Economic Model Predictive Control: State of the Art and Open Problems

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Universität Bayreuth · Universität Stuttgart · KIT Karlsruhe Institute of Technology

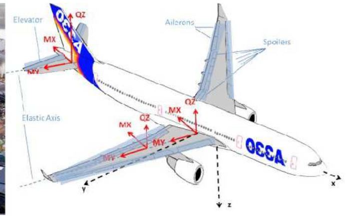
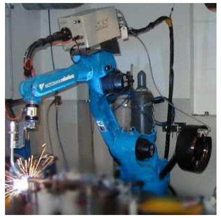
Model Predictive Control

Why Model Predictive Control?

Model Predictive Control (MPC) "[...] *is the only advanced control technique—that is, more advanced than standard PID control—to have had a significant and widespread impact on industrial process control."*

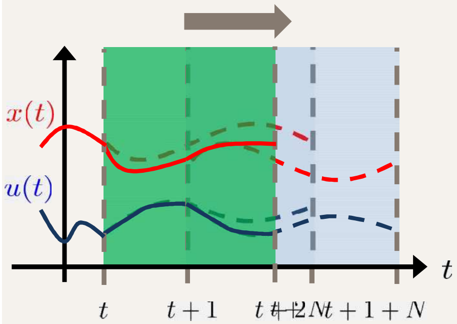
J. Maciejowski (Univ. Cambrige, UK). *Predictive control: with constraints*. Pearson Ed. Limited, 2002

Industrial applications of MPC include

Model Predictive Control – Main Idea

Model predictive control = repeated optimal control



1. State measurement $x(t)$ at $t \in \mathbb{N}$

2. Solve

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t))$$

subject to

$$\begin{aligned} x(k+1|t) &= f(x(k|t), u(k|t)), \quad & k = 0, \dots, N-1 \\ x(0|t) &= x(t) & \\ (x(k|t), u(k|t))^\top &\in \mathbb{X} \times \mathbb{U}, & k = 0, \dots, N-1 \\ x(N|t) &\in \mathbb{X}_f & \end{aligned}$$

3. Apply $\mu_N(x(t)) = u^\star(0|t)$

Historic origins

Predictive control for nonlinear systems

Lee and Markus, Foundations of Optimal Control Theory, 1967, p. 423:

One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to ***measure the current*** *control process* ***state*** *and* ***then compute very rapidly*** *for the open-loop control function. The* ***first portion of this function*** *is then* ***used during a short time*** *interval, after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated.*

Main objective of control design

Morari, Arkun, Stephanopoulos, Studies in the synthesis of control structures for chemical processes: Part I. *AIChE* Journal, 1980, pp. 220-232:

[In] attempting to synthesize a feedback optimizing control structure, our main objective is to translate the economic objective into process control objectives.

Classical tracking NMPC

Control task (e.g. stabilization / setpoint tracking)

Task-specific Optimal Control Problem ← Process or system model

Implementation ← Efficient numerical algorithms

NMPC controller ← State estimation

Setpoints

Control hierarchy in nowadays process systems

Plant-wide economic considerations

Real Time Optimization
- Setpoints for control
- Static models

Advanced Control
- Enforces setpoints
- Often MPC or NMPC

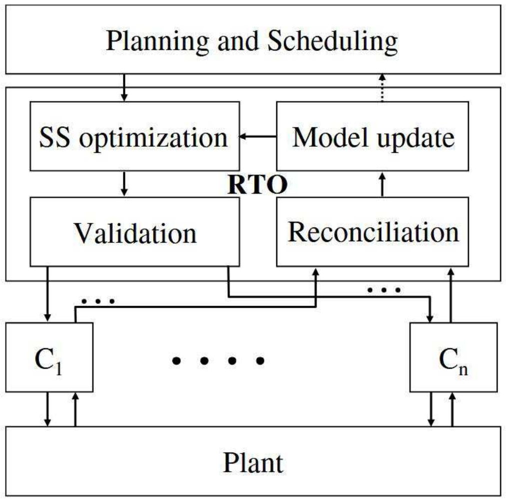


Figure taken from:
Engell, Sebastian. "Feedback control for optimal process operation." *Journal of Process Control* 17.3 (2007): 203-219.

Main idea of economic MPC

Performance specification (i.e. cost function)

Task-specific Optimal Control Problem ← Process or system model

Implementation ← Efficient numerical algorithms

Economic NMPC controller ← State estimation

Outline of the workshop

Part	Topic	Speaker
I.	**Welcome / Introduction** **Revisiting stabilizing NMPC**	Timm Faulwasser
II.	**Economic MPC with terminal constraints** • Optimal operation at steady state • Stability using dissipativity and terminal constraints	Matthias Müller
III.	**Economic MPC without terminal constraints** • Dissipativity and turnpike properties • Recursive feasibility and stability	Timm Faulwasser
	Coffee break	
IV.	**Economic MPC without dissipativity** • Lyapunov-based EMPC • Multi-objective EMPC	Lars Grüne
V.	**Advanced topics and open problems** • Extension to periodic solutions • Discounted problems • Time-varying problems	Lars Grüne
VI.	• Economic MPC for uncertain systems	Matthias Müller
VII.	**Summary and wrap up**	Matthias Müller

In terms of notation, presentation and examples, the workshop mostly follows along the lines of:
Faulwasser, T.; Grüne, L. & Müller, M. Economic Nonlinear Model Predictive Control: Stability, Optimality and Performance. *Foundations and Trends in Systems and Control,* **2018.**

NMPC in discrete and continuous time

Discrete-Time NMPC

1. State measurement $x(t)$ at $t \in \mathbb{N}$

2. Solve discrete-time OCP

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t))$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}$$
$$x(N|t) \in \mathbb{X}_f$$

3. Apply $\mu_N(x(t)) = u^\star(0|t)$

Sampled-Data Continuous-Time NMPC

1. State measurement $x(t_k)$ at $t_k \in \mathbb{R}_0^+$

2. Solve OCP

$$\min_{u(\cdot|t)} \int_0^T \ell(x(\tau|t), u(\tau|t)) d\tau + V_f(x(T|t))$$

subject to

$$\frac{dx(\tau|t)}{d\tau} = f(x(\tau|t), u(\tau|t)), \quad x(0|t) = x(t_k)$$
$$(x(\tau|t), u(\tau|t))^\top \in \mathbb{X} \times \mathbb{U}$$
$$x(T|t) \in \mathbb{X}_f$$

3. Apply $u(t) = u^\star(\tau|t)$, $\tau \in [t_k, t_k + \delta)$

Here: Focus on discrete-time setting.

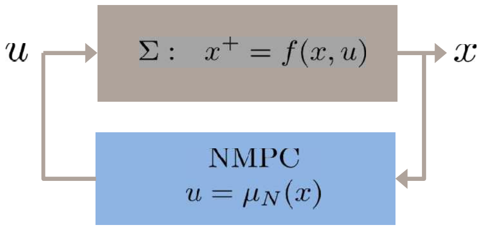
Revisiting Tracking/Stabilizing NMPC

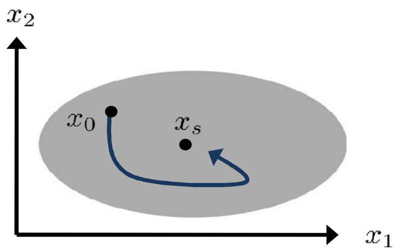
Considered control problem

Setpoint Stabilization

- Reference = setpoint $x_s \in \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Constraint satisfaction: $\forall t \in \mathbb{N} : u(t) \in \mathbb{U}$ and $x(t; x_0, u(\cdot)) \in \mathbb{X}$
- Stability: $\forall \varepsilon > 0 \; \exists \delta > 0$ such that

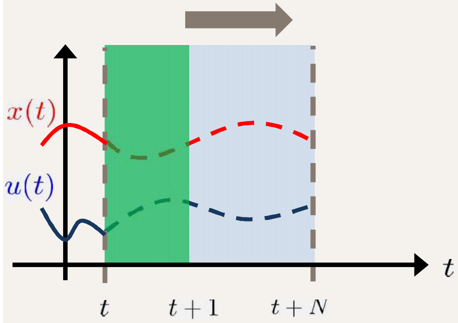
$$\|x(0) - x_s\| > \delta \quad \Rightarrow \quad \|x(t; x_0, u(\cdot)\| < \varepsilon \quad \forall t \geq 0$$





Model Predictive Control – Main idea

Model predictive control = repeated optimal control



1. State measurement $x(t)$ at $t \in \mathbb{N}$

2. Solve

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t))$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad k = 0, \dots, N-1$$
$$x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}, \quad k = 0, \dots, N-1$$
$$x(N|t) \in \mathbb{X}_f$$

3. Apply $\mu_N(x(t)) = u^\star(0|t)$

Notation

- State trajectory predicted at time t: $x(\cdot|t)$
- Input trajectory predicted at time t: $u(\cdot|t)$

Main ingredients for design of tracking NMPC

Discrete-Time NMPC

1. State measurement $x(t)$ at $t \in \mathbb{N}$

2. Solve discrete-time OCP

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t))$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}$$
$$x(N|t) \in \mathbb{X}_f$$

3. Apply $\mu_N(x(t)) = u^\star(0|t)$

Ingredients

- System model: $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$
- State constraints: $\mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Input constraints: $\mathbb{U} \subseteq \mathbb{R}^{n_u}$
- State feedback / state estimate $x(t)$

$\to$ Assumed to be exactly known.

- Stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$
- Terminal penalty $V_f : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$
- Terminal constraint $\mathbb{X}_f \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Prediction horizon $N \in \mathbb{N}$

$\to$ To be designed/chosen!

Closed-loop system

Discrete-Time NMPC

1. State measurement $x(t)$ at $t \in \mathbb{N}$

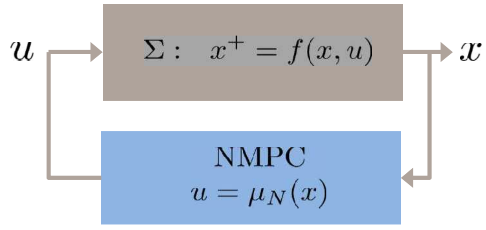
2. Solve discrete-time OCP

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t))$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}$$
$$x(N|t) \in \mathbb{X}_f$$

3. Apply $\mu_N(x(t)) = u^\star(0|t)$



$$x^+ = f(x, \mu_N(x))$$

→Recursive feasibility?
→Stability of closed loop?

Recursive feasibility

Considered NMPC scheme

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + V_f(x(N|t))$$

subject to (1)

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}$$
$$x(N|t) \in \mathbb{X}_f$$

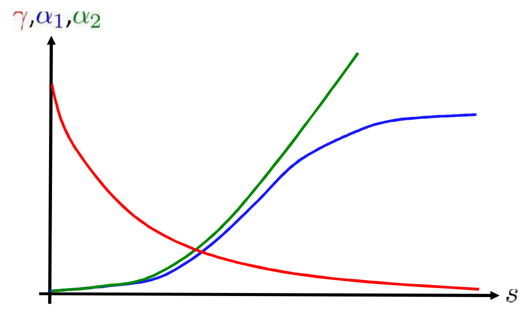
Definition (Recursive feasibility).
Let $\mathbb{X}_0 \subseteq \mathbb{X}$ denote a set of initial conditions $x(0) = x_0$ for which OCP (1) admits a feasible solution. OCP (1) is said to be *recursively feasible with respect to* $\mathbb{X}_0$, if for all $x(0) = x_0 \in \mathbb{X}_0$ the inclusion

$$f(x_0, \mu_N(x_0)) \in \mathbb{X}_0$$

holds.

Comparison functions

- $\mathcal{L} := \left\{\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \gamma \text{ continuous and decreasing with } \lim_{s\to\infty} \gamma(s) = 0\right\}$
- $\mathcal{K} := \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \alpha \text{ continuous and strictly increasing with } \alpha(0) = 0\}$
- $\mathcal{K}_\infty := \{\alpha \in \mathcal{K} \,|\, \alpha \text{ unbounded}\}$
- $\mathcal{KL} := \{\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+ \,|\, \beta(\cdot, k) \in \mathcal{K}, \beta(r, \cdot) \in \mathcal{L}\}$.



Kellett, C. A compendium of comparison function results. *Mathematics of Control, Signals, and Systems, Springer*, **2014**, *26*, 339-374

Main assumptions for stabilizing NMPC with terminal constraints

Considered NMPC scheme

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t),u(k|t)) + V_f(x(N|t))$$

subject to

$$x(k+1|t) = f(x(k|t),u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t),u(k|t))^\top \in \mathbb{X}\times\mathbb{U}$$
$$x(N|t)\in\mathbb{X}_f \tag{1}$$

Assumption 1 (Lower boundedness of ℓ).
The stage cost satisfies $\ell(0,0)=0$. Furthermore, there exists $\alpha_1\in\mathcal{K}_\infty$ such that for all $(x,u)\in\mathbb{X}\times\mathbb{U}$

$$\alpha_1(\|x\|)\le\ell(x,u).$$

Assumption 2 (Local bound on the cost-to-go).
For all $x\in\mathbb{X}_f$, there exist an input $u=\kappa_f(x)\in\mathbb{U}$ s.t. $f(x,\kappa_f(x))\in\mathbb{X}_f$ holds and

$$V_f(f(x,\kappa_f(x)))+\ell(x,\kappa_f(x))\le V_f(x).$$

Furthermore, $V_f(0)=0$ and $V_f(x)\ge 0$ for all $x\in\mathbb{X}_f$.

Stability of NMPC with terminal constraints

Theorem (Stability of tracking NMPC with terminal constraints).
Let Assumptions 1 and 2 hold. Suppose that $0\in$ int$(\mathbb{X}_f)$ and that there exists $\alpha_3\in\mathcal{K}_\infty$ such that, for all $x\in\mathbb{X}_f$, $V_f(x)\le\alpha_3(\|x\|)$.

Then the closed-loop system $x^+=f(x,\mu_N(x))$ arising from the NMPC scheme has the following properties:

1. If OCP (1) is feasible for $t=0$, then it is feasible for all $t\in\mathbb{N}$.
2. The origin $x=0$ is an asymptotically stable equilibrium of $x^+=f(x,\mu_N(x))$.
3. The region of attraction of $x=0$ is given by the set of all initial conditions x_0 for which OCP (1) is feasible.

References

- Chen, H. & Allgöwer, F. A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica,* **1998***, 34*, 1205-121
- Mayne, D.; Rawlings, J.; Rao, C. & Scokaert, P. Constrained model predictive control: Stability and optimality. *Automatica,* **2000***, 36*, 789-814
- Rawlings, J. & Mayne, D. Model Predictive Control: Theory & Design. *Nob Hill Publishing, Madison, WI,* **2009**
- Grüne, L. & Pannek, J. Nonlinear Model Predictive Control: Theory and Algorithms. *Springer Verlag,* **2017**

Blue print for NMPC stability proofs with terminal constraints

- Step 1: Recursive feasibility: append terminal control law

$$u(k|t+1)=\begin{cases}u^\star(k+1|t), & k=0,\dots,N-2\\ \kappa_f(x^\star(N|t)), & k=N-1\end{cases}$$

No plant-model mismatch:

- $x(t+1)=f(x(t),u^\star(1|t))=x^\star(1|t)$
- $x^\star(N|t)\in\mathbb{X}_f$

Assumption 2:

- $f(x^\star(N|t),\kappa_f(x^\star(N|t))\in\mathbb{X}_f$

Blue print for NMPC stability proofs with terminal constraints

- Step 2: Consider the optimal value function as a *Lyapunov* function
  - Optimal value function $V_N:\mathbb{R}^{n_x}\to\mathbb{R}_0^+$

$$V_N(x(t)) := \sum_{k=0}^{N-1}\ell(x^\star(k|t),u^\star(k|t))+V_f(x^\star(N|t))$$

  - Performance of feasible input $u(\cdot|t+1)$ applied at $x(t+1)=x^\star(1|t)$

$$J_N(x(t+1),u(\cdot|t+1)) := \sum_{k=0}^{N-1}\ell(x(k|t+1),u(k|t+1))+V_f(x(N|t+1))$$

  - Decrease of $V_N(x)$?

$$V_N(x(t+1))-V_N(x(t))\le J_N(x(t+1),u(\cdot|t+1))-V_N(x(t))$$

Blue print for NMPC stability proofs with terminal constraints

- Step 2: Consider the optimal value function as a *Lyapunov* function
  - Optimal value function $V_N : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$

$$V_N(x(t)) := \sum_{k=0}^{N-1} \ell(x^\star(k|t), u^\star(k|t)) + V_f(x^\star(N|t))$$

  - Performance of feasible input $u(\cdot|t+1)$ applied at $x(t+1) = x^\star(1|t)$

$$J_N(x(t+1), u(\cdot|t+1)) := \sum_{k=0}^{N-1} \ell(x(k|t+1), u(k|t+1)) + V_f(x(N|t+1))$$

  - Decrease of $V_N(x)$?

$$V_N(x(t+1)) - V_N(x(t)) \leq J_N(x(t+1), u(\cdot|t+1)) - V_N(x(t))$$

$$J_N(x(t+1), u(\cdot|t+1)) - V_N(x(t)) \leq -\alpha_1(\|x(t)\|) + \underbrace{\ell(x^\star(N|t), \kappa_f(x^\star(N|t))) + V_f(f(x^\star(N|t), \kappa_f(x^\star(N|t)))) - V_f(x^\star(N|t))}_{\text{Assumption 2 } \leq 0}$$

  - $V_N(x(t+1)) - V_N(x(t)) \leq -\alpha_1(\|x(t)\|)$

Tracking NMPC without terminal constraints?

