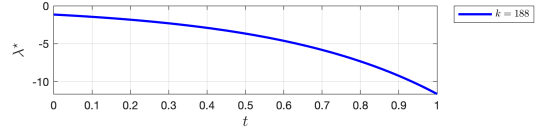
Example

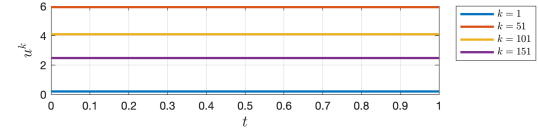
Indirect shooting as sketched above: $\nu^0 = 0, \lambda_0^0 = -0.1, x_0 = -1, x_1 = 0.75$

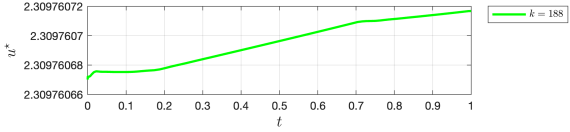












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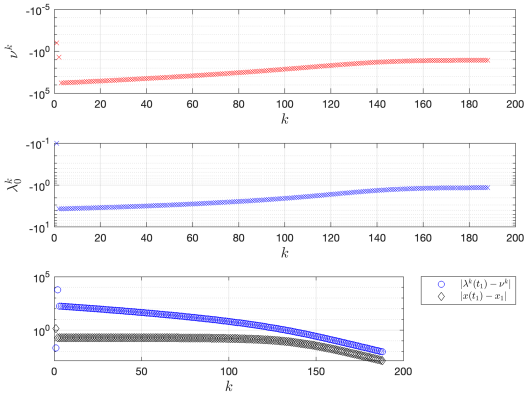

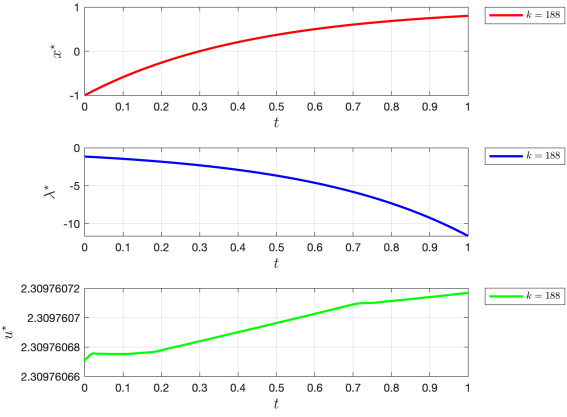

$\nu^\star = -11.51$; $\lambda_0^\star = -1.1543$, $|\lambda^\star(t_f) - \nu^\star| = 9.4 \cdot 10^{-3}$, $|x^\star(t_f) - x_1| = 1.43 \cdot 10^{-3}$

Remarks on indirect methods

- Requires to set up the NCOs from the PMP

[13] T. Englert, A. Völz, F. Mesmer, S. Rhein, K. Graichen. "A software framework for embedded nonlinear model predictive control using a gradient-based augmented Lagrangian approach (GRAMPC)". In: Optimization and Engineering 20.3 (2019), pp. 769–809

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- Either adjoint dynamics or state dynamics are unstable

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Direct Solution Methods

Direct solution approaches – Main idea

"First discretize, then optimize"

Convert OCP

$$\min_{\mathbf{u}(\cdot)} J(x_0, \mathbf{u}(\cdot))$$

subject to

$$\dot{x}(t) = f(t, x(t), \mathbf{u}(t)), \quad x(t_0) = x_0$$
$$\forall t \in [t_0, t_f] : \mathbf{u}(t) \in \mathbb{U} \subseteq \mathbb{R}^m$$
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into nonlinear opt. problem (NLP)

$$\min_{\xi \in \mathbb{R}^{n_\xi}} \mathrm{f}(\xi)$$

subject to

$$\mathrm{h}(\xi) = 0$$
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Optimization in a function space

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Optimization in a finite-dimensional vector space

- How to discretize $\mathbf{u}(\cdot)$? How to obtain a solution to $\dot{x} = f(x, u)$?

Input discretization?

Parametrize $\mathbf{u}(\cdot)$ by finitely many parameters u_k, $k = 1, \ldots, N_{opt}$

$$\mathbf{u}(\cdot) \in L_\infty([t_0, t_f], \mathbb{U}) \quad \rightsquigarrow \quad \mathbf{u}(t) = \sum_{k=1}^{N_{opt}} u_k \phi_k(t)$$

with basis functions $\phi_k(\cdot)$

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Which basis functions to use?

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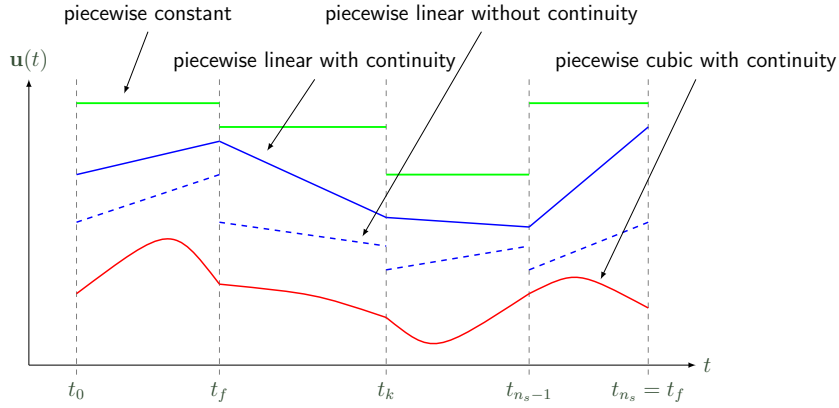

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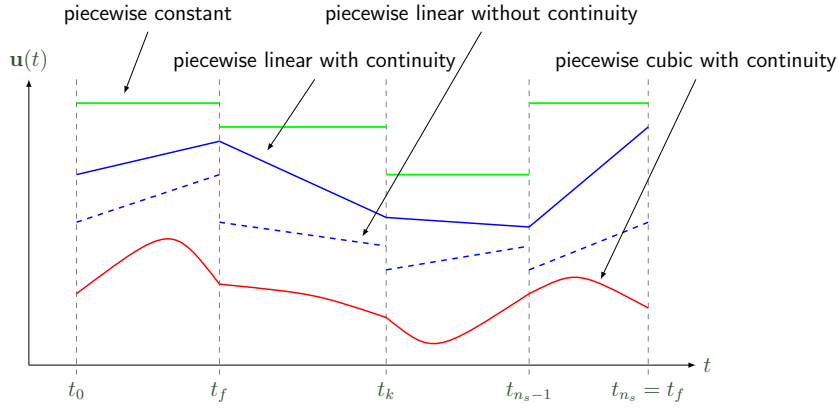

Very often, one simply uses piecewise constant input parametrizations

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Simplest choice: Euler scheme $g(t, x, u) = x + hf(t, x, u)$

(but more sophisticated schemes like Heun, classical Runge-Kutta, or implicit methods may be advantageous)

Considered OCP

$$\min_{\mathbf{u}(\cdot),p} L(t_f, x(t_f))$$

subject to (OCP)

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For the illustration of the procedure we consider a problem without terminal conditions. This can be adapted to most common settings, such as

- free end time problems
- periodic boundary conditions
- free initial conditions
- ...

Direct single shooting (simultaneous)

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$$\min_{u(0),\ldots,u(N-1),x(N),p} L(N, x(N)) \qquad \text{subject to}$$
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- Unstable systems lead to highly sensitive dependence of the solution from the control input $\rightsquigarrow$ NLP difficult to solve

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- Introduce $N+1$ (vector valued) additional decision variables $\xi(0), \ldots, \xi(N-1)$
- Solve ODE separately on N small intervals, enforce continuity through additional constraints upon convergence

$$\min_{u(0),\ldots,u(N-1),\xi(0),\ldots,\xi(N-1),p} L(N, x(N)) \quad \text{subject to}$$

$$\forall k \in \{0, \ldots, N-1\} : x(k+1) = g(k, \xi(k), u(k)),$$
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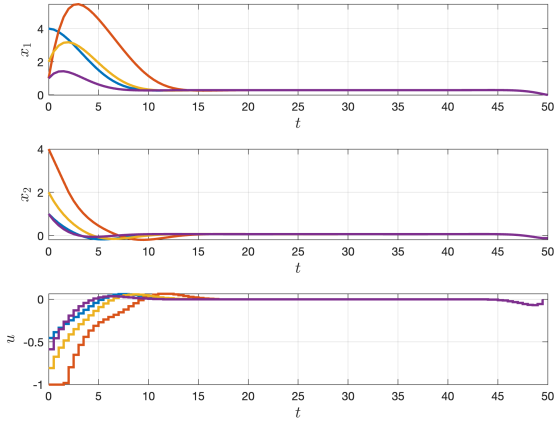
Example – Direct multiple shooting



Revisit the optimal investment problem: $x_1 =$ invested capital, $x_2 =$ investment, $\mathbf{u}$ $=$ change of investment

Solved with CasADi[13] and ipopt

[14] Joel AE Andersson, Joris Gillis, Greg Horn, James B Rawlings, Moritz Diehl. "CasADi: a software framework for nonlinear optimization and optimal control". In: Mathematical Programming Computation 11 (2019), pp. 1–36

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- Workhorse method for many MPC implementations, see Part 4 of the course

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- For further introductory reading, see Chapter 5 of [Chachuat '09][16]

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The last two parts of this course present techniques that overcome this limitation for optimal control problems with particular properties

Outline of the course

Part 1: Optimal Control Problems — An Introduction

Part 2: Solution Concepts

Part 3: Numerical Solution Methods

Part 4: Model Predictive Control

Part 5: Deep Neural Networks for High-Dimensional Problems

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Contents of this part

Part 4: Model Predictive Control

- Model Predictive Control
- The Turnpike Property
- Main Performance Result
- Use Case: Optimal Startup of a Combined Cycle Power Plant

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with fixed $N \in \mathbb{N}$. How do we get a feedback law F?

MPC from the trajectory point of view

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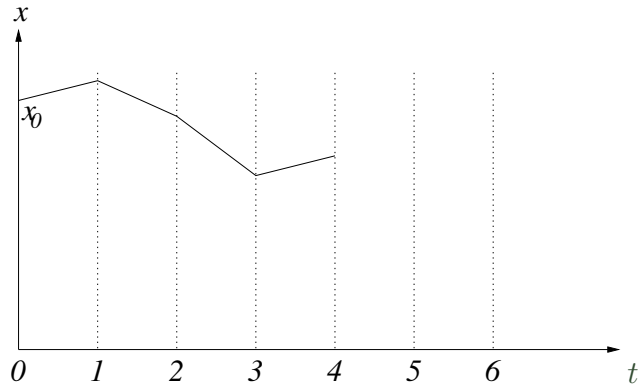

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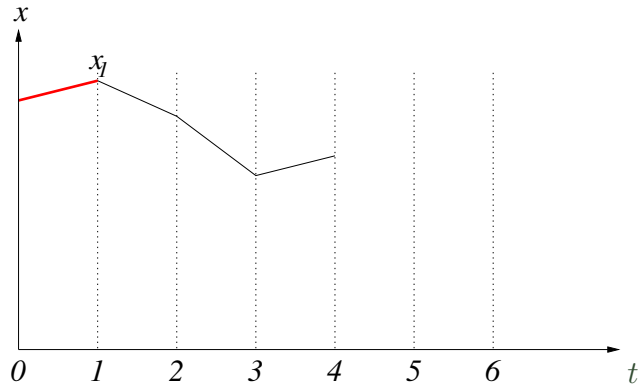

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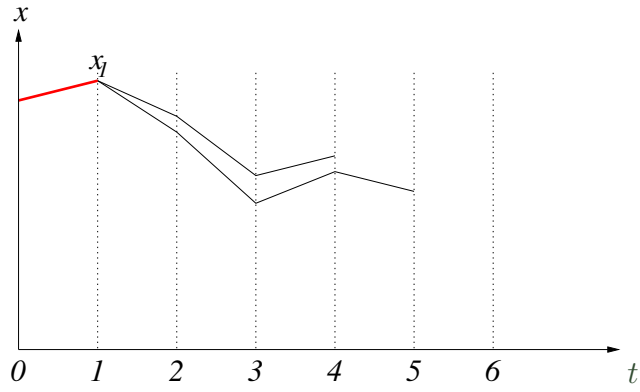

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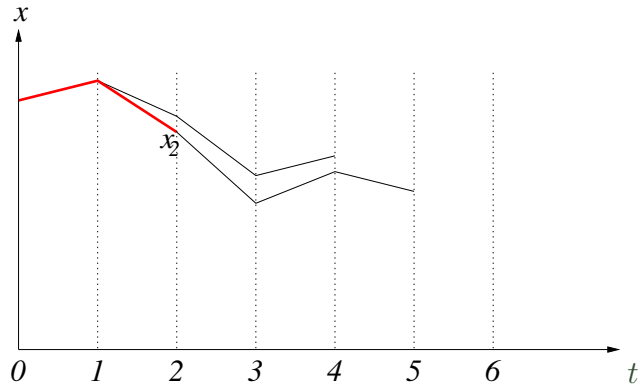

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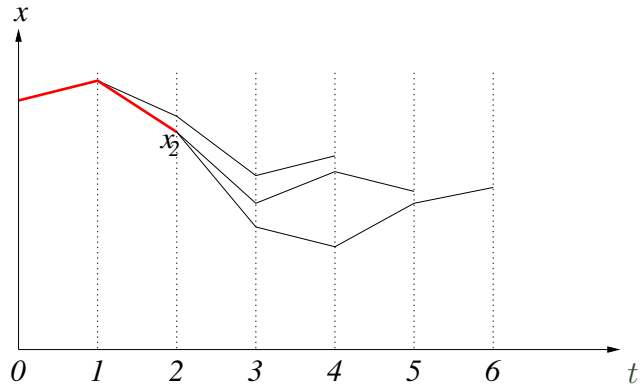

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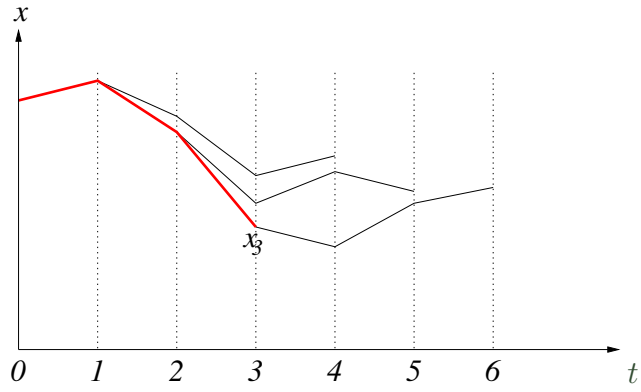