1. Replace $V_f(x)$ by scaled terminal penalty $\beta V_f(x)$.
   Limon, D.; Alamo, T.; Salas, F. & Camacho, E. F. On the stability of constrained MPC without terminal constraint. *IEEE Trans. Automat. Contr.*, 2006, 51, 832-836

2. Use a control Lyapunov function as terminal penalty.
   Jadbabaie, A.; Yu, J. & Hauser, J. Unconstrained receding-horizon control of nonlinear systems. *IEEE Trans. Automat. Contr.*, 2001, 46, 776-783

3. Use a sufficiently long prediction horizon.
   Jadbabaie, A. & Hauser, J. On the stability of receding horizon control with a general terminal cost. *IEEE Trans. Automat. Contr.*, 2005, 50, 674-678

4. Consider so-called cost-controllability conditions.
   Grüne, L. & Pannek, J. Nonlinear Model Predictive Control: Theory and Algorithms. *Springer Verlag*, 2011

What changes in economic NMPC?

Tracking NMPC

- Objective: solve control task
- Stability with & without terminal constraints/penalties
- Stage cost $\ell : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$
- Terminal penalty $V_f : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$
- Terminal constraint $\mathbb{X}_f \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Prediction horizon $N \in \mathbb{N}$

$\rightarrow$ To be designed/chosen!

Economic NMPC

- Objective: optimize performance; i.e.

$$\min_{u(\cdot)} \sum_{t=0}^{\infty} \ell(x(t), u(t)) \quad \text{s.t. } \ldots$$

- Stability?
- Stage cost ℓ is given
- Terminal penalty $V_f : \mathbb{R}^{n_x} \to \mathbb{R}_0^+$
- Terminal constraint $\mathbb{X}_f \subseteq \mathbb{X} \subseteq \mathbb{R}^{n_x}$
- Prediction horizon $N \in \mathbb{N}$

$\rightarrow$ To be designed/chosen!

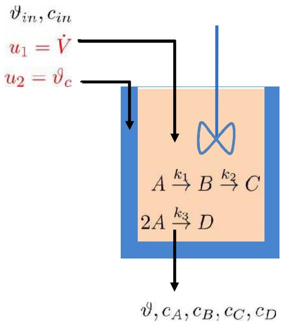
Motivating Examples

Example – Van de Vusse reactor

Van de Vusse reactor $A \overset{k_1}{\to} B \overset{k_2}{\to} C, \quad 2A \overset{k_3}{\to} D$

Dynamics (partial model)

$$\dot{c}_A = r_A(c_A, \vartheta) + (c_{in} - c_A)u_1$$
$$\dot{c}_B = r_B(c_A, c_B, \vartheta) - c_B u_1$$
$$\dot{\vartheta} = h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,$$
$$r_A(c_A, \vartheta) = -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2$$
$$r_B(c_A, c_B, \vartheta) = k_1(\vartheta)c_A - k_2(\vartheta)c_B$$
$$h(c_A, c_B, \vartheta) = -\delta\Big(k_1(\vartheta)c_A \Delta H_{AB} + k_2(\vartheta)c_B \Delta H_{BC} + 2k_3(\vartheta)c_A^2 \Delta H_{AD}\Big)$$
$$k_i(\vartheta) = k_{i0} \exp \frac{-E_i}{\vartheta + \vartheta_0}, \quad i = 1, 2, 3.$$



Constraints

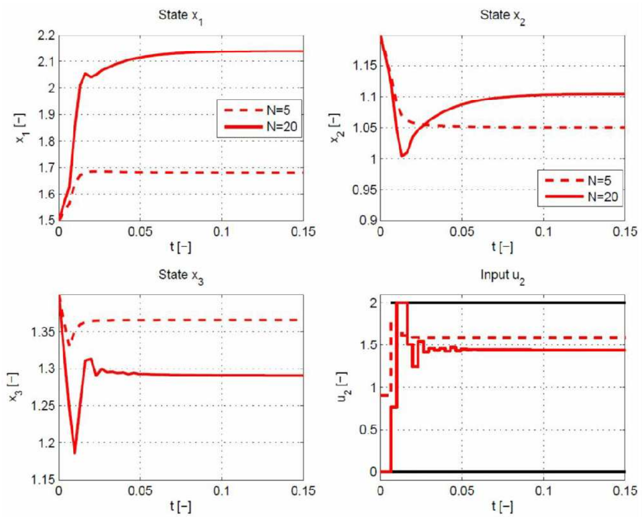
$c_A \in [0, 6]\frac{mol}{l} \quad c_B \in [0, 4]\frac{mol}{l} \quad \vartheta \in [70, 150]°C$
$u_1 \in [3, 35]\frac{1}{h} \quad u_2 \in [0, 200]°C.$

Objective = maximize produced amount of B

$$J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(t) u_1(t) dt, \qquad \beta > 0$$

Rothfuß, R.; Rudolph, J. & Zeitz, M. Flatness based control of a nonlinear chemical reactor model. *Automatica,* **1996***, 32*, 1433-1439

Example – Van de Vusse reactor



- $x_1 = \sigma_1 c_A, x_2 = \sigma_2 c_b, x_3 = \sigma_3 \vartheta$
- Discretized with Runge-Kutta 8(7), $N = 20$, sampling rate $\delta = 0.0033$

Example – Reactor with parallel reaction

- Chemical reaction: $R \longrightarrow P_1, \ R \longrightarrow P_2$
- States: $x_1 \approx$ concentration of R, $x_2 \approx$ concentration of P_1, $x_3 \approx$ dimensionless temperature
- Input: $u \approx$ heat flux through cooling jacket
- Constraints: $\mathbb{U} = [0.049, 0.449]$, $\mathbb{X} = \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$
- Dynamics

$$\dot{x}_1 = 1 - r_1(x_1, x_3) - x_1$$
$$\dot{x}_2 = r_2(x_1, x_3) - x_2$$
$$\dot{x}_3 = u - x_3$$

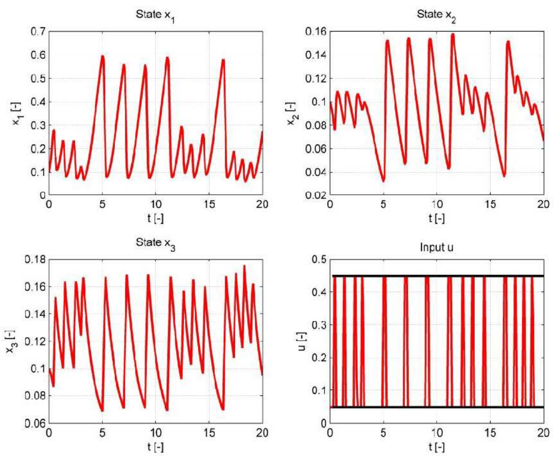
- $r_1 : \mathbb{R}^2 \to \mathbb{R}$ and $r_2 : \mathbb{R}^2 \to \mathbb{R}$:

$$r_1(x_1, x_3) = 10^4 x_1^2 e^{-\frac{1}{x_3}} + 400 x_1 e^{-\frac{0.55}{x_3}} \text{ and } r_2(x_1, x_3) = 10^4 x_1^2 e^{-\frac{1}{x_3}}.$$

- Stage cost $\ell(x) = -x_2$

Bailey, J.; Horn, F. & Lin, R. Cyclic operation of reaction systems: Effects of heat and mass transfer resistance. *AIChE Journal, Wiley Online Library*, **1971***, 17*, 818-825

Example – Reactor with parallel reaction



- Discretized with Runge-Kutta 5(4), $N = 50$, sampling rate $\delta = 0.1$

Economic model predictive control:
state of the art and open problems

Timm Faulwasser, Lars Grüne, Matthias A. Müller

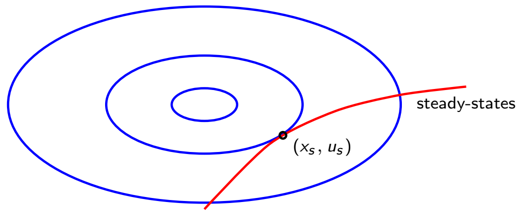
Pre-conference workshop CDC 2017

Optimal steady-state operation

Definition - optimal operation at steady-state

- Optimal steady-state: $(x_s, u_s) = \underset{x \in \mathbb{X}, u \in \mathbb{U}, x = f(x,u)}{\arg\min} \ell(x,u)$
- A system is **optimally operated at steady-state** if for each feasible state and input sequences $x(\cdot)$ and $u(\cdot)$ the following holds:

$$\liminf_{T\to\infty} \sum_{t=0}^{T-1} \frac{\ell(x(t),u(t))}{T} \geq \ell(x_s,u_s).$$



Optimal steady-state operation

Definition - Dissipativity [Willems '72, Byrnes & Lin '94]

A system is strictly **dissipative** with respect to the **supply rate** s if there exists a **storage function** $\lambda : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ it holds that

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u) - \alpha_\ell(\|(x - x_s, u - u_s)\|), \qquad \alpha_\ell \in \mathcal{K}_\infty.$$

Dissipativity and optimal steady-state operation

additional controllability condition
[Müller,Angeli,Allgöwer '15]
$\Longrightarrow$

Optimal operation at steady-state

Dissipativity w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$

$\Longleftarrow$
[Angeli,Amrit,Rawlings '12]

Optimal steady-state operation

Theorem [Angeli,Amrit,Rawlings '12]

A system is optimally operated at steady-state if it is dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$.