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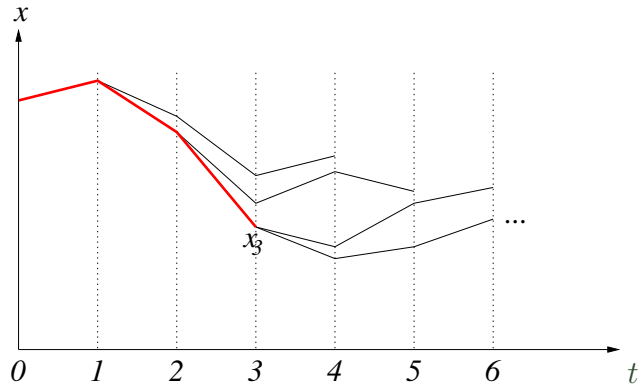

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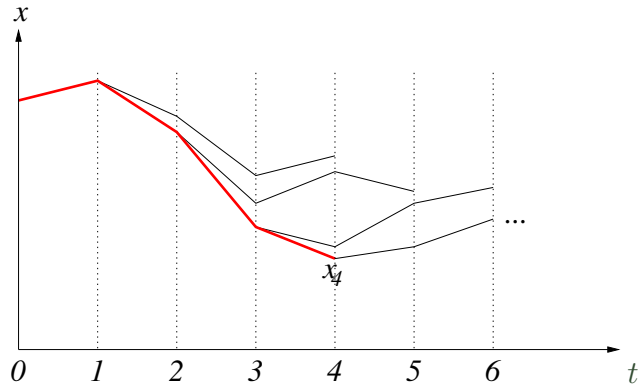

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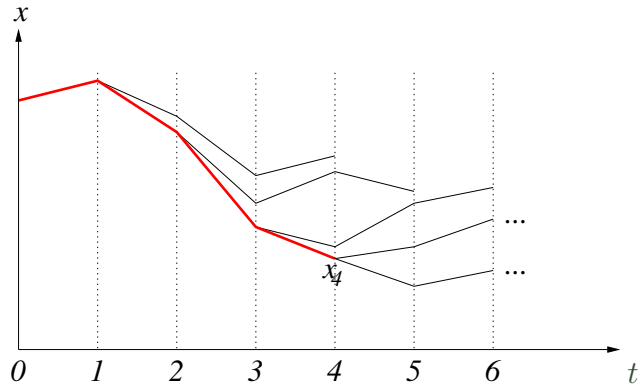

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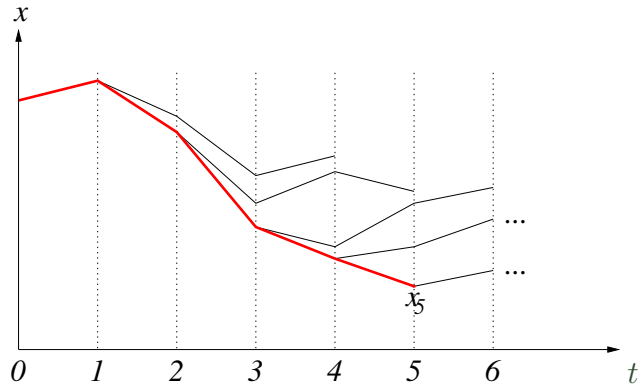

black = predictions (open loop optimisation)

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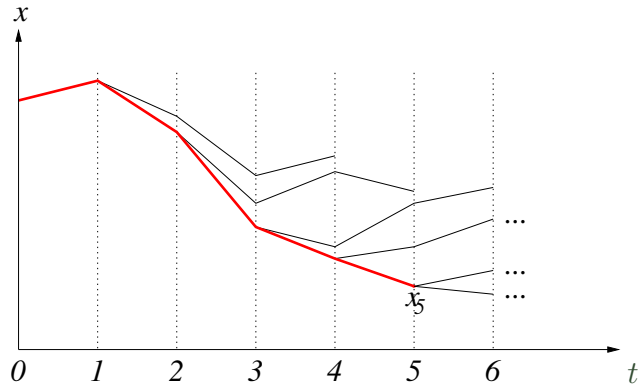

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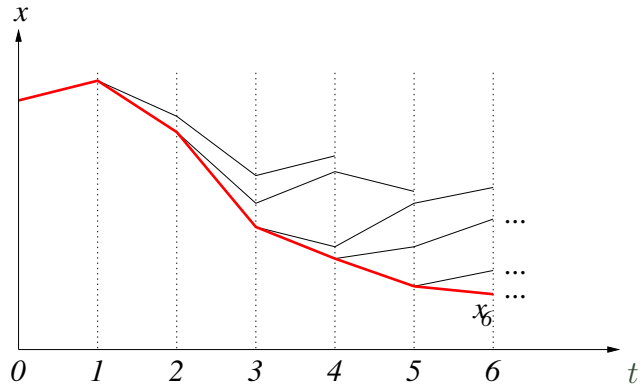

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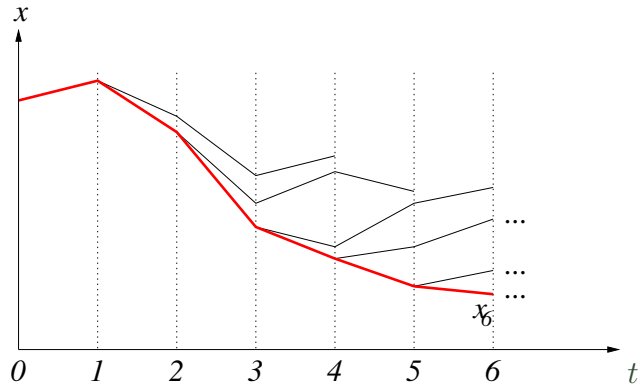

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The feedback control value $F(x_{MPC}(t))$ is evaluated online as the first element of the finite horizon optimal control sequence

The turnpike property

The turnpike property

—

a temporal redundancy property

The turnpike property

The turnpike property describes a behaviour of (approximately) optimal trajectories for a finite horizon optimal control problem

$$\underset{\mathbf{u}}{\text{minimise}} \quad J_N(x, \mathbf{u}) = \sum_{t=0}^{N-1} \ell(x_{\mathbf{u}}(t), \mathbf{u}(t))$$

with a cost function $\ell : X \times U \to \mathbb{R}$
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We illustrate the property by two simple examples

Example: minimum energy control

Example: Keep the state of the system inside a given interval X minimising the quadratic control effort

$$\ell(x,u) = u^2$$

with dynamics

$$x^+ = 2x + u$$

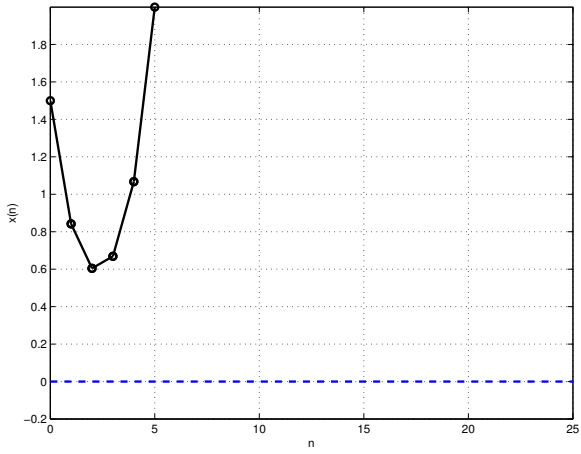
and constraints $\mathbb{X} = [-2,2]$, $\mathbb{U} = [-3,3]$

Quiz

Which is the "cheapest" state for keeping the system inside $\mathbb{X} = [-2, 2]$?

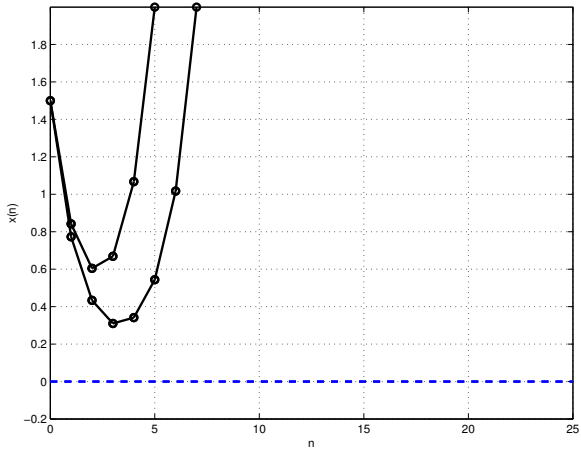
- $x = 2$
- $x = 0$
- Any $|x| < \varepsilon$ for $\varepsilon > 0$ sufficiently small

Example: optimal trajectories



Optimal trajectory for $N = 5$

Example: optimal trajectories



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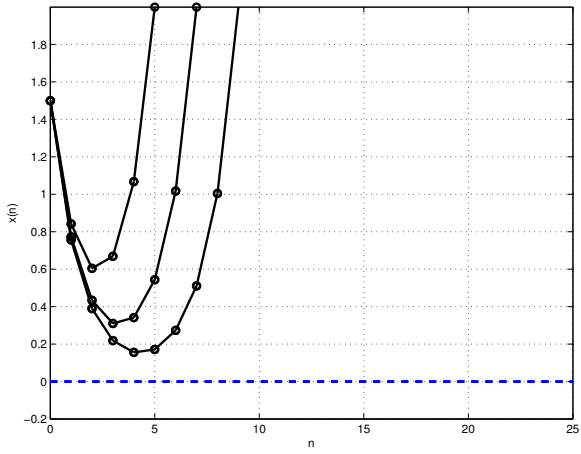

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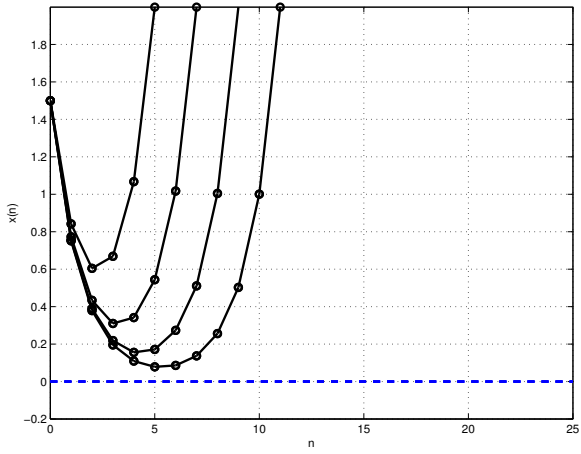

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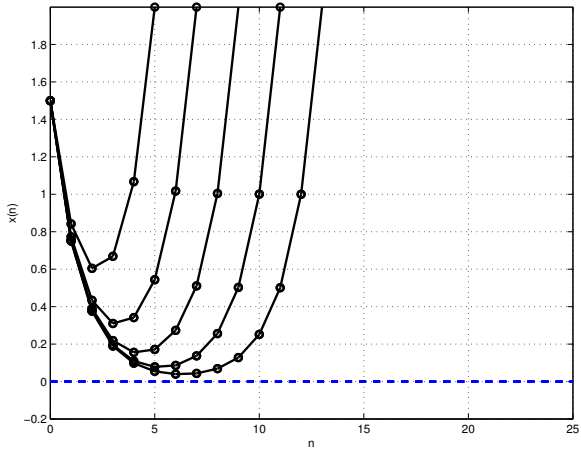

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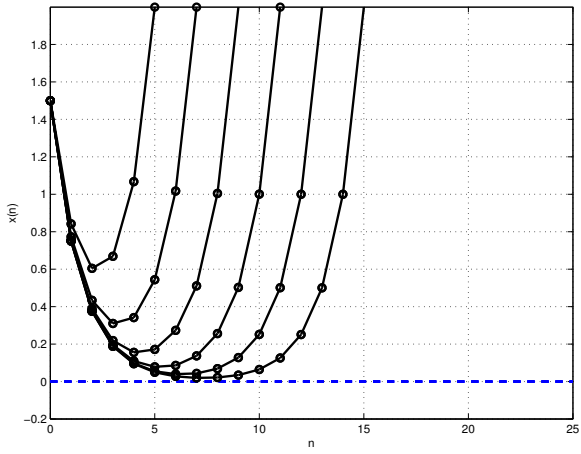

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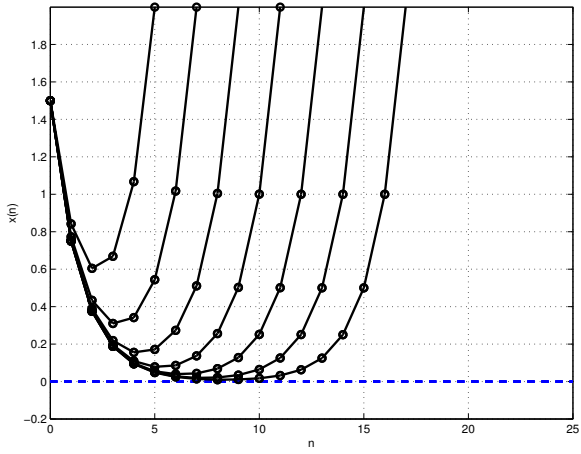

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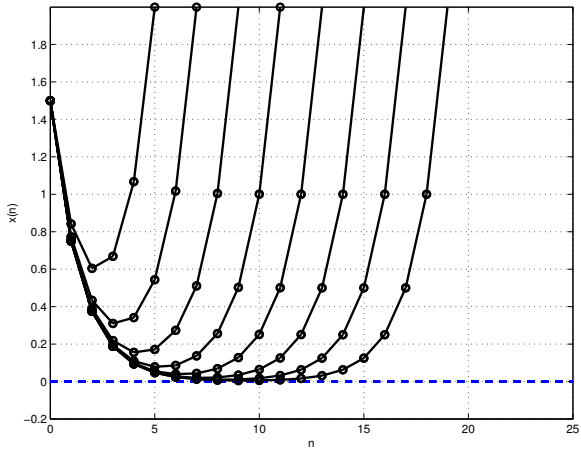

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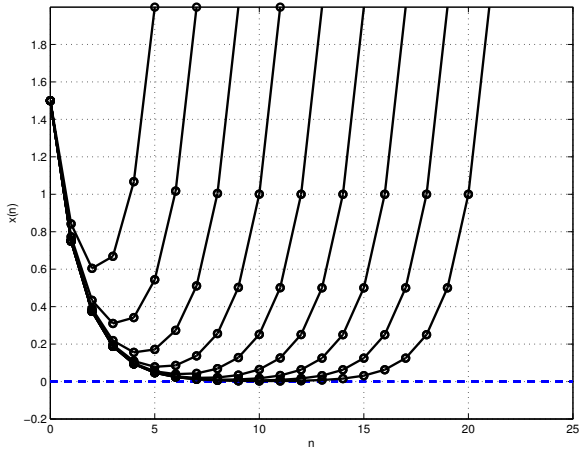

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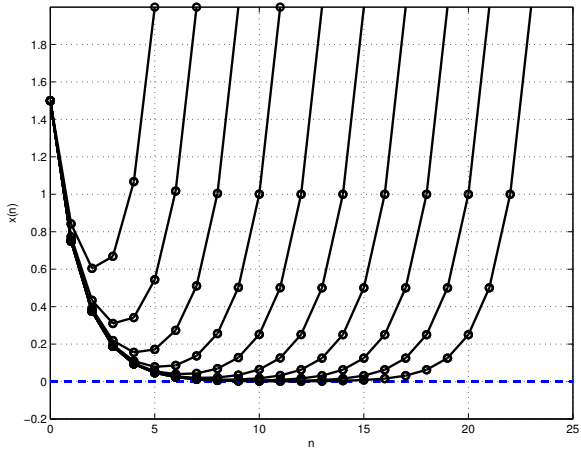