Sketch of proof: By dissipativity, we have

$$0 \leq \lim_{T\to\infty} \frac{\lambda(x(T)) - \lambda(x(0))}{T} \leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} [\ell(x(k),u(k)) - \ell(x_s,u_s)]}{T}$$

Optimal steady-state operation

Theorem [Willems '72]

A system is dissipative with respect to the supply rate s if and only if the available storage S_a is bounded for all x. Moreover, S_a is a possible storage function.

$$S_a(x) := \sup_{\substack{T\geq 0 \\ z(0)=x,\ z(k+1)=f(z(k),v(k)) \\ (z(k),v(k))\in\mathbb{X}\times\mathbb{U}}} \sum_{k=0}^{T-1} -s(z(k),v(k)) \qquad (1)$$

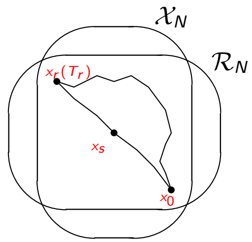
Optimal steady-state operation

Definitions

- $\mathcal{X}_N$: set of states which can be controlled to x_s in N steps
- $\mathcal{R}_N$: set of states which can be reached from x_s in N steps
- $\mathbb{Z}_N$: set of state/input pairs which are part of a feasible trajectory staying inside $\mathcal{X}_N \cap \mathcal{R}_N$

Theorem [Müller,Angeli,Allgöwer '15]

Suppose that a system is optimally operated at steady-state. Then it is dissipative on $\mathbb{Z}_N$ with supply rate $s(x,u) := \ell(x,u) - \ell(x_s,u_s)$ for each $N \geq 0$.



Sketch of proof (by contradiction):

- For each $r \geq 0$, there exist sequences with $\sum_{k=0}^{T_r-1} \ell(x_r(k),u_r(k)) \leq -r$.
- Can steer the system to x_s and from there to x_0 in N steps at a time.
- We have $\sum_{k=0}^{T_r+2N-1} \ell(x_r(k),u_r(k)) < 0$.
- $\Rightarrow \liminf_{T\to\infty} \sum_{k=0}^{T-1} \frac{\ell(\hat{x}(k),\hat{u}(k))}{T} < 0$
- This contradicts optimal steady-state operation.

Optimal steady-state operation

Definition - Dissipativity [Willems '72, Byrnes & Lin '94]

A system is **dissipative** with respect to the **supply rate** s if there exists a **storage function** $\lambda : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$ it holds that

$$\lambda(f(x,u)) - \lambda(x) \leq s(x,u).$$

Dissipativity and optimal steady-state operation

additional controllability condition

[Müller,Angeli,Allgöwer '15]

$\Longrightarrow$

Optimal operation at steady-state

Dissipativity w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$

$\Longleftarrow$

[Angeli,Amrit,Rawlings '12]

Stability using terminal constraints

If steady-state operation is optimal, does closed-loop system converge to x_s?

$$V_N(x(t)) := \min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t),u(k|t))$$

subject to

$$\begin{aligned} x(k+1|t) &= f(x(k|t),u(k|t)), \quad k = 0,\dots,N-1 \\ x(0|t) &= x(t) \\ (x(k|t),u(k|t))^\top &\in \mathbb{X}\times\mathbb{U}, \qquad k = 0,\dots,N-1 \\ x(N|t) &= x_s \end{aligned}$$

Remark: Can be extended to framework including terminal region and cost.

Theorem [Angeli,Amrit,Rawlings '12]

Assume
- strict dissipativity w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$,
- V_N and λ are continuous at x_s.

Then x_s is an asymptotically stable equilibrium of the resulting closed-loop system.

- Main idea for stability proof in stabilizing MPC: use optimal value function as Lyapunov function

$$V_N(x(t+1)) - V_N(x(t)) \le -\ell(x(t),u(t)) + \ell(x_s,u_s) \le -\alpha(\|x(t)-x_s\|)$$

- In economic MPC: second inequality **not** satisfied!

Stability using terminal constraints

Theorem [Angeli,Amrit,Rawlings '12]

Assume
- strict dissipativity w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$,
- V_N and λ are continuous at x_s.

Then x_s is an asymptotically stable equilibrium of the resulting closed-loop system.

- Define rotated cost function

$$\tilde{\ell}(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$$

- If system is strictly dissipative: $\tilde{\ell}(x,u) \ge \alpha_\ell(\|x-x_s\|)$

Original optimization problem

$$V_N(x(t)) = \min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t),u(k|t))$$

s.t. $x(0|t) = x(t), \quad x(k+1|t) = f(x(k|t),u(k|t)), \quad k = 0,\dots,N-1$

$(x(k|t),u(k|t))^\top \in \mathbb{X}\times\mathbb{U}, \quad k=0,\dots,N-1, \quad x(N|t) = x_s$

Stability using terminal constraints

Theorem [Angeli,Amrit,Rawlings '12]

Assume
- strict dissipativity w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$,
- V_N and λ are continuous at x_s.

Then x_s is an asymptotically stable equilibrium of the resulting closed-loop system.

- Define rotated cost function

$$\tilde{\ell}(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$$

- If system is strictly dissipative: $\tilde{\ell}(x,u) \ge \alpha_\ell(\|x-x_s\|)$

Modified optimization problem

$$\widetilde{V}_N(x(t)) = \min_{u(\cdot|t)} \sum_{k=0}^{N-1} \tilde{\ell}(x(k|t),u(k|t))$$

s.t. $x(0|t) = x(t), \quad x(k+1|t) = f(x(k|t),u(k|t)), \quad k = 0,\dots,N-1$

$(x(k|t),u(k|t))^\top \in \mathbb{X}\times\mathbb{U}, \quad k=0,\dots,N-1, \quad x(N|t) = x_s$

Stability using terminal constraints

Theorem [Angeli,Amrit,Rawlings '12]

Assume
- strict dissipativity w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$,
- V_N and λ are continuous at x_s.

Then x_s is an asymptotically stable equilibrium of the resulting closed-loop system.

- Define rotated cost function

$$\tilde{\ell}(x,u) = \ell(x,u) - \ell(x_s,u_s) + \lambda(x) - \lambda(f(x,u))$$

- If system is strictly dissipative: $\tilde{\ell}(x,u) \ge \alpha_\ell(\|x-x_s\|)$

- Key step: original and modified optimization problem have **same** solution
- Can use $\widetilde{V}_N$ as Lyapunov function:

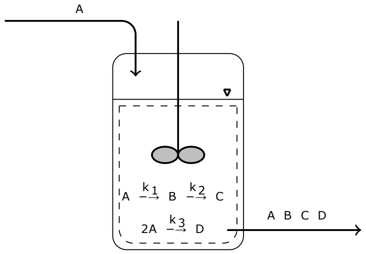
$$\widetilde{V}_N(x(t+1)) - \widetilde{V}_N(x(t)) \le -\tilde{\ell}(x(t),u(t)) \le -\alpha_\ell(\|x(t)-x_s\|)$$

Example - chemical reactor with dissipativity

Van de Vusse reactor:

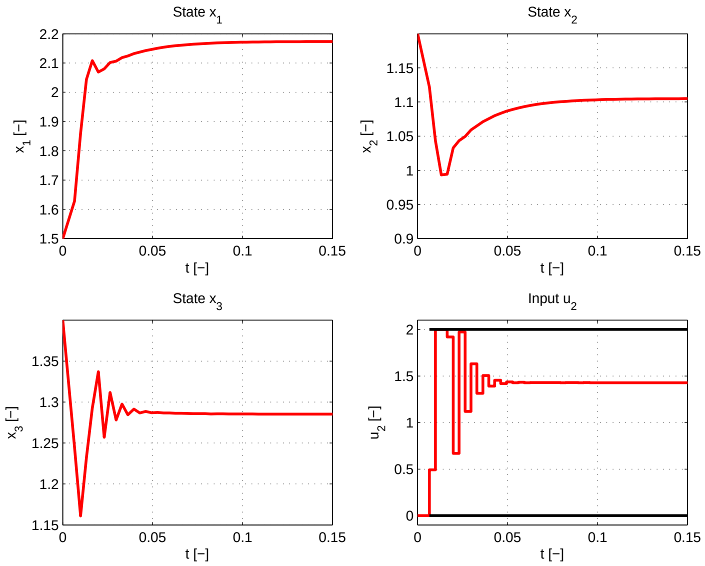
- Reactions $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ and $2A \xrightarrow{k_3} D$, with A: reactant, B: desired product, C, D: waste products

$$\dot c_A = r_A(c_A, \vartheta) + (c_{in} - c_A)u_1$$
$$\dot c_B = r_B(c_A, c_B, \vartheta) - c_B u_1$$
$$\dot\vartheta = h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,$$



  ϑ: temperature in the reactor, u_1: normalized flow rate of A, u_2: temperature in cooling jacket
- Control objective: maximize production rate of $B \rightarrow \ell(x,u) = -c_B u_1$
- System is strictly dissipative w.r.t. supply rate $s(x,u) = \ell(x,u) - \ell(x_s,u_s)$

Example - chemical reactor with dissipativity

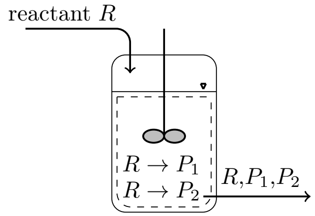


Example - chemical reactor without dissipativity

Continuous flow stirred-tank reactor with parallel reactions

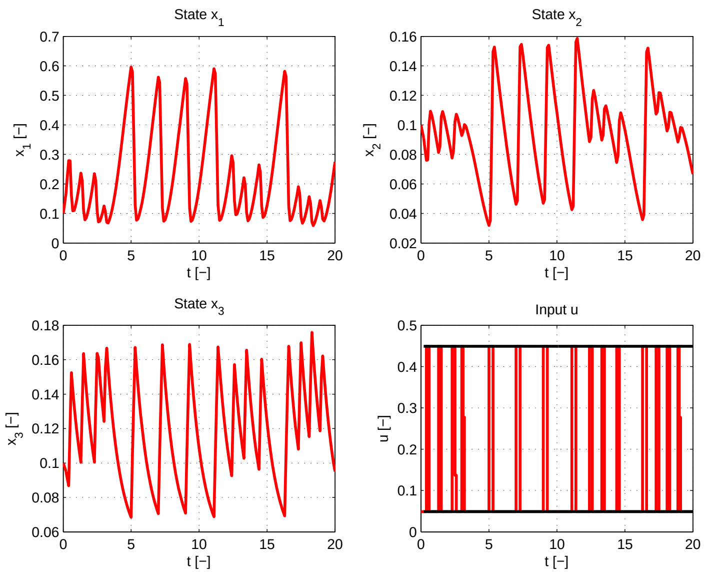
- Reactions $R \rightarrow P_1$ and $R \rightarrow P_2$, with R: reactant, P_1: desired product, P_2: waste product

$$\dot x_1 = 1 - 10^4 x_1^2 e^{-1/x_3} - 400 x_1 e^{-0.55/x_3} - x_1$$
$$\dot x_2 = 10^4 x_1^2 e^{-1/x_3} - x_2$$
$$\dot x_3 = u - x_3$$



  x_1: concentration of R, x_2: concentration of P_1, x_3: temperature in the reactor, u: proportional to heat flux through cooling jacket
- Control objective: maximize product $P_1 \rightarrow \ell(x,u) = -x_2$

Example - chemical reactor without dissipativity

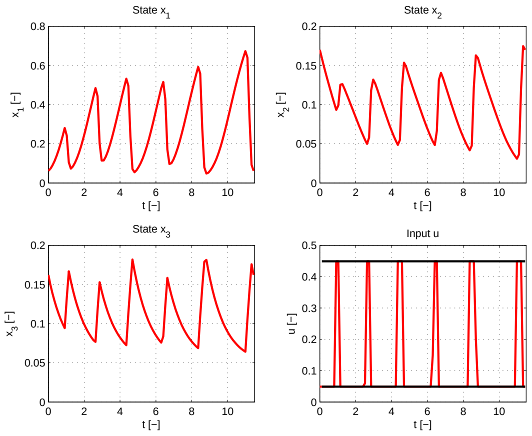


Example - chemical reactor without dissipativity

Optimal periodic orbit length: $T^\star \approx 11.444$

$$\min_{u(\cdot),\,T} \frac{1}{T}\int_0^T -x_2(\tau)\mathrm{d}\tau \tag{2}$$

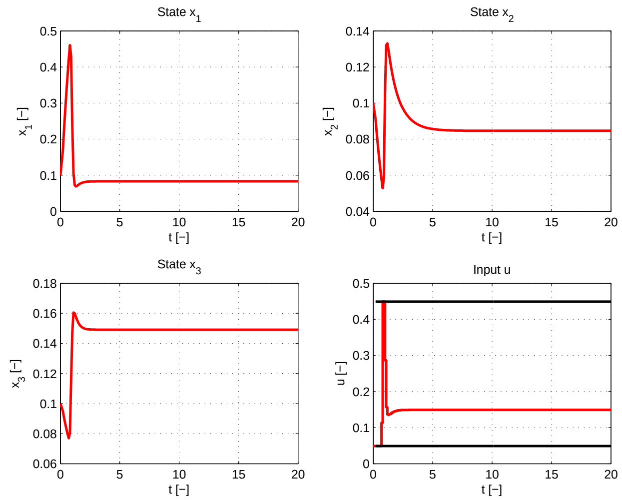
subject to $\quad x(0) = x(T), \quad T \in [5, 20].$



Example - chemical reactor without dissipativity

Recovering steady-state optimality through regularization:

$$\ell(x, u) = -x_2 + \omega(u - u_s)^2, \quad \omega > 0$$



Asymptotic average and transient performance

What can be said about closed-loop performance?

- Infinite horizon averaged performance:
  $\overline{J}^{cl}_\infty(x_0, \mu_N) := \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ell(x(t), \mu_N(x(t)))$

Theorem [Angeli,Amrit,Rawlings '12]

$$\overline{J}^{cl}_\infty(x_0, \mu_N) \le \ell(x_s, u_s)$$

Sketch of proof:

- $V_N(x(t+1)) - V_N(x(t)) \le -\ell(x(t), u(t)) + \ell(x_s, u_s)$
- Iterate this inequality, divide by T and take $\liminf$

Remark: This bound is valid independent of dissipativity.

Asymptotic average and transient performance

What can be said about closed-loop performance?

- Infinite horizon non-averaged performance:
  $J^{cl}_\infty(x_0, \mu_N) := \limsup_{T\to\infty} \sum_{t=0}^{T-1} \ell(x(t), \mu_N(x(t)))$
- Finite horizon non-averaged performance:
  $J^{cl}_T(x_0, \mu_N) := \sum_{t=0}^{T-1} \ell(x(t), \mu_N(x(t)))$
- **Assumption:** Strict dissipativity plus technical (continuity) assumptions on storage and optimal value function.
  $\Rightarrow$ Closed loop satisfies $\|x(t) - x_s\| \le \beta(\|x_0 - x_s\|, t)$ with $\beta \in \mathcal{KL}$.
- Define $\mathbb{U}^T_{\kappa(x_0)} := \{u \in \mathbb{U}^T \mid u \text{ admissible and } \|x(T, x_0, u) - x_s\| \le \kappa\}$

Theorem [Grüne & Panin '15]

The following performance bounds hold:

- $J^{cl}_\infty(x_0, \mu_N) \le V_\infty(x_0) + \delta(N)$ with $\delta \in \mathcal{L}$
- $J^{cl}_T(x_0, \mu_N) \le \inf_{u\in\mathbb{U}^T_{\kappa(x_0)}} J_T(x_0, u) + \delta_1(N) + \delta_2(T)$ with $\kappa = \beta(\|x - x_0\|, T)$ and $\delta_1, \delta_2 \in \mathcal{L}$

D. Angeli, R. Amrit and J. B. Rawlings, On average performance and stability of economic model predictive control, IEEE TAC 57(7), pp. 1615-1626, 2012.

C. I. Byrnes and W. Lin, Losslessness, Feedback Equivalence, and the Global Stabilization of Discrete-Time Nonlinear Systems, IEEE TAC 39(1), pp. 83-98, 1994.

L. Grüne and A. Panin, On non-averaged performance of economic MPC with terminal conditions, IEEE CDC, pp. 4332-4337, 2015.

M. A. Müller, D. Angeli and F. Allgöwer, On necessity and robustness of dissipativity in economic model predictive control, IEEE TAC 60(6), pp. 1671-1676, 2015.