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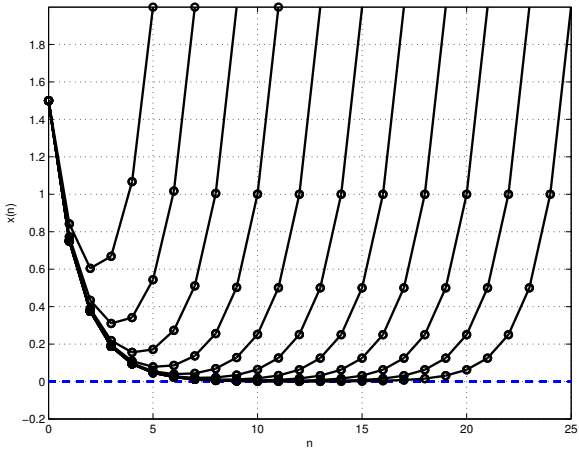

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Example: a macroeconomic model

Consider a classical 1d macroeconomic model

[Brock/Mirman '72]

Minimise the finite horizon objective $\sum_{t=0}^{N-1} \ell(x_{\mathbf{u}}(t), \mathbf{u}(t))$ with

$$\ell(x,u) = -\ln(Ax^\alpha - u), \quad A = 5,\ \alpha = 0.34$$

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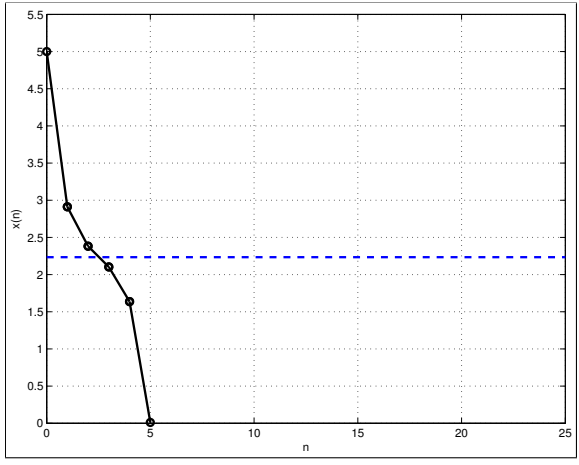
On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e, u^e) \approx 1.4673$$

Example: optimal trajectories



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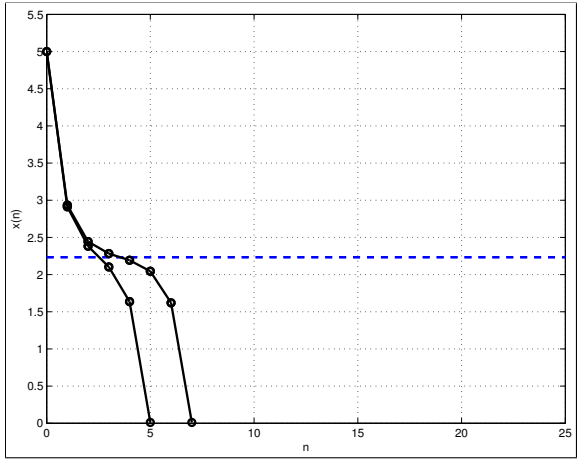

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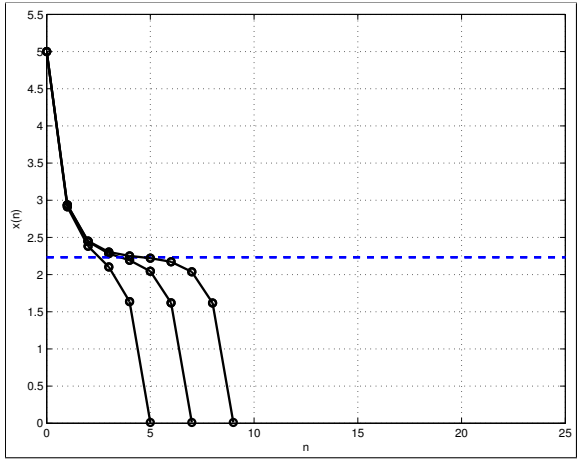

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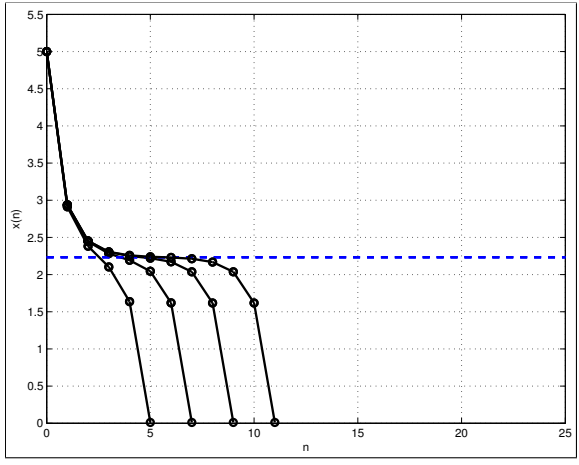

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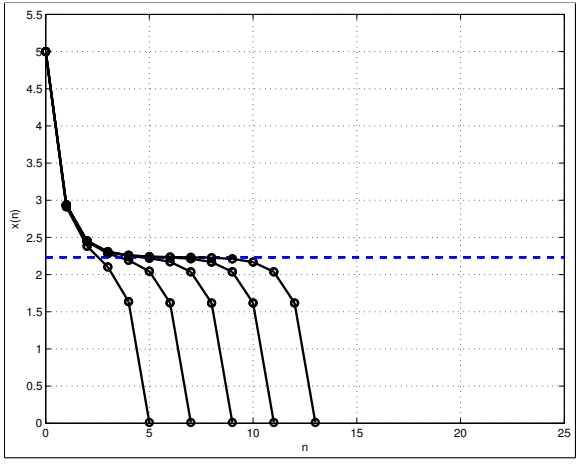

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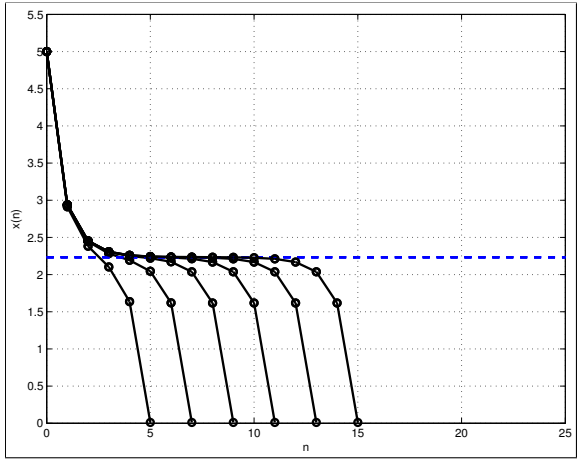

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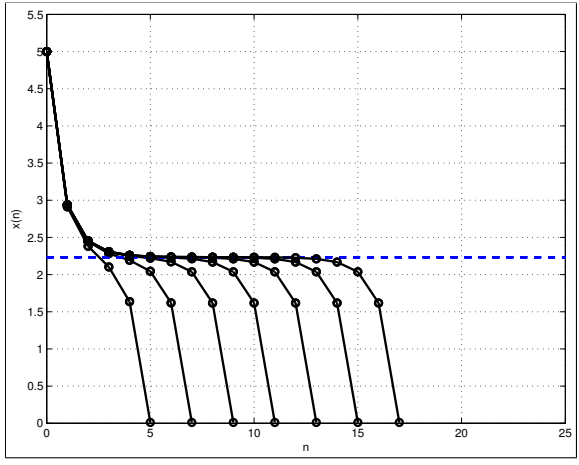

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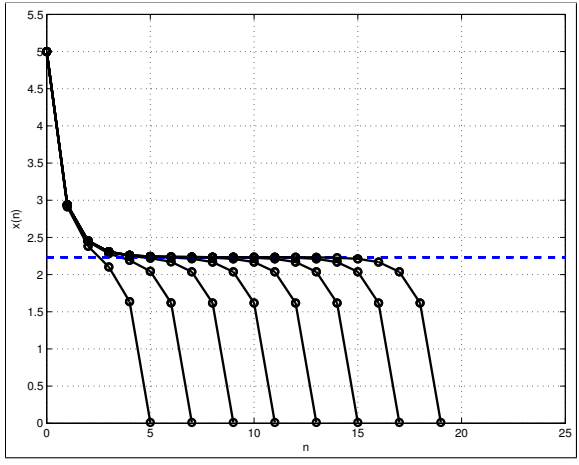

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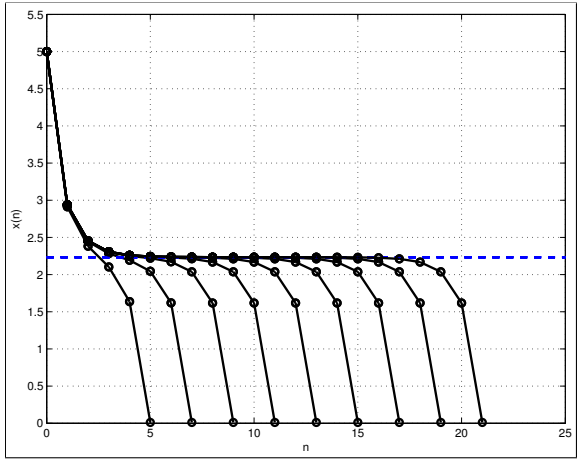

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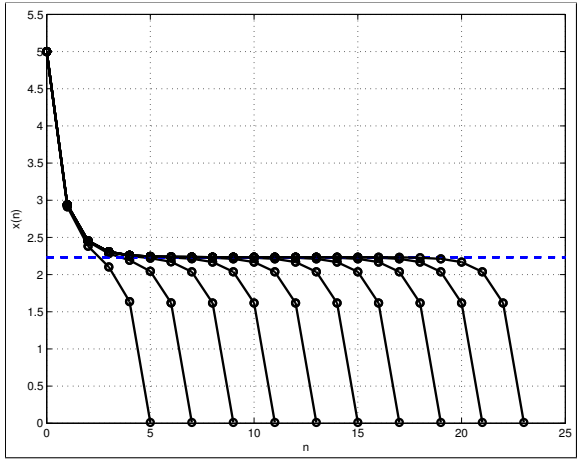

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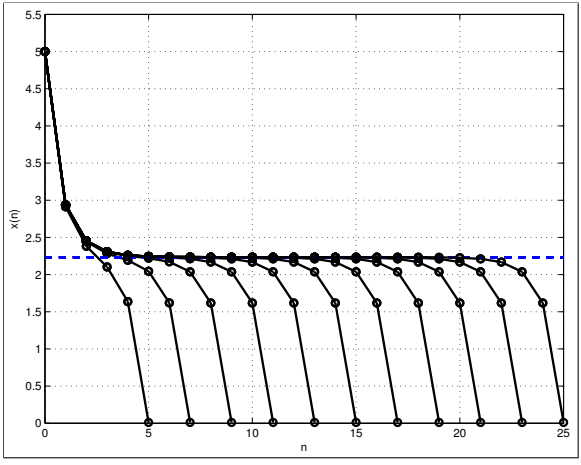

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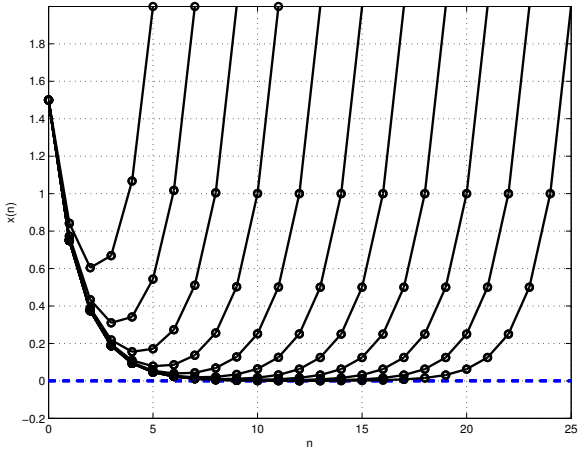
How to formalise the turnpike property?



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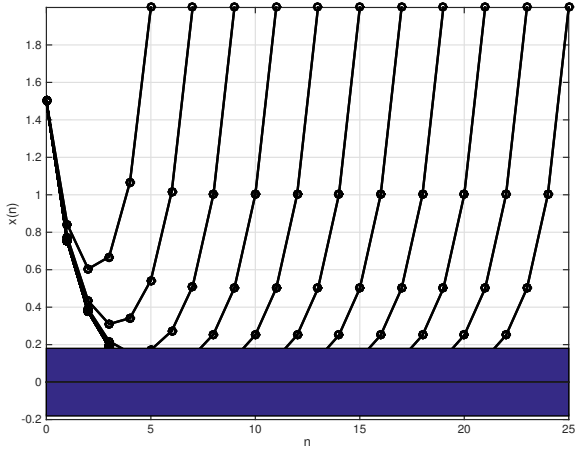

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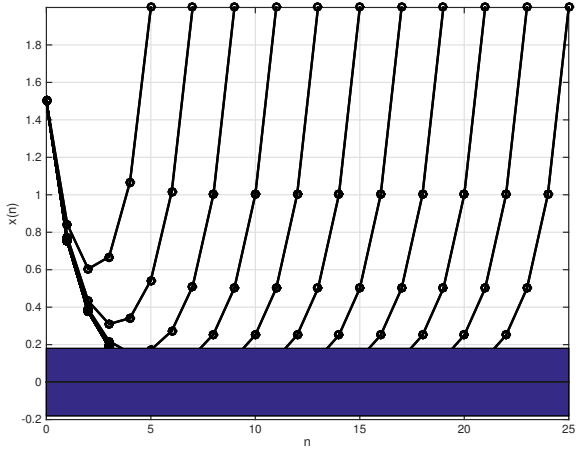

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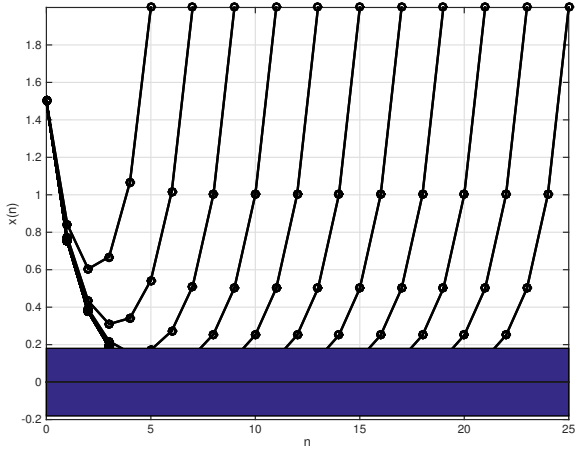