J. C. Willems, Dissipative Dynamical Systems - Part I: General Theory, Archive for Rational Mechanics and Analysis 45(5), pp. 321-351, 1972.

Turnpike Properties

Example – Optimal fish harvest

$$\min_{u(\cdot)} \int_0^T ax(t) + bu(t) - cx(t)u(t)dt$$

subject to

$$\dot{x} = x(x_S - x - u), \quad x(0) = x_0$$
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$

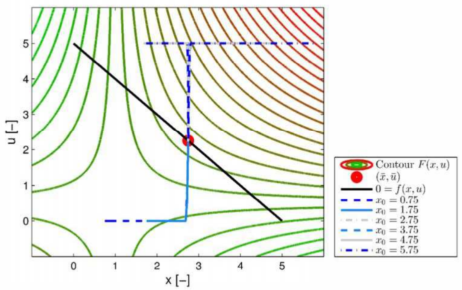
- x fish density
- u fishing rate
- $x_s = 5$ highest sustainable fish density
- $a = 1, b = c = 2, u_{max} = 5$

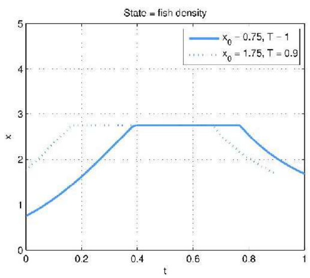
Example – Optimal fish harvest (bilinear objective)

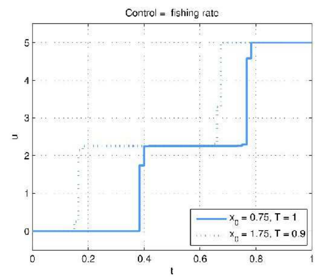
$$\min_{u(\cdot)} \int_0^T ax(t) + bu(t) - cx(t)u(t)dt$$

subject to

$$\dot{x} = x(x_S - x - u), \quad x(0) = x_0$$
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$







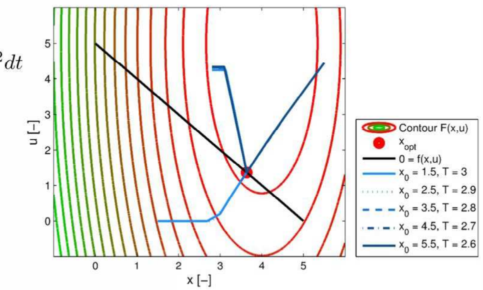
Cliff, E. & Vincent, T. An optimal policy for a fish harvest. *Journal of Optimization Theory and Applications*, **1973**, *12*, 485-496

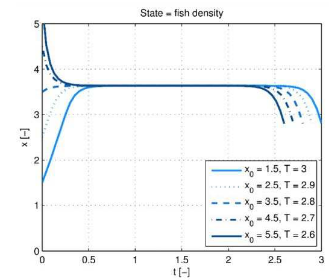
Example – Optimal fish harvest (quadratic objective)

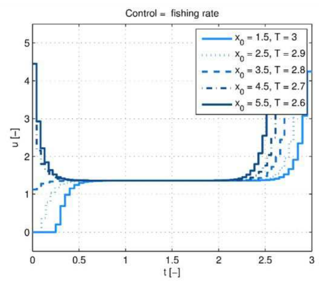
$$\min_{u(\cdot)} \int_0^T \tfrac{1}{2}q(x(t) - x_C)^2 + \tfrac{1}{2}r(u(t) - u_C)^2 dt$$

subject to

$$\dot{x} = x(x_S - x - u), \quad x(0) = x_0$$
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$
$$u_{max} = 5, x_S = 5$$
$$q = 10, r = 1, x_C = 4, u_C = 5$$





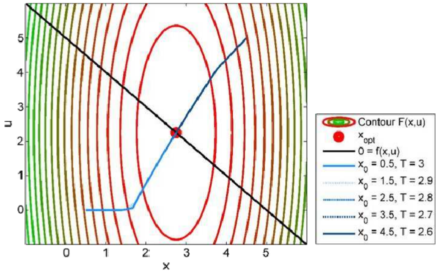


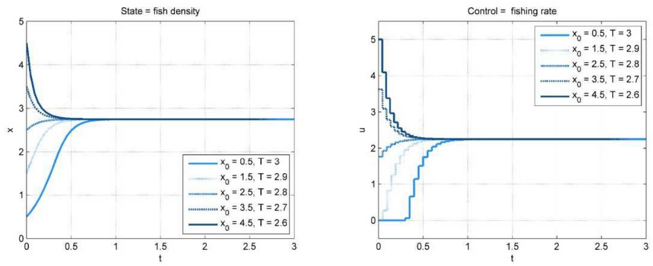
Example – Optimal fish harvest (quad. objective cont'd)

$$\min_{u(\cdot)} \int_0^T \tfrac{1}{2}q(x(t)-x_C)^2 + \tfrac{1}{2}r(u(t)-u_C)^2 dt$$

subject to

$$\dot{x} = x(x_S - x - u), \quad x(0) = x_0$$
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$
$$u_{max} = 5, x_S = 5$$
$$q = 10, r = 1, x_C = 2.75, u_C = 2.25$$





→ Similar behavior for different initial conditions and horizon lengths.
→ Similarity properties of solutions of parametric OCPs.

Turnpike property!

Assumptions for economic NMPC without terminal constraints

Considered OCP

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) \tag{2}$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}$$

Assumption 3 (Strict dissipativity of OCP (2)).
There exists a bounded non-negative storage function $\lambda : \mathbb{X} \to \mathbb{R}_0^+$ such that OCP (2) is strictly dissipative with respect to $(x_s, u_s) \in \text{int}(\mathbb{X} \times \mathbb{U})$ in the sense of the Definition on slide II.3.

Assumption 4 (Exponential reachability of x_s).
For all $x_0 \in \mathbb{X}_0$, there exists an infinite-horizon admissible input $u(\cdot; x_0)$, $c > 0$, $\rho \in [0, 1)$, such that

$$\|(x(k; x_0, u(\cdot; x_0)), u(k; x_0)) - (x_s, u_s)\| \le c\rho^k,$$

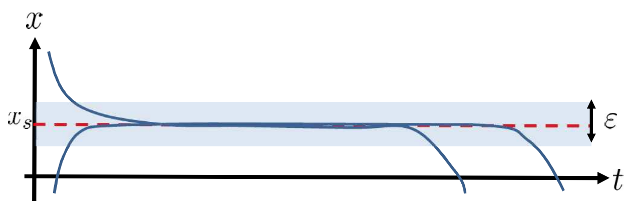
i.e. the steady state x_s is exponentially reachable.

Turnpike properties in OCPs

Proposition (Turnpike in OCP (2)).
Let Assumptions 3 and 4 hold, and suppose that the storage function λ is bounded on $\mathbb{X}$. Then there exists $C < \infty$, such that, for all $x_0 \in \mathbb{X}_0$, we have

$$\#\mathbb{Q}_\varepsilon \ge N - \frac{C}{\alpha_\ell(\varepsilon)}$$

where $\mathbb{Q}_\varepsilon := \{k \in \{0, \dots, N-1\} \,|\, \|(x^\star(k; x_0), u^\star(k; x_0)) - (x_s, u_s)\| \le \varepsilon\}$, $\#\mathbb{Q}_\varepsilon$ is the cardinality of $\mathbb{Q}_\varepsilon$— i.e., the amount of time an optimal pair spends inside an ε-ball centered at (x_s, u_s)— , and $\alpha_\ell \in \mathcal{K}_\infty$ is from the dissipation inequality on slide II.3.