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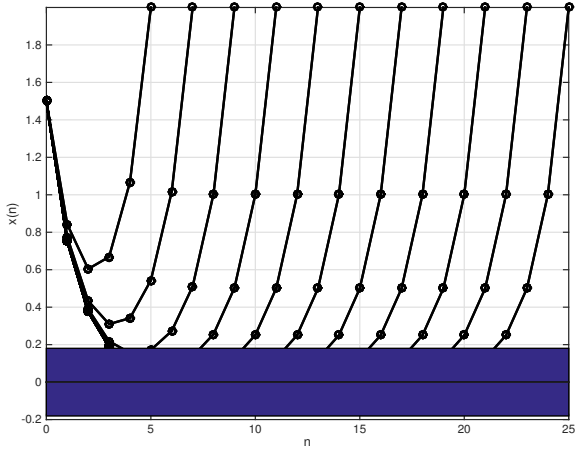

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In continuous time: The Lebesgue measure replaces the counting of points

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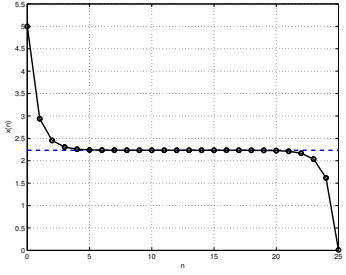

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- Selected applications:
  - synthesis of optimal trajectories [Anderson/Kokotovic '87]
  - learning in neural ODEs [Esteve-Yagüe/Geshkovski/Pighin/Zuazua '21ff, Püttschneider/Faulwasser '24f]
  - optimization based estimation [Schiller/Gr./Müller '24f]
  - model predictive control [will be explained in a few minutes]

Tracking type problems

For a stabilizable equilibrium x^s with control u^s, i.e., $g(x^s, u^s) = x^s$, the tracking cost

$$\ell(x^s, u^s) := \|x - x^s\| + \mu\|u - u^s\|$$

for $\mu \geq 0$ defines an optimal control problem with turnpike property at $x^e = x^s$

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Strict Dissipativity is a key to understanding how large it is

Strict dissipativity

The optimal control problem is called strictly dissipative on X, if there exists a storage function $\lambda : X \to \mathbb{R}$, bounded from below, and $\alpha \in \mathcal{K}$ such that for all $x \in X$, $u \in U$ with $g(x,u) \in X$:

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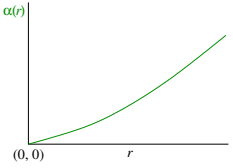
$\alpha \in \mathcal{K}$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$



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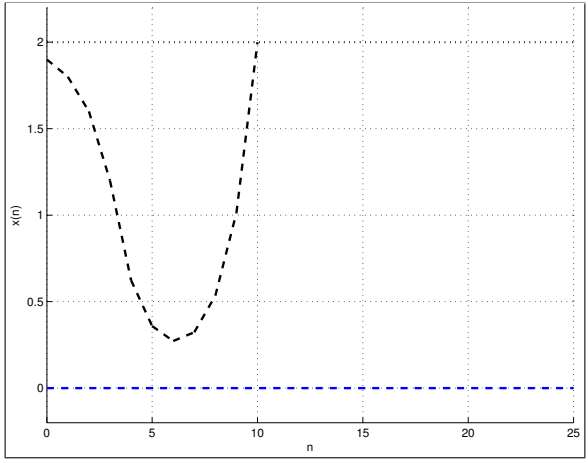
Main Performance Result

MPC in presence of the turnpike property



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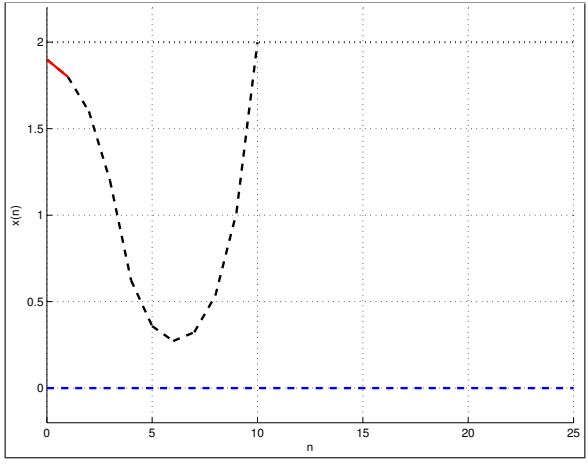

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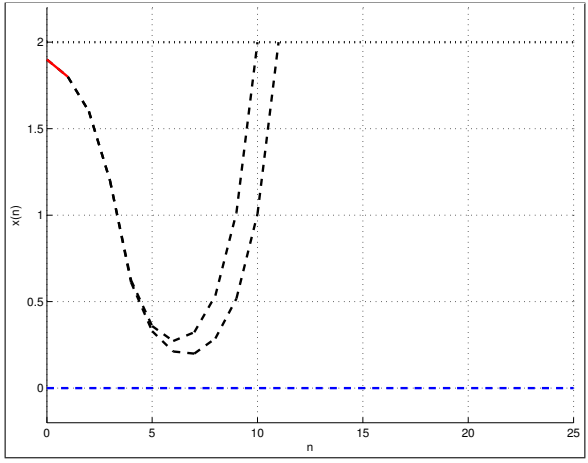

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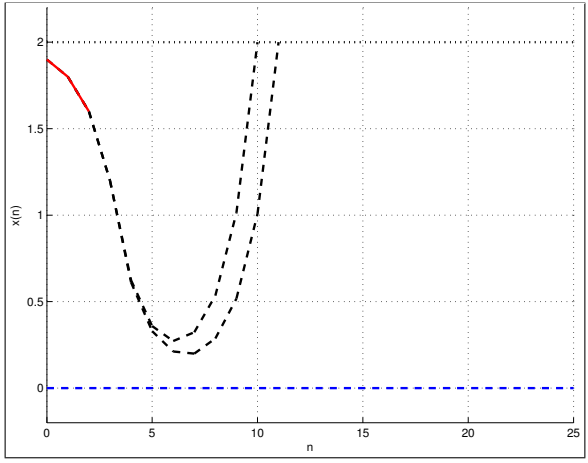

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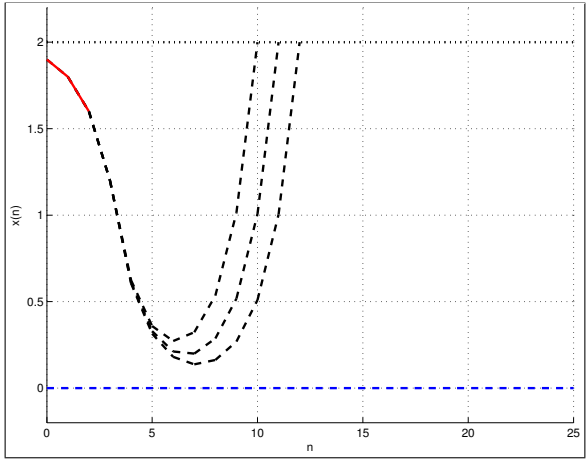

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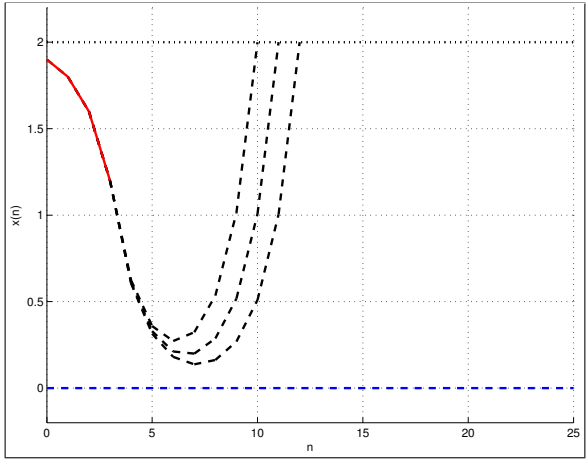

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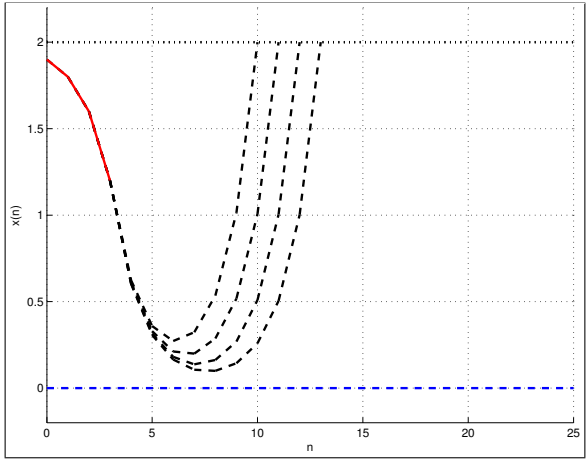

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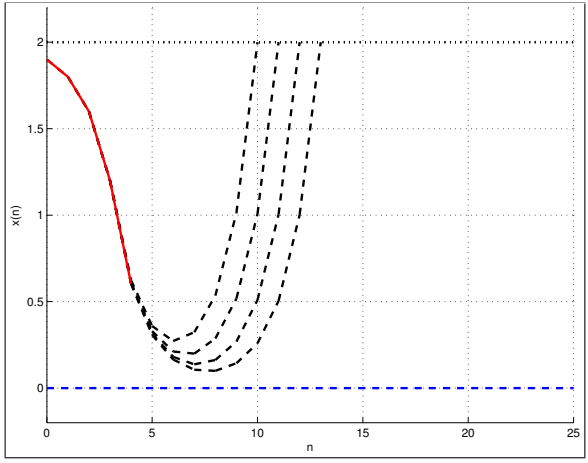

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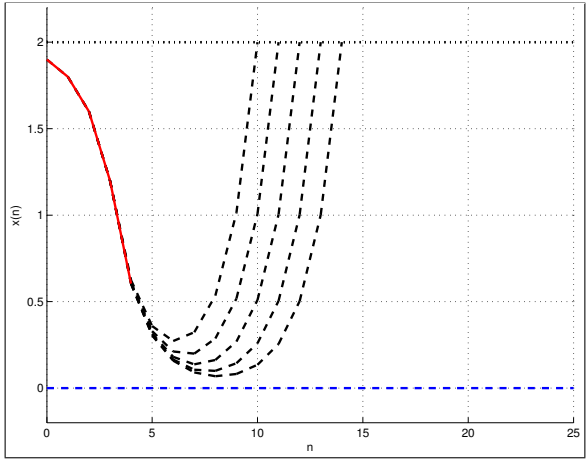

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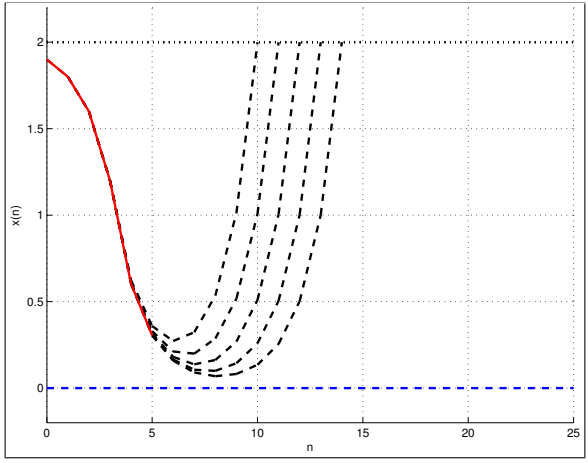

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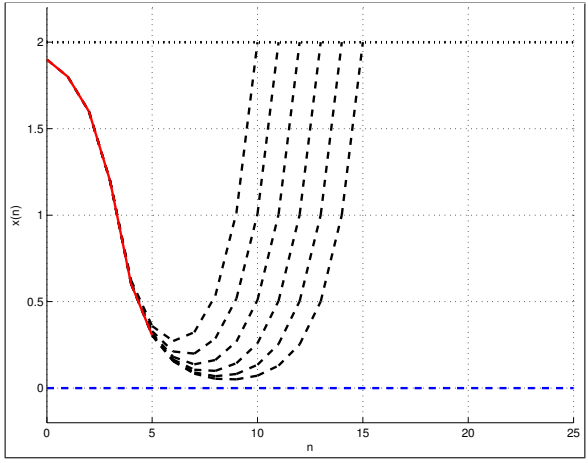

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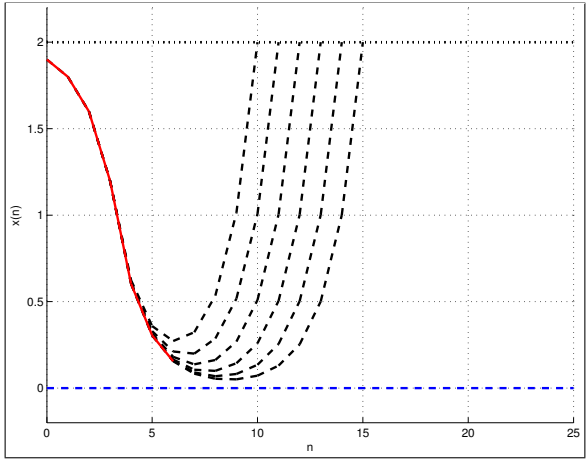

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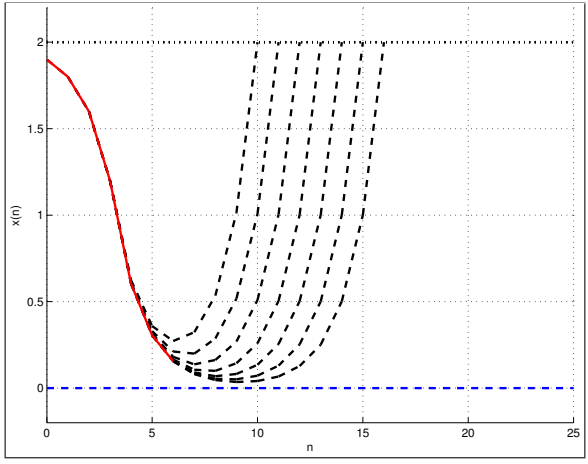

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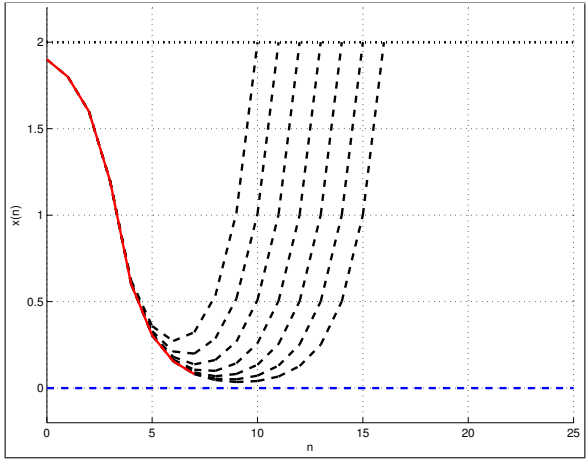

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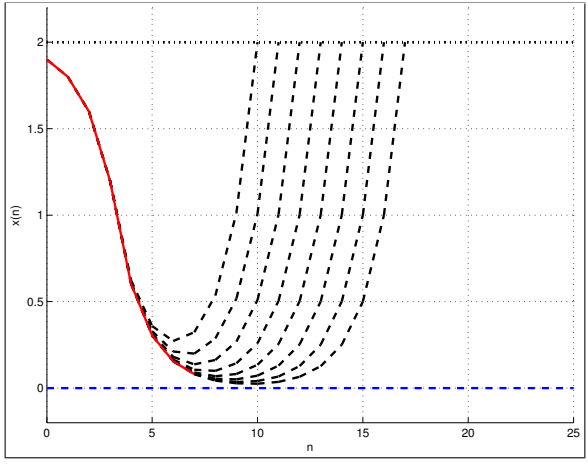

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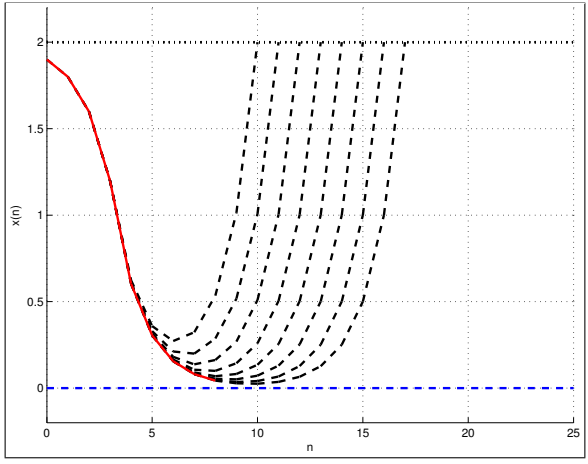

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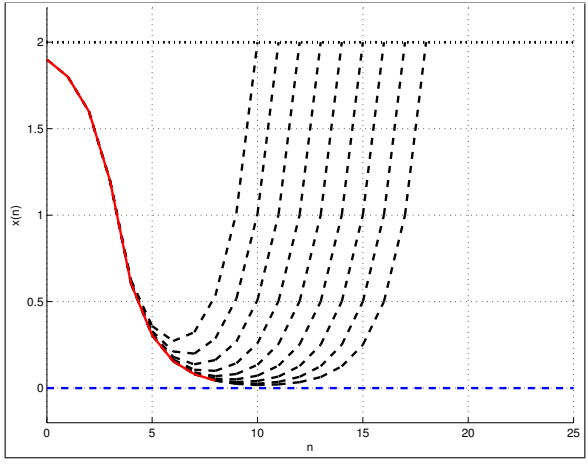

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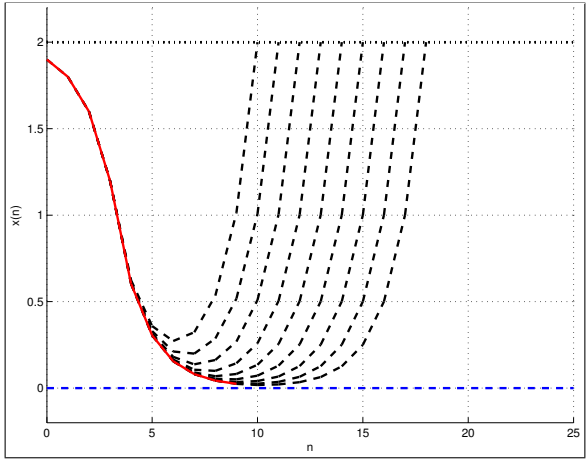

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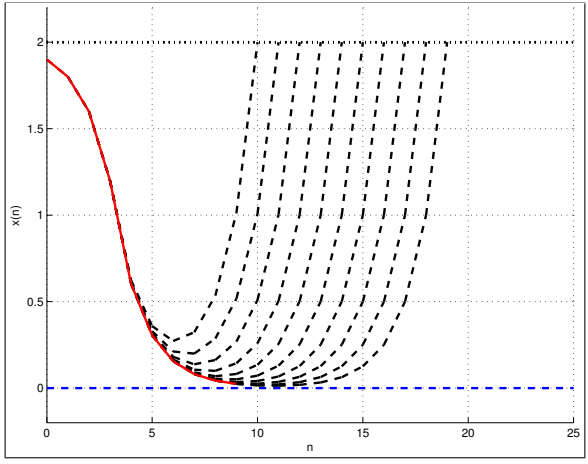

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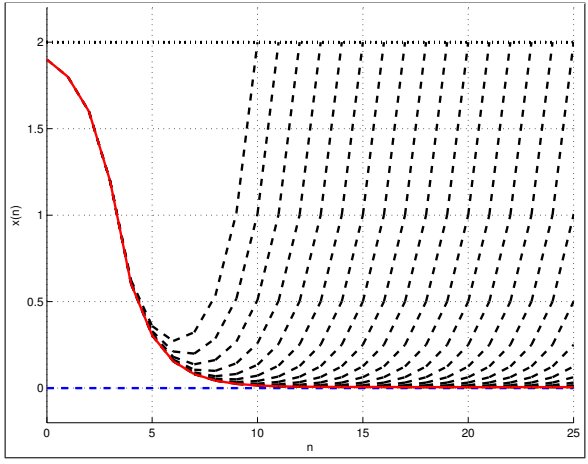

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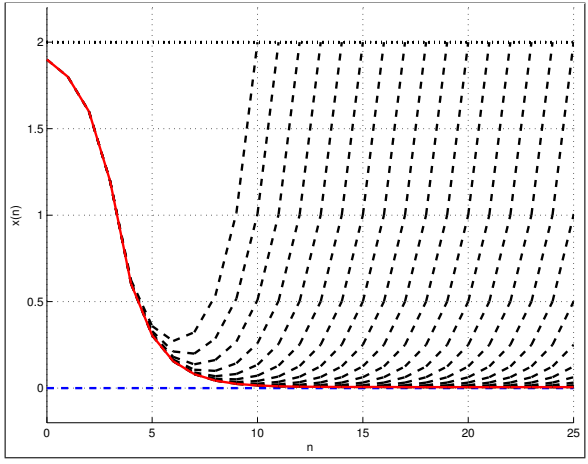

This behaviour allows to prove the following performance theorem

MPC via strict dissipativity — Theorem

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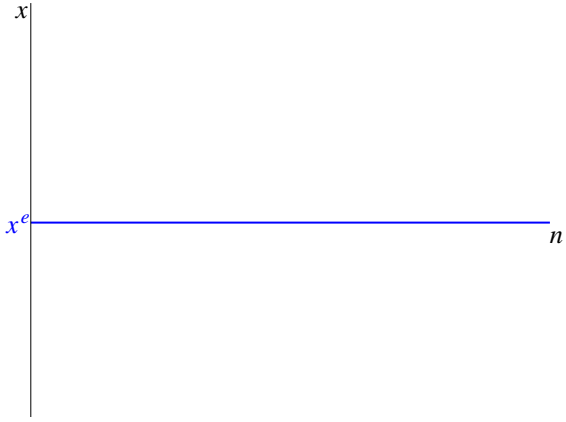
We explain the first two properties graphically

Illustration of transient optimality



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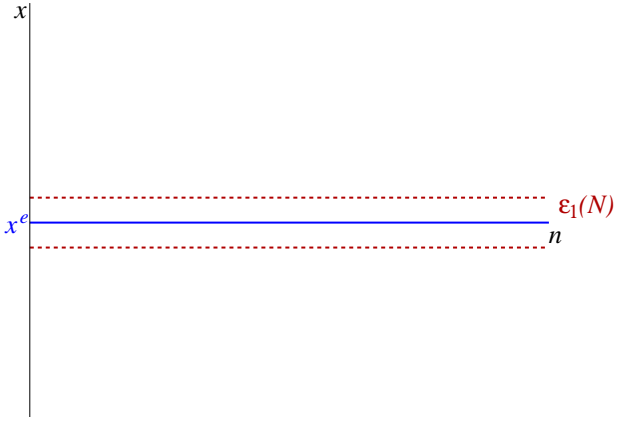

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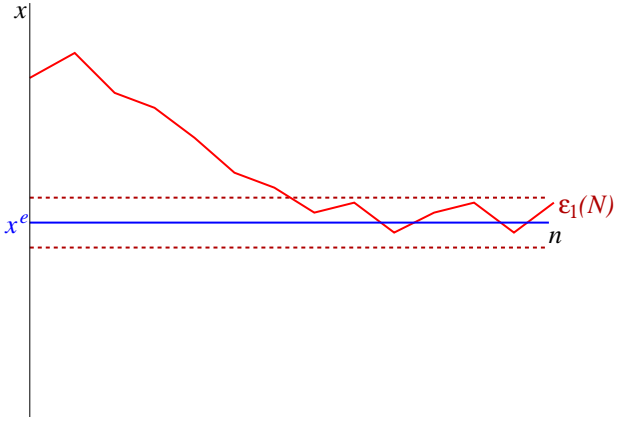

Practical asymptotic stability:

$x_{MPC}(n)$ converges to the $\varepsilon_1(N)$-ball around x^e

(with $\varepsilon_1(N) \to 0$ as $N \to \infty$)

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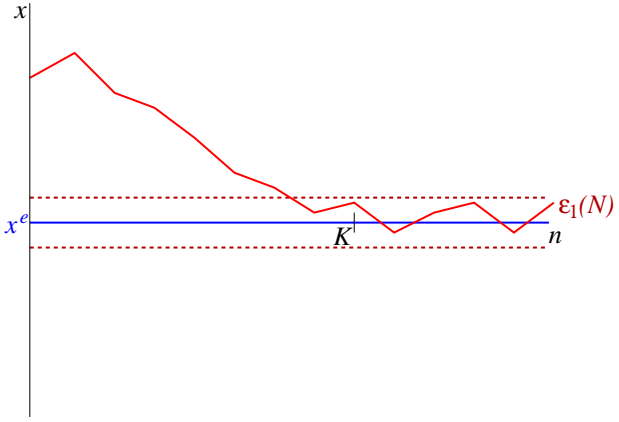

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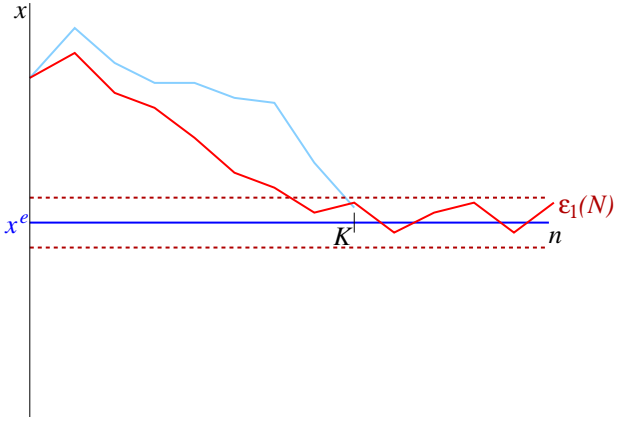

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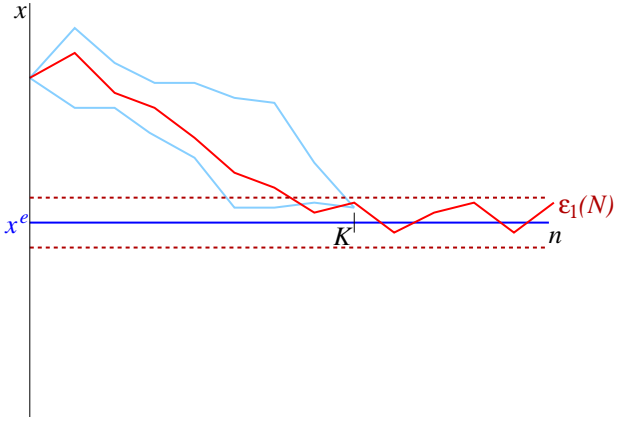

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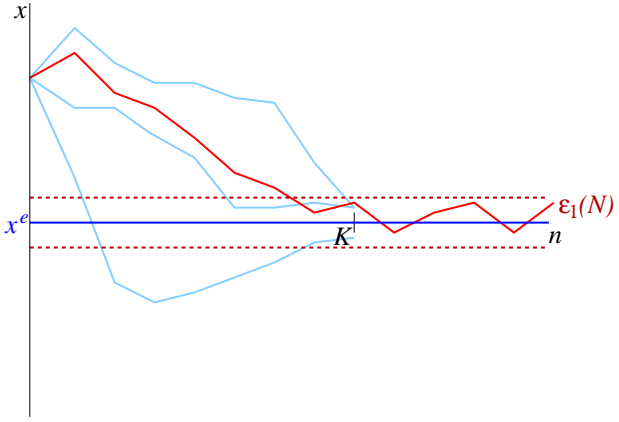

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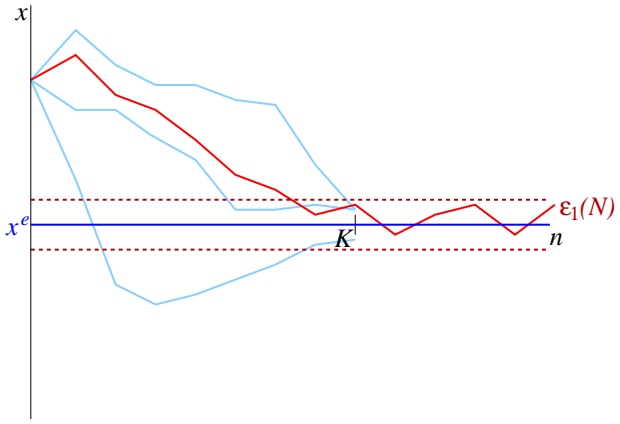

Transient optimality:

cost of all other trajectories reaching the ball at time K is higher than that of $x_{MPC}(n)$ up to an error $K\varepsilon_1(N) + \varepsilon_2(K)$

(with $\varepsilon_1(N), \varepsilon_2(K) \to 0$ as $N, K \to \infty$)

Use case

Use case

—

optimal startup of a combined cycle power plant

Combined Cycle Power Plants

C. Stadler/Bwag, wikipedia

Combined cycle power plants generate electrical energy via a combination of gas and steam turbines

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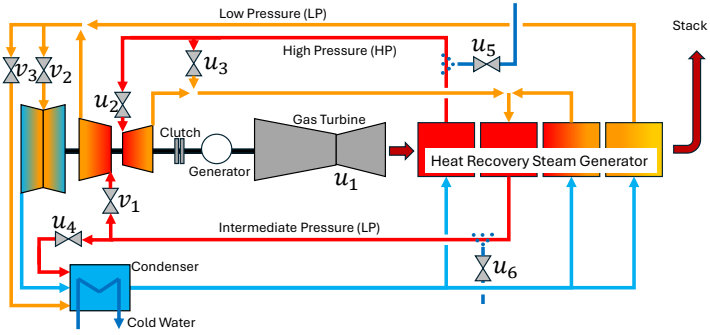
Combined cycle power plants generate electrical energy via a combination of gas and steam turbines