Turnpike properties in OCPs

Proof sketch

- $V_N(x_0)$ is the optimal value function of OCP (2).
- $\ell(x_s, u_s) = 0$
- The strict dissipation inequality implies

$$V_N(x_0) \ge \underbrace{\lambda(x^\star(N, x_0)) - \lambda(x_0)}_{\ge -2\bar{\lambda} := \sup_{x \in \mathbb{X}} |\lambda(x)|} + \sum_{k=0}^{N-1} \alpha_\ell(\|(x^\star(k; x_0), u^\star(k; x_0)) - (x_s, u_s)\|)$$

Turnpike properties in OCPs

Proof sketch

- $V_N(x_0)$ is the optimal value function of OCP (2).
- $\ell(x_s, u_s) = 0$
- The strict dissipation inequality implies

$$V_N(x_0) \geq \underbrace{\lambda(x^\star(N,x_0)) - \lambda(x_0)}_{\geq -2\bar{\lambda} := \sup_{x\in\mathbb{X}} |\lambda(x)|} + \sum_{k=0}^{N-1} \alpha_\ell(\|(x^\star(k;x_0), u^\star(k;x_0)) - (x_s,u_s)\|)$$

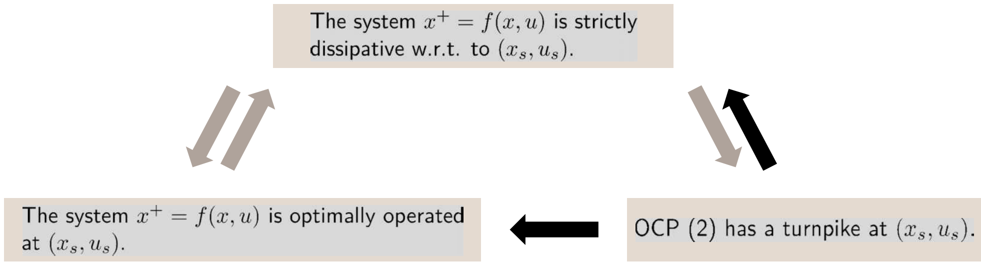
- Exp. reachability implies: $V_N(x_0) \leq \dfrac{L_\ell c}{1-\rho}$
- $\displaystyle\sum_{k=0}^{N-1} \alpha_\ell(\|(x^\star(k;x_0), u^\star(k;x_0)) - (x_s,u_s)\|) \geq (N - \#\mathbb{Q}_\varepsilon)\alpha_\ell(\varepsilon)$

$\Rightarrow$

$$\#\mathbb{Q}_\varepsilon \geq N - \frac{L_\ell c(1-\rho)^{-1} + 2\bar{\lambda}}{\alpha_\ell(\varepsilon)}$$

Relation of dissipativity and turnpike properties

Under suitable technical assumptions, additional relations (black arrows) can be established:



→ Turnpike and dissipativity properties of OCPs are *essentially, i.e.almost,* equivalent.

References

- Faulwasser et al. On Turnpike and Dissipativity Properties of Continuous-Time Optimal Control Problems. *Automatica,* **2017**, *81*, 297-304
- Grüne, L. & Müller, M. On the relation between strict dissipativity and turnpike properties. *Sys. Contr. Lett.*, **2016**, *90*, 45 - 53

Practical Stability without Terminal Constraints

Recursive feasibility

Assumption 5 (Local controllability around (x_s, u_s)).
The Jacobian linearization of $x^+ = f(x,u)$ at (x_s, u_s) is n_x-step reachable.

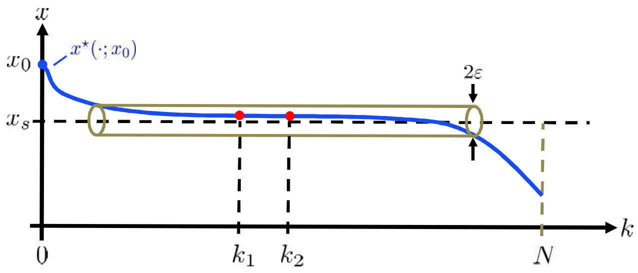
Proposition (Recursive feasibility of OCP (2)).
Let Assumptions 3–5 hold. Then, there exists a finite horizon $N \in \mathbb{N}$ such that, for all $x_0 \in \mathbb{X}_0$, OCP (2) is recursively feasible.

Faulwasser, T. & Bonvin, D. On the Design of Economic NMPC based on an Exact Turnpike Property. *9th IFAC International Symposium on Advanced Control of Chemical Processes,* **2015**
Faulwasser, T. & Bonvin, D. On the Design of Economic NMPC based on Approximate Turnpike Properties. *54th IEEE Conference on Decision and Control,* **2015**, 4964 - 4970

Recursive feasibility

Proof sketch

- The turnpike property of OCP (2) implies that, for any $\varepsilon > 0$, there exists a finite N such that k_1, k_2, with $k_1 + 2n_x \le k_2 \le N$, such that $x_1^\varepsilon := x^\star(k_1; x_0) \in \mathcal{B}_\varepsilon(x_s)$ and $x_2^\varepsilon := x^\star(k_2; x_0) \in \mathcal{B}_\varepsilon(x_s)$.



Recursive feasibility

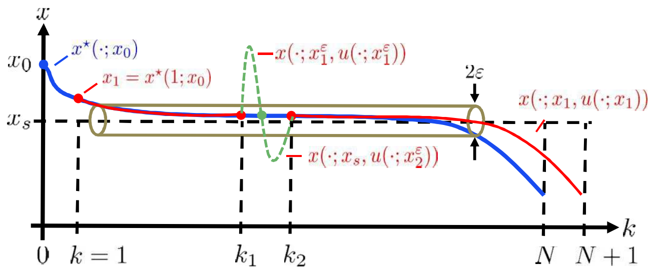
Proof sketch (cont'd)

- Controllability of the linearization at (x_s, u_s) guarantees existence of $u_{1,2}^\varepsilon(\cdot)$ close to (x_s, u_s) such that

$$x(n_x; x_1^\varepsilon, u_1^\varepsilon(\cdot; x_1^\varepsilon)) = x_s \quad \text{and} \quad x(n_x; x_s, u_2^\varepsilon(\cdot; x_2^\varepsilon)) = x_2^\varepsilon$$
$$x(k; x_1^\varepsilon, u_1^\varepsilon(\cdot; x_1^\varepsilon)) \in \mathbb{X},\ x(k; x_s, u_2^\varepsilon(\cdot; x_2^\varepsilon)) \in \mathbb{X},$$

- Consider

$$u(k; x_1) = \begin{cases} u^\star(k+1; x_0) & k = 0, \dots, k_1 - 2 \\ u_1^\varepsilon(k; x_1^\varepsilon) & k = k_1 - 1, \dots, k_1 - 1 + n_x \\ u_2^\varepsilon(k; x_2^\varepsilon) & k = k_1 + n_x, \dots, k_1 - 1 + 2n_x \\ u^\star(k; x_0) & k = k_1 + 2n_x, \dots, N - 1 \end{cases}.$$



Recap – Rotated OCP

- Rotated sage cost

$$\widetilde{\ell}(x, u) = \ell(x, u) - \ell(x_s, u_s) + \lambda(x) - \lambda(f(x, u))$$

- Rotated OCP

$$\widetilde{V}_N(x(t)) := \min_{u(\cdot|t)} \sum_{k=0}^{N-1} \widetilde{\ell}(x(k|t), u(k|t))$$

subject to

$$x(k+1|t) = f(x(k|t), u(k|t)), \quad x(0|t) = x(t)$$
$$(x(k|t), u(k|t))^\top \in \mathbb{X} \times \mathbb{U}$$

Stability of economic NMPC without terminal constraints

Theorem (Practical stability of economic NMPC with terminal constraints).
Let Assumptions 3–5 hold and suppose that $\mathbb{X}$ is compact. Then, for sufficiently large horizon $N \in \mathbb{N}$ the closed-loop system $x^+ = f(x, \mu_N(x))$ satisfies:

(i) If, for the horizon $N \in \mathbb{N}$, OCP (2) is feasible for $t = 0$ and $x(0) \in \mathbb{X}_0$, then it is feasible for all $k \in \mathbb{N}$.

(ii) There exist $\rho \in \mathbb{R}^+$ and $\beta \in \mathcal{KL}$ such that, for all $x(0) \in \mathbb{X}_0$, the closed-loop trajectories generated by $x^+ = f(x, \mu_N(x))$ satisfy

$$\|x(t) - x_s\| \le \max\{\beta(\|x_0 - x_s\|, t),\ \rho\}.$$

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$$\|x(t) - x_s\| \leq \max\{\beta(\|x_0 - x_s\|, t), \rho\}.$$

(iii) If additionally

(a) there exist $\gamma_V \in \mathcal{K}$ such that for each $N \in \mathbb{N}$ and all $x \in \mathbb{X}_0$ $|\widetilde{V}_N(x) - \widetilde{V}_N(x_s)| \leq \gamma_{\widetilde{V}}(\|x - x_s\|)$,

(b) and the storage function λ is continuous at $x = x_s$,

then (ii) holds with $\rho = \rho(N)$ where $\rho(N) \to 0$ for $N \to \infty$.

Stability of economic NMPC without terminal constraints

Theorem (Practical stability of economic NMPC with terminal constraints).
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(ii) There exist $\rho \in \mathbb{R}^+$ and $\beta \in \mathcal{KL}$ such that, for all $x(0) \in \mathbb{X}_0$, the closed-loop trajectories generated by $x^+ = f(x, \mu_N(x))$ satisfy

$$\|x(t) - x_s\| \leq \max\{\beta(\|x_0 - x_s\|, t), \rho\}.$$

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(a) there exist $\gamma_V \in \mathcal{K}$ such that for each $N \in \mathbb{N}$ and all $x \in \mathbb{X}_0$ $|\widetilde{V}_N(x) - \widetilde{V}_N(x_s)| \leq \gamma_{\widetilde{V}}(\|x - x_s\|)$,

(b) and the storage function λ is continuous at $x = x_s$,

then (ii) holds with $\rho = \rho(N)$ where $\rho(N) \to 0$ for $N \to \infty$.