The exhaust gas from the gas turbine produces steam for a steam turbine

Combined Cycle Power Plants



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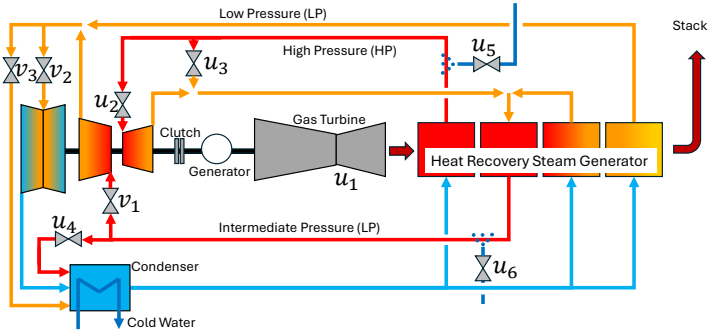

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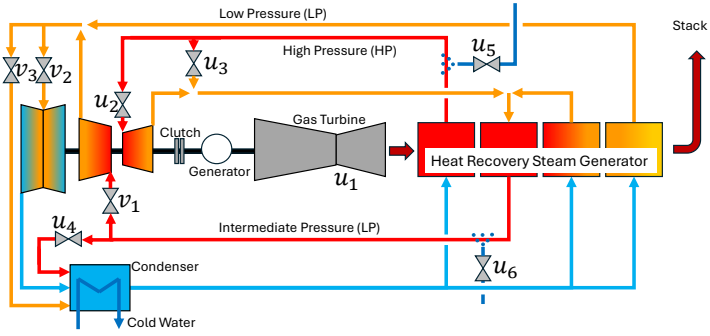
This way, up to 64% efficiency can be reached (and even more if the remaining heat is used for district heating)

Combined Cycle Power Plants



u_1: exhaust gas temperature	u_4: IP bypass valve	v_1: IP turb. control valve
u_2: HP turbine control valve	u_5: HP spray valve	v_2: LP turb. control valve
u_3: HP bypass valve	u_6: RH spray valve	v_3: LP bypass

Combined cycle power plants generate electrical energy via a combination of gas and steam turbines

The exhaust gas from the gas turbine produces steam for a steam turbine

This way, up to 64% efficiency can be reached (and even more if the remaining heat is used for district heating)

Goal: Develop control strategies for flexible, fast, and economic start-up

Control Objective and Constraints

Objective:

- reach a desired load level with minimal fuel consumption $\rightsquigarrow$ economic MPC

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- slower ramp up of gas turbine

Model

The model is derived from

- conservation of mass

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$$\dot{m} = \sum_{i \in I_m} \dot{m}_i \qquad [kg \cdot s^{-1}]$$

m: system mass $[kg]$
m_i: mass of medium entering (> 0) / leaving (< 0) the system $[kg]$
I_m: set of indices for medium entering/leaving the system

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$$\frac{d}{dt}(mu) = \sum_{i \in I_Q} \dot{Q}_i - \sum_{i \in I_W} \dot{W}_i + \sum_{i \in I_m} h_{m_i} \dot{m}_i \qquad [W]$$

m: system mass $[kg]$
u: specific internal energy of system $[J \cdot kg^{-1}]$
Q_i: heat transfer into (> 0) / out (< 0) of the system $[J]$
W_i: work performed by (> 0) / on (< 0) the system $[J]$
m_i: mass of medium entering (> 0) / leaving (< 0) the system $[kg]$
h_i: specific enthalpy of medium entering / leaving the system $[J \cdot kg^{-1}]$
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Convective heat transfer, e.g., steam $\leftrightarrow$ wall:

$$\dot{Q} = \alpha \cdot A \cdot (T - T_f) \qquad [W]$$

Q: heat transfer $[J]$
α: heat transfer coefficient $[W \cdot m^{-2} \cdot K^{-1}]$
A: contact area between fluid and material $[m^2]$
T: material temperature $[K]$
T_f: fluid temperature $[K]$

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Conductive heat transfer, inside material:

$$\dot{Q} = \lambda \cdot A \cdot \frac{dT}{dx} \qquad [W]$$

Q: heat transfer $[J]$
λ: conductivity $[W \cdot m^{-1} \cdot K^{-1}]$
A: cross-sectional area $[m^2]$
T: material temperature $[K]$
x: location $[m]$

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The economic optimization objective is

$$\int_{t_0}^{\infty} \mu(t) \cdot \varphi(t) dt$$

with $\mu(t) = 1$ if the desired load level is not yet reached and $\mu(t) = 0$ otherwise, $\varphi(t) =$ fuel consumption at time t

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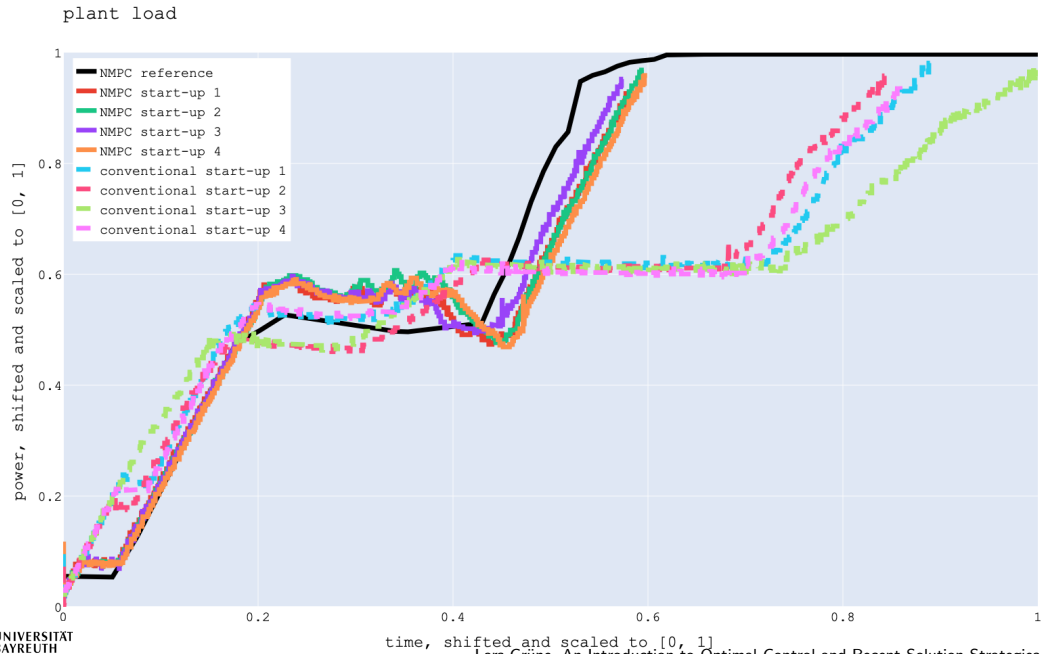
- For tracking MPC, the reference trajectory is the turnpike. If the system state is sufficiently close, a prediction horizon of a few minutes is suitable
- Since in the online phase the initial state at the beginning of the start up is unknown, a library of optimal references with different initial states is computed in advance

Start-up runs on the plant



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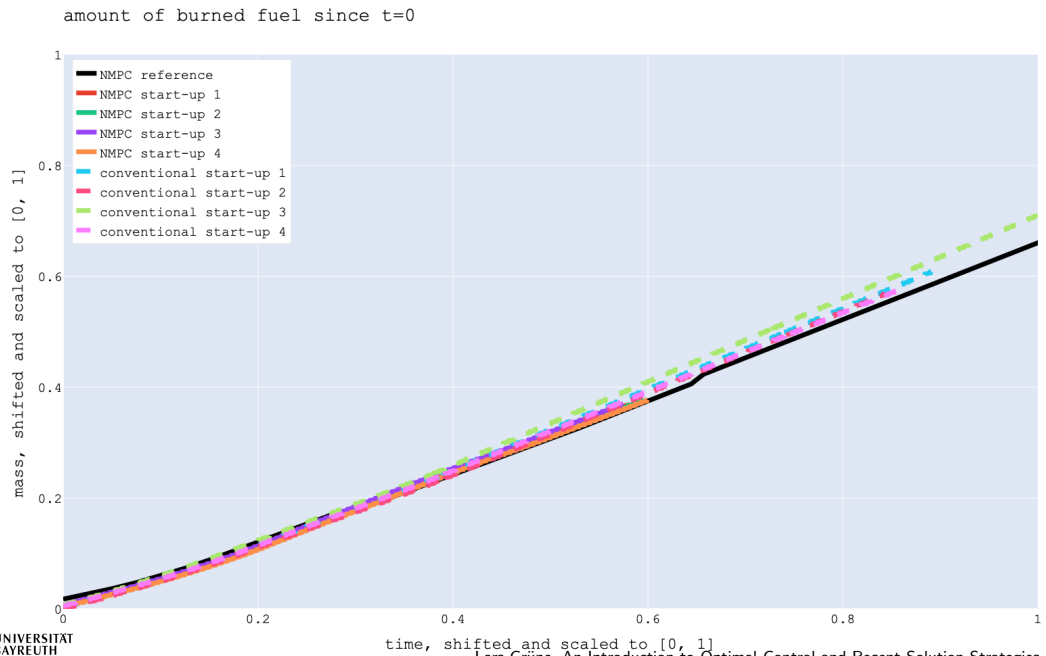

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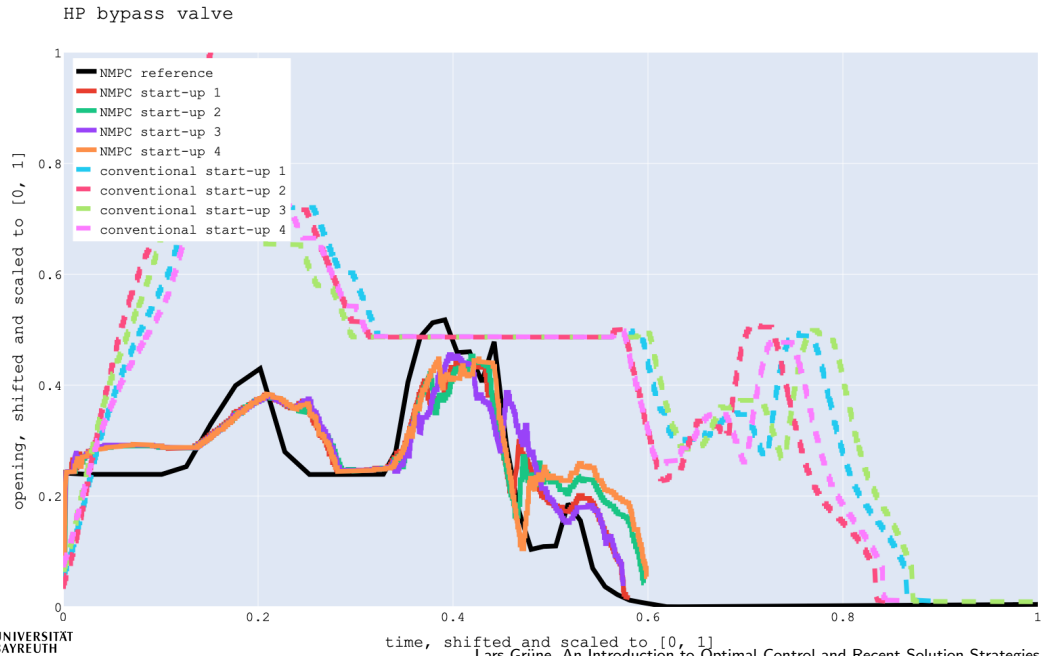

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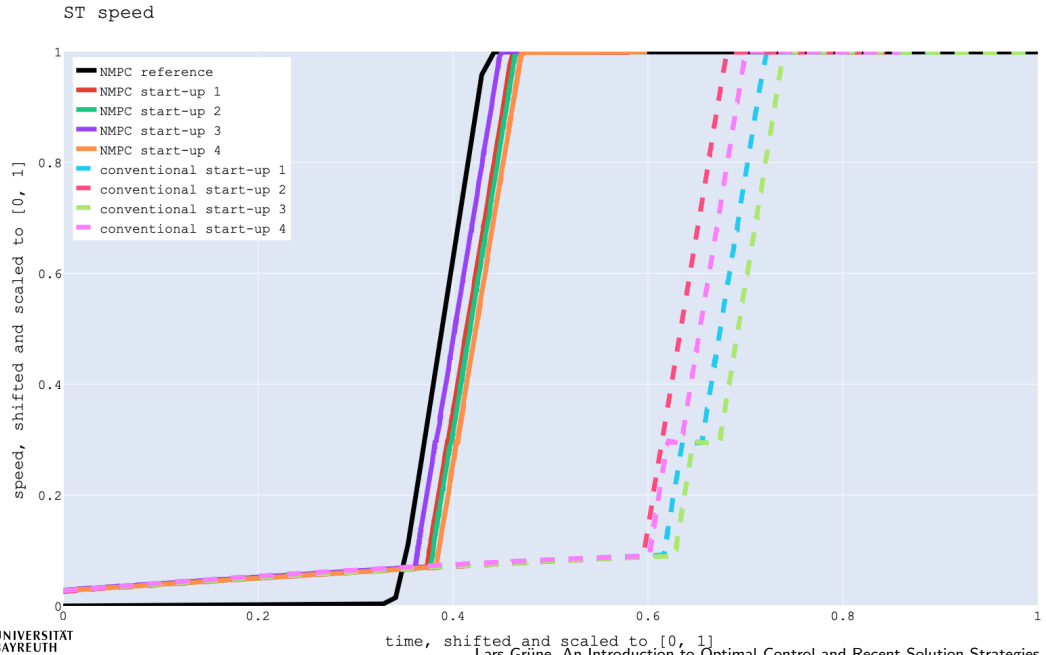

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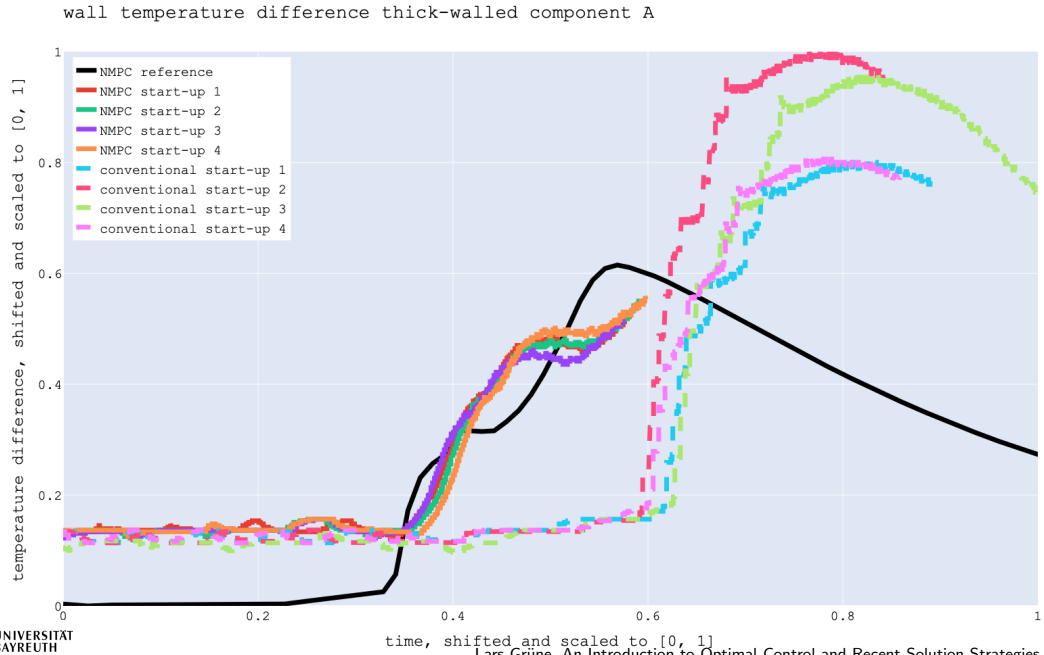

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- It generates a feedback law by online optimization
- If the turnpike property and dissipativity holds, near-optimal performance can be shown
- MPC is highly successful in industrial applications

Outline of the course

Part 1: Optimal Control Problems — An Introduction

Part 2: Solution Concepts

Part 3: Numerical Solution Methods

Part 4: Model Predictive Control

Part 5: Deep Neural Networks for High-Dimensional Problems

Purpose of this part

Identify structural properties of optimal control problems, under which deep neural networks can indeed avoid the curse of dimensionality

Use ideas from distributed optimal control to describe these properties

Give rigorous estimates for the approximation error

Contents of this part

Part 5: Deep Neural Networks for High-Dimensional Problems

- Reminder: Deep Reinforcement Learning
- Separable functions
- Distributed optimal control
- Decaying Sensitivity

Deep Reinforcement Learning

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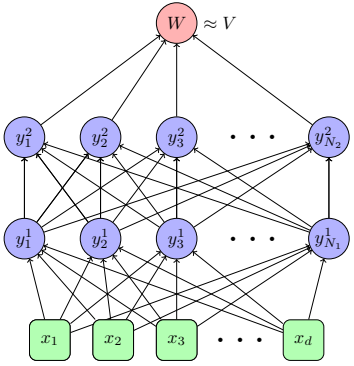
Deep neural network with 2 hidden layers



$$W(x;\theta) = a \cdot y^2 + c$$

$$y_k^2 = \sigma^2(w_k^2 \cdot y^1 + b_k^2)$$

$$y_k^1 = \sigma^1(w_k^1 \cdot x + b_k^1)$$

w_k^1, w_k^2, a = vectors of weights, " $\cdot$ " = scalar product

b_k^1, b_k^2, c = scalar parameters, $\sigma^1, \sigma^2 : \mathbb{R} \to \mathbb{R}$ = activation functions

Examples: $\sigma(r) = r$, $\sigma(r) = \max\{r, 0\}$, $\sigma(r) = \ln(e^r + 1)$, $\sigma(r) = \frac{1}{1+e^{-r}}$

θ = vector of all parameters $(w_k^\ell, b_k^\ell, a, c)$

$W(x;\theta^*) \approx V(x)$, approximating function for "trained" θ^*

Deep Reinforcement Learning

The network is then "trained" such that the function $W(x;\theta^*)$ with optimized parameters θ^* satisfies the Hamilton-Jacobi-Bellman equation

$$\inf_{u\in U}\{\ell(x,u)+DW(x;\theta^*)f(x,u)\}\approx 0$$

or the Bellman equation

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Here, the following necessary property will be discussed:

Can DNNs provide approximations $W(\cdot\,;\theta^*)\approx V$ for large space dimensions d?

Universal approximation theorem

Let $K = [-\kappa, \kappa]^n$, $\kappa > 0$ fixed, $d \in \mathbb{N}$ varying

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Theorem [Cybenko '89, Mhaskar '96, Poggio et al. '17]:
Let $\sigma^1 : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and not polynomial. Then, for any $\varepsilon > 0$, a neural network with one hidden layer provides an approximation

$$\inf_{\theta \in \mathbb{R}^P} \|W(x;\theta) - V(x)\|_{\infty,K} \leq \varepsilon$$

for any $V \in \mathcal{W}_1^n$ with a minimal number of neurons

$$N = \mathcal{O}\left(\varepsilon^{-n}\right)$$

Quiz

Assume the required number of neurons for achieving the accuracy $\varepsilon > 0$ is $\mathcal{O}\left(\varepsilon^{-n}\right)$.

By how much does the number increase if we want to reduce the accuracy ε to $\varepsilon/5$?

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E.g., for $n = 10$: $\quad 5^{10} \approx 10^7 > 10^5$

Separable functions

What are separable functions and why are they beneficial?

Separable function:

$$V(x) = \sum_{j=1}^{s} V_j(w_j), \quad w_j = \begin{pmatrix} x_{i_{j,1}} \\ \vdots \\ x_{i_{j,d_j}} \end{pmatrix}$$

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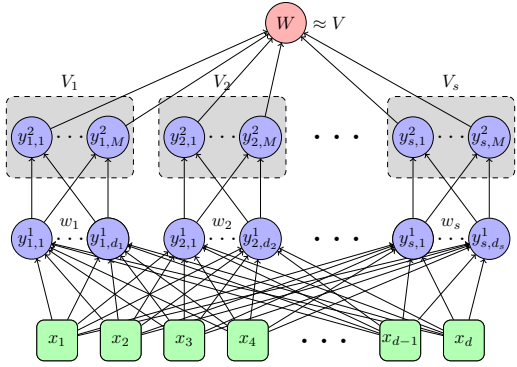
We approximate the individual V_j by the grey blocks, whose number grows linearly with the dimension d



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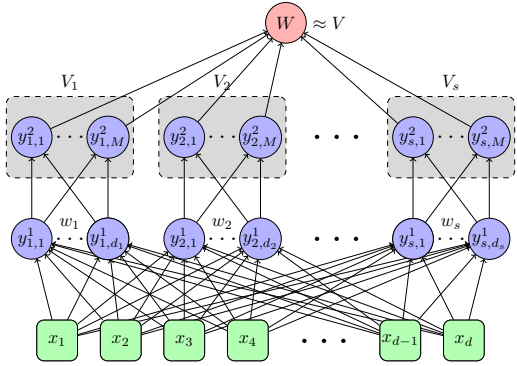

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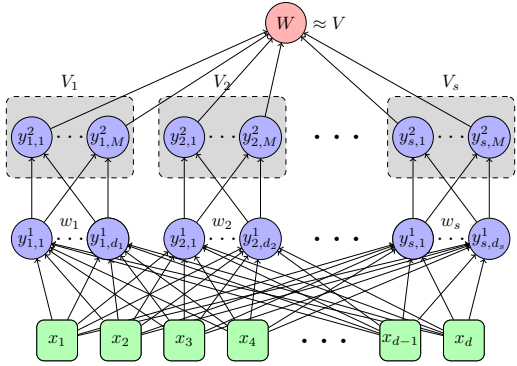

Theorem [Gr. 21]: Functions $V(x) = \sum_{j=1}^{s} V_j(w_j)$ with $V_j \in \mathcal{W}_1^{d_j}$ and $d_j \leq d_{\max}$ independent of d can be approximated on K with any accuracy $\varepsilon > 0$ with a number of neurons growing only polynomially in d

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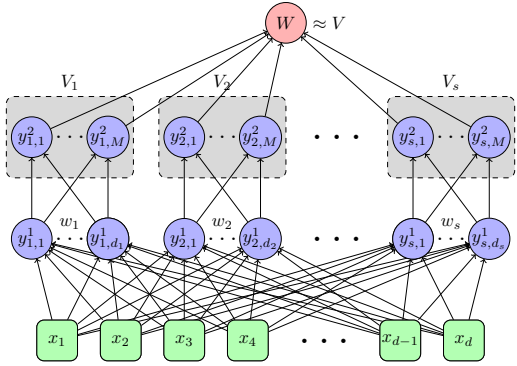

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More precisely, the number of required neurons is $\mathcal{O}\left(\varepsilon^{-d_{\max}}\right) \mathcal{O}\left(d^{d_{\max}+1}\right)$

Distributed optimal control

Setting for distributed control

We assume that the system can be decomposed into s subsystems

$$\dot{z}_i = f_i(z_i, z_{-i}, u_i), \quad i = 1, \ldots, s, \qquad z_{-i} = \begin{pmatrix} z_1 \\ \vdots \\ z_{i-1} \\ z_{i+1} \\ \vdots \\ z_s \end{pmatrix}$$

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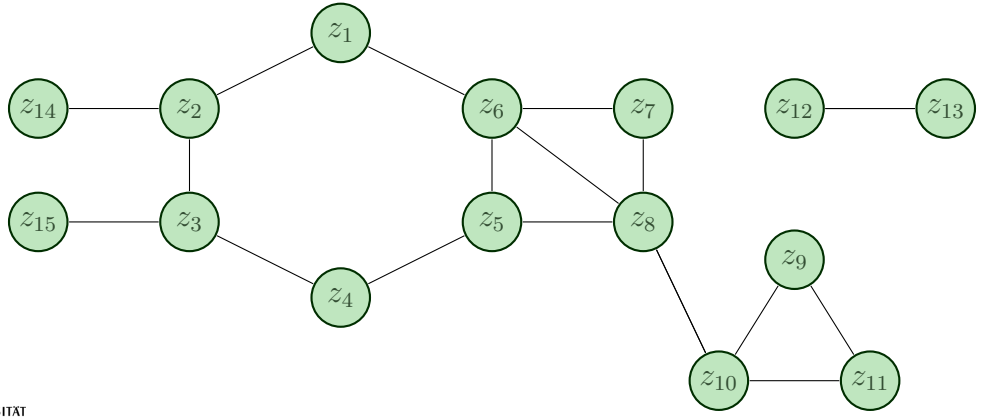

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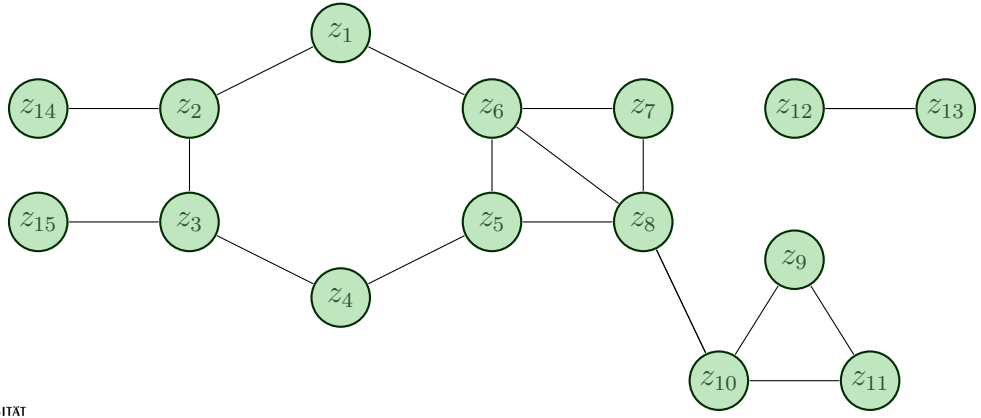

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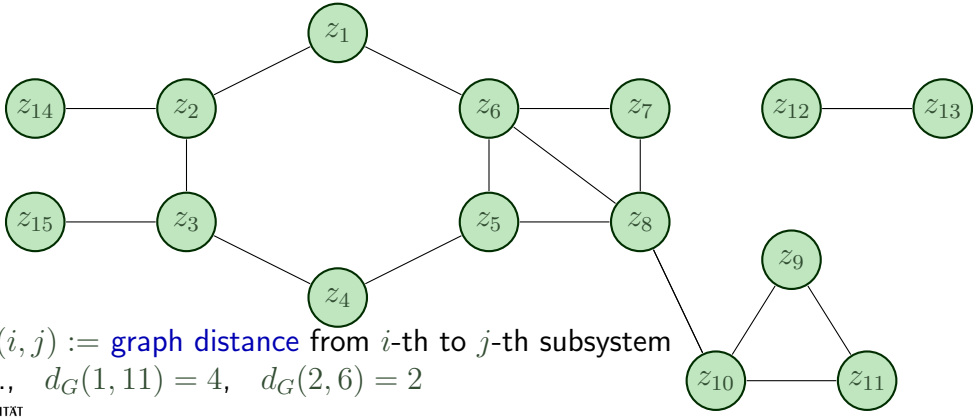
with $z_i \in \mathbb{R}^{n_i}$, whose interconnection is expressed via an undirected graph: Whenever f_i depends on z_j, then there is an edge from z_i to z_j



$d_G(i,j) :=$ graph distance from i-th to j-th subsystem

e.g., $d_G(1,11) = 4$, $d_G(2,6) = 2$

Setting for distributed control

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Question: How does the state of a subsystem influence the optimal trajectory of another subsystem far away?

Decaying sensitivity

Convoy example

Example: Convoy of vehicles

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It is known that a perturbation in the first vehicle (e.g., a braking manoeuvre) may amplify while propagating through the convoy

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However, the perturbation will decrease quickly, if the vehicles are controlled optimally

Convoy example

Consider a convoy of $i = 1, \ldots, N$ vehicles on a road with state $z_i = (x_i, v_i)^T$ and dynamics

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i$$

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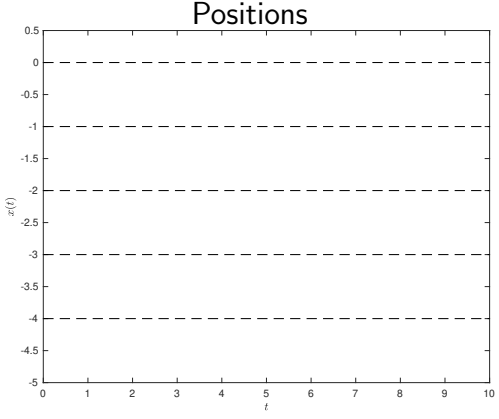
Convoy example

Optimal solutions for $N = 100$ vehicles, shown $i = 1, \ldots, 5$, $x_{ref} \equiv 0$, $v_{ref} \equiv 0$



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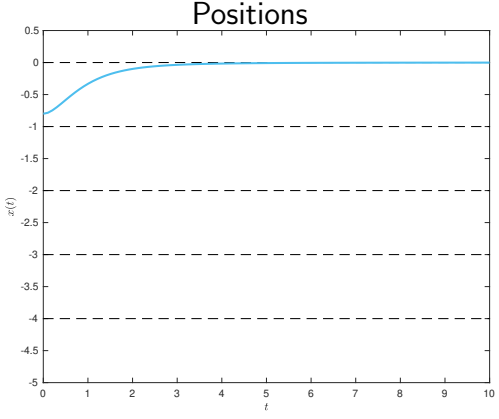

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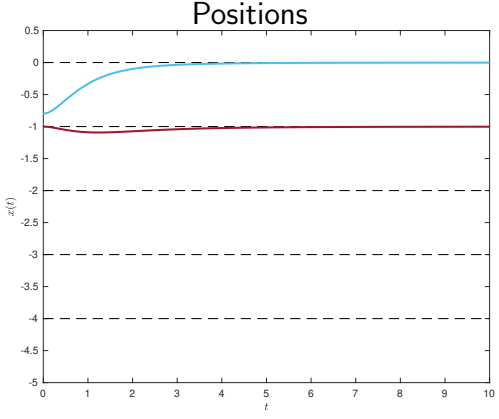

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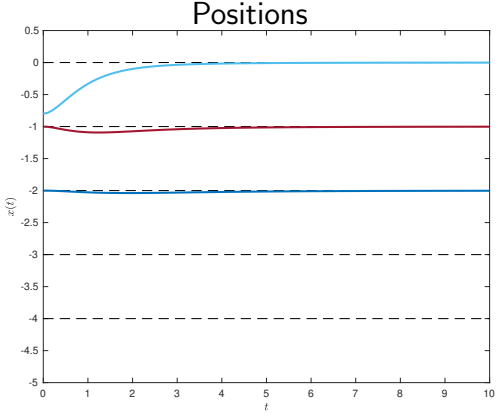

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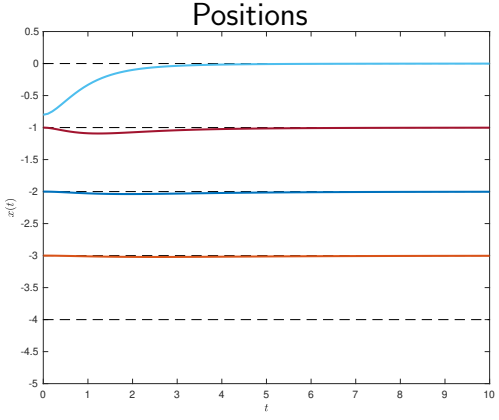

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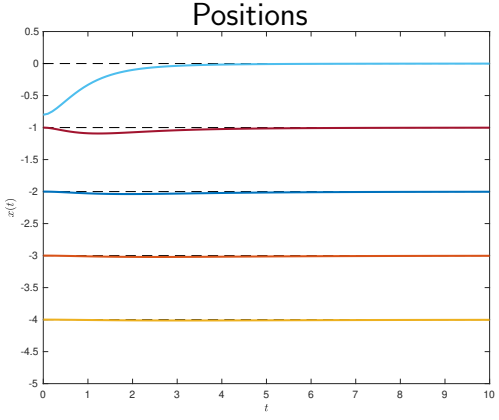

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Decaying sensitivity for LQ optimal control

Consider linear quadratic optimal control problems, i.e., with linear dynamics $f(x,u) = Ax + Bu$ and with quadratic cost $\ell(x,u) = x^T Q x + u^T R x$

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"Exponentially decaying sensitivity"

Construction of the separable approximations

Structure of the separable approximation

We approximate V by a separable function

$$V(x) \approx \sum_{k=1}^{s} \Psi_k^l(z_{i_{k,1}}, z_{i_{k,2}}, \ldots, z_{i_{k,l}})$$

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Note: It is in general unrealistic to expect

$$V(x) = \sum_{k=1}^{s} \Psi_k^l(z_{i_{k,1}}, z_{i_{k,2}}, \ldots, z_{i_{k,l}}) \qquad \text{or} \qquad V(x) \approx \sum_{k=1}^{s} \Psi_k^1(z_{i_{k,1}})$$

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Theorem [Sperl/Saluzzi/Kalise/Gr. '25]: When the sensitivity decays exponentially

$$\left| V(x) - V(0) - \sum_{k=1}^{s} \Psi_k^l(z_k,\dots,z_{\min\{k+l,s\}}) \right| \le c\rho^l \quad \text{on } L_2 \text{ balls}$$

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If $d_G(z_i,z_j) \neq |i-j|$, then for any node the number of nodes with graph distance l must grow slower than ρ^{-l}

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Numerical test

We test the approach on the linear-quadratic problem with

$$\dot{x} = Ax + u, \qquad \ell(x,u) = \|x\|_2^2 + \|u\|_2^2,$$

and $A \in \mathbb{R}^{200\times 200}$ a randomly generated banded matrix

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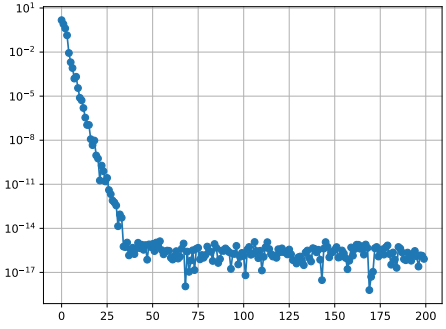

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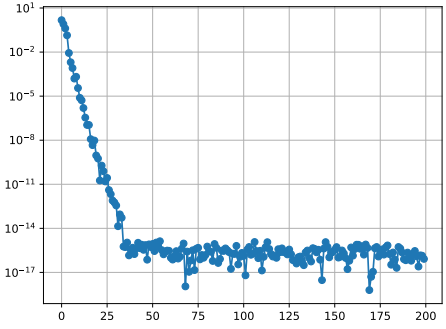
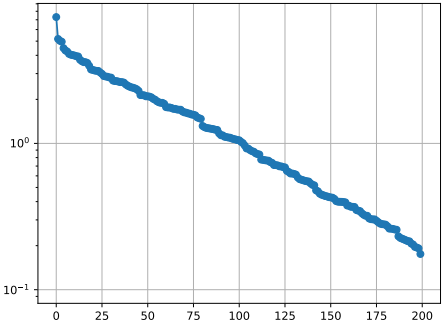


$\|P_{j1}\|$, $j = 1, \ldots, 200$

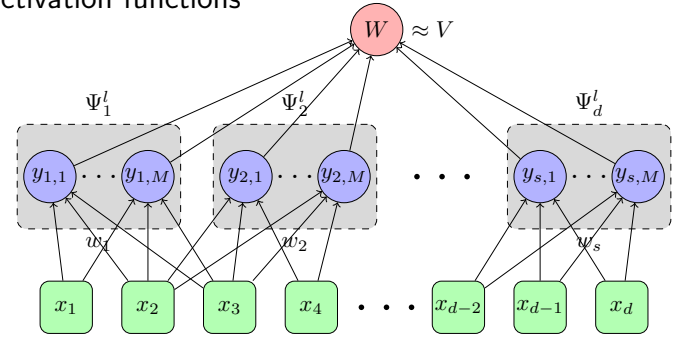


singular values of P

Numerical test

Separable network structure with

- $d_j = 1 \quad \Rightarrow \quad z_k = x_k$
- $l = 10$ (number of inputs for each Ψ_k^l)
- $M = 16$ (number of neurons for each Ψ_k^l)
- sigmoid activation functions



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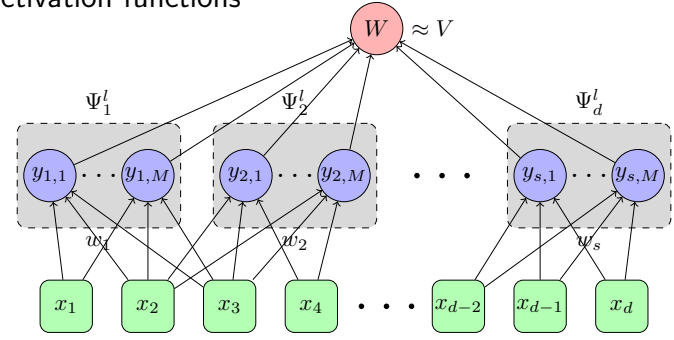
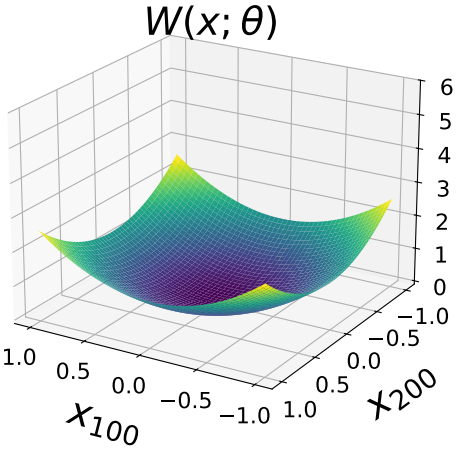
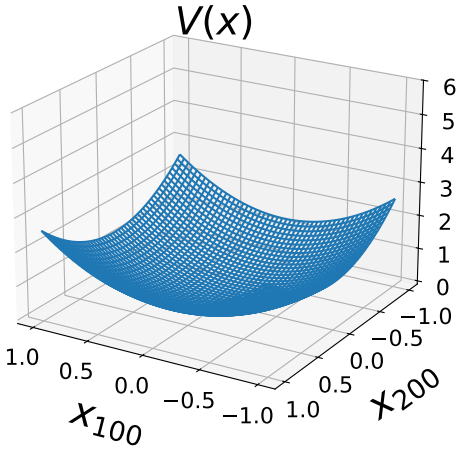
In order to avoid effects due to incomplete learning in reinforcement learning (exploration vs. exploitation), we use supervised learning to learn V



Numerical results

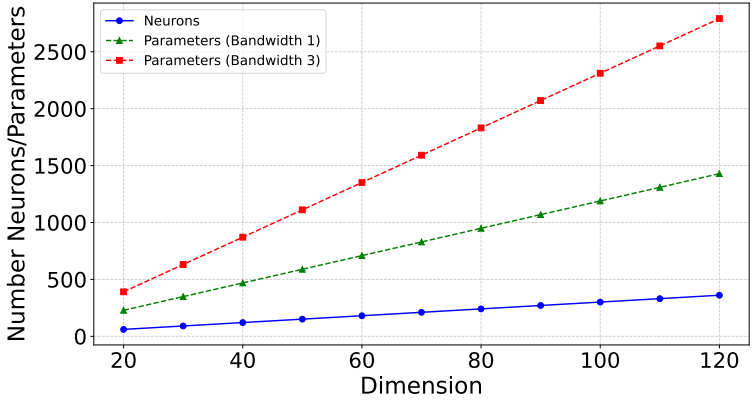
For the 200-dim problem we reach a mean-square error $< 10^{-3}$ in the test data





Numerical results

We have numerically evaluated the minimal number of neurons and parameters in the network for achieving a mean-square error in the test data of 10^{-3} for varying dimensions



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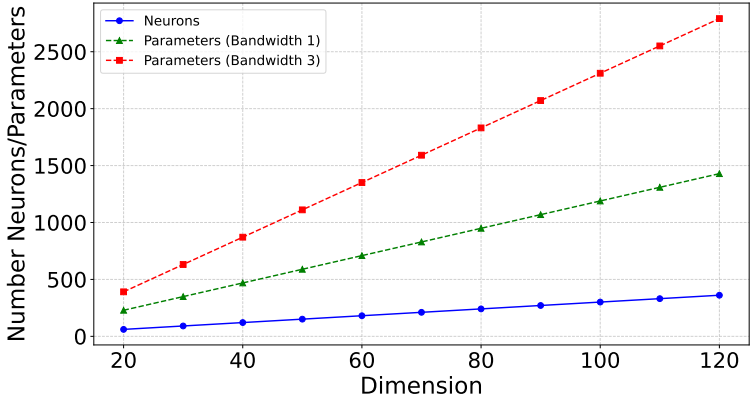
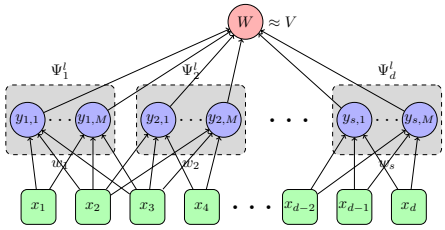


Interestingly, the growth is linear (and not polynomial with degree ≥ 2)

Separable vs. fully connected networks

Obviously, one can always embed a separable network into a fully connected network



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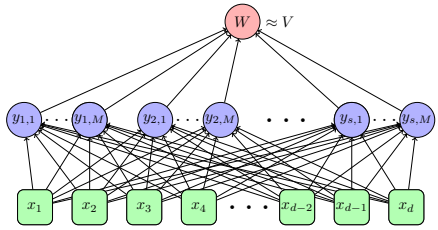

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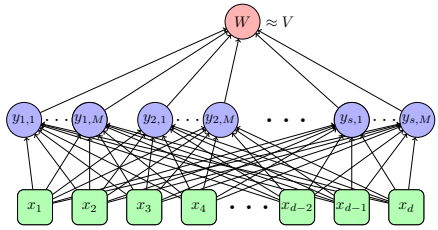

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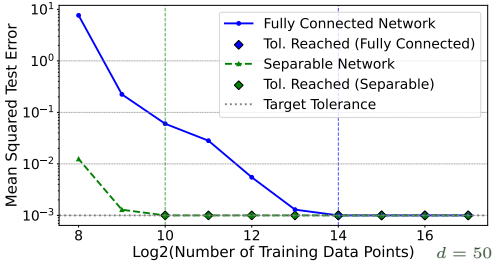

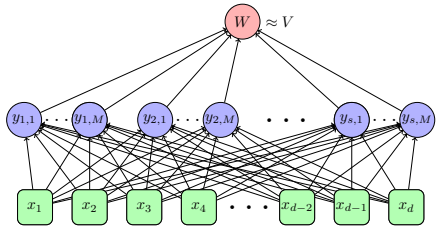

Yes: the amount of data and the number of training epochs needed for training is significantly smaller

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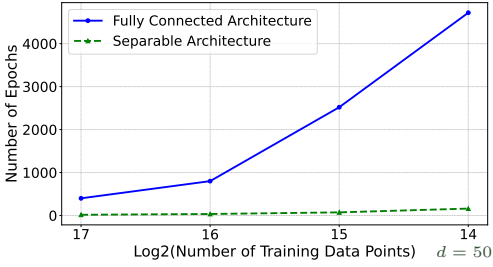

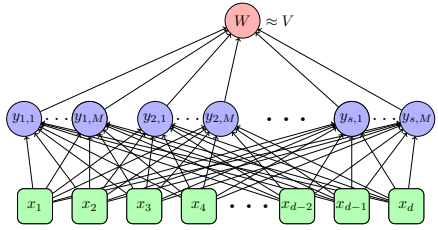

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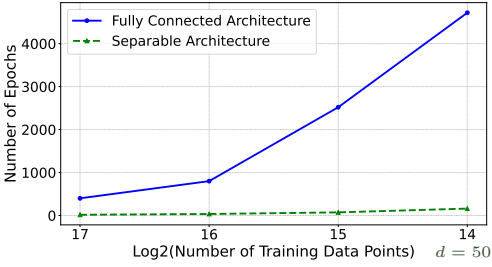

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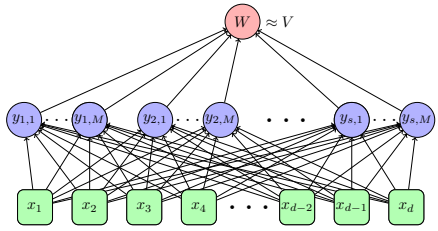

We conjecture that this can be explained using anchored decompositions [Sobol '69, Kuo/Sloan/Wasilikowski/Woźniakowski '09, Rieger/Wendland '24]

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- Using Deep Neural Networks for approximating optimal value function does in general not remove the curse of dimensionality
- However, the existence of separable approximations of optimal value functions allows for a curse-of-dimensionality-free approximation via deep neural networks
- This existence can be established via decaying sensitivity
- Open questions:
  - Exponential sensitivity for nonlinear problems
  - Analysis of sampling efficiency

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