Grüne, L. Economic receding horizon control without terminal constraints. *Automatica* , **2013**, *49*, 725-734
Grüne, L. & Stieler, M. Asymptotic stability and transient optimality of economic MPC without terminal conditions. *Journal of Process Control*, **2014**, *24*, 1187-1196
Faulwasser, T. & Bonvin, D. On the Design of Economic NMPC based on Approximate Turnpike Properties. *54th IEEE Conference on Decision and Control*, **2015**, 4964 - 4970

Proof sketch

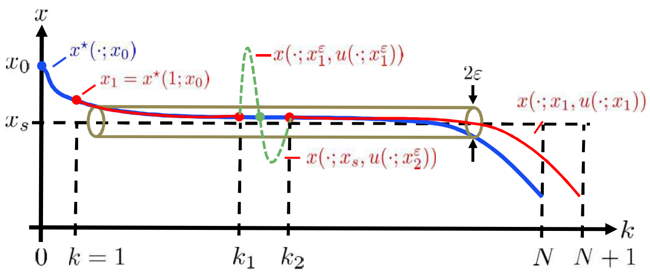
Part (i): already shown.

Part (ii): w.l.o.g. $l(x_s, u_s) = 0$

- Consider shifted value function $\hat{V}_N(x) := \lambda(x) + V_N(x) - V_N(x_s)$

- Decrease condition:

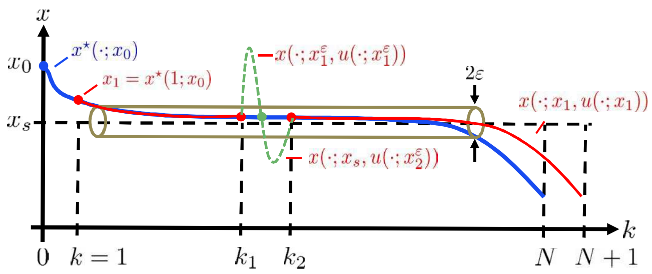
$$\hat{V}_N(x(t+1)) - \hat{V}_N(x(t)) \leq \underbrace{\lambda(x(t+1)) + J_N(x(t+1), u(\cdot|t+1)) - V_N(x_s) - \hat{V}_N(x(t))}_{=:\,\Delta(t)}$$



Proof sketch

Part (ii) (cont'd):

- $$\begin{aligned}\Delta(t) &= \lambda(x(t+1)) - \lambda(x(t)) - \ell(x(t), u^\star(0|t)) \\ &+ \sum_{k=0}^{k_1-1} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=1}^{k_1} \ell(x^\star(k|t), u^\star(k|t)) \\ &+ \sum_{k=k_1}^{k_1-1+2n_x} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+1}^{k_1-1+2n_x} \ell(x^\star(k|t), u^\star(k|t)) \\ &+ \sum_{k=k_1+2n_x}^{N-1} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+2n_x}^{N-1} \ell(x^\star(k|t), u^\star(k|t))\end{aligned}$$



Proof sketch

Part (ii) (cont'd):

- $$\begin{aligned}\Delta(t) &= \lambda(x(t+1)) - \lambda(x(t)) - \ell(x(t), u^\star(0|t)) \\ &+ \sum_{k=0}^{k_1-1} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=1}^{k_1} \ell(x^\star(k|t), u^\star(k|t)) \\ &+ \sum_{k=k_1}^{k_1-1+2n_x} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+1}^{k_1-1+2n_x} \ell(x^\star(k|t), u^\star(k|t)) \\ &+ \sum_{k=k_1+2n_x}^{N-1} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+2n_x}^{N-1} \ell(x^\star(k|t), u^\star(k|t))\end{aligned}$$

$$\sum_{k=k_1}^{k_1-1+2n_x} \ell(x(k|t+1), u(k|t+1)) - \sum_{k=k_1+1}^{k_1-1+2n_x} \ell(x^\star(k|t), u^\star(k|t)) \leq \ell(x(k_1|t+1), u(k_1|t+1)) + 2n_x L_\ell c(\varepsilon) \leq (2n_x+1) L_\ell c(\varepsilon)$$

$$\Rightarrow \quad \hat{V}_N(x(t+1)) - \hat{V}_N(x(t)) \leq \Delta(t) \leq -\alpha_\ell(\|x(t) - x_s\|) + (2n_x+1) L_\ell c(\varepsilon)$$

Proof sketch

Part (iii):

Lemma (Relation between $\widetilde{V}_N$ and V_N).
Let Assumptions 3–5 hold. Moreover,

1. let there exist $\gamma_{\widetilde{V}} \in \mathcal{K}$ such that for each $N \in \mathbb{N}$ and all $x \in \mathbb{X}_0$ $|\widetilde{V}_N(x) - \widetilde{V}_N(x_s)| \leq \gamma_{\widetilde{V}}(\|x - x_s\|)$,
2. and let the storage function λ be continuous at $x = x_s$.

Then
$$\widetilde{V}_N(x) = V_N(x) + \lambda(x) - V_N(x_s) + R(x, N)$$
with $|R(x,N)| \leq \nu(\|x - x_s\|) + \omega(N)$, $\nu \in \mathcal{K}, \omega \in \mathcal{L}$.

$$\Rightarrow \quad \hat{V}_N(x(t+1)) - \hat{V}_N(x(t)) \leq -\alpha_\ell(\|x(t) - x_s\|) + \omega(N) \text{ with } \omega \in \mathcal{L}$$

Grüne, L. & Pannek, J. Nonlinear Model Predictive Control: Theory and Algorithms. *Springer Verlag*, **2017**
Grüne, L. & Stieler, M. Asymptotic stability and transient optimality of economic MPC without terminal conditions. *Journal of Process Control*, **2014**, *24*, 1187-1196

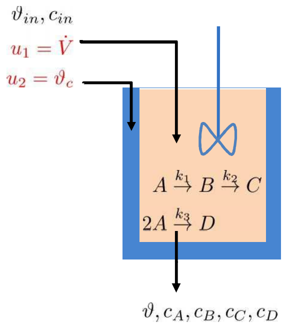
Example – Van de Vusse reactor (revisited)

Van de Vusse reactor $\quad A \xrightarrow{k_1} B \xrightarrow{k_2} C, \qquad 2A \xrightarrow{k_3} D$

Dynamics (partial model)

$$\begin{aligned}\dot{c}_A &= r_A(c_A, \vartheta) + (c_{in} - c_A)u_1 \\ \dot{c}_B &= r_B(c_A, c_B, \vartheta) - c_B u_1 \\ \dot{\vartheta} &= h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,\end{aligned}$$

$$\begin{aligned}r_A(c_A, \vartheta) &= -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2 \\ r_B(c_A, c_B, \vartheta) &= k_1(\vartheta)c_A - k_2(\vartheta)c_B \\ h(c_A, c_B, \vartheta) &= -\delta\Big(k_1(\vartheta)c_A \Delta H_{AB} + k_2(\vartheta)c_B \Delta H_{BC} + 2k_3(\vartheta)c_A^2 \Delta H_{AD}\Big) \\ k_i(\vartheta) &= k_{i0} \exp \frac{-E_i}{\vartheta + \vartheta_0}, \quad i = 1, 2, 3.\end{aligned}$$



Constraints

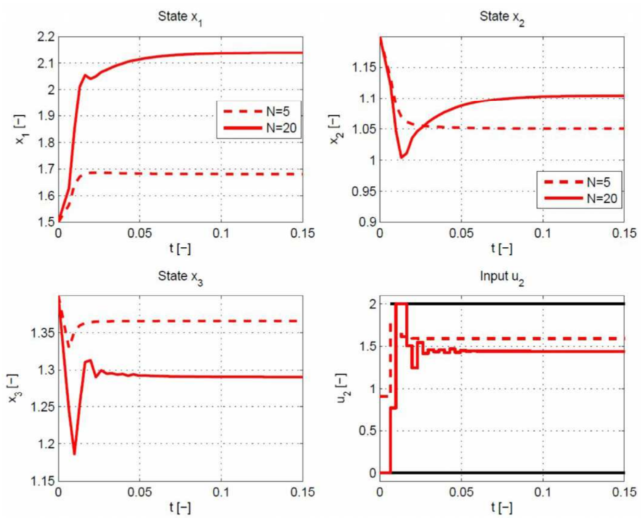
$$c_A \in [0, 6]\tfrac{mol}{l} \quad c_B \in [0, 4]\tfrac{mol}{l} \quad \vartheta \in [70, 150]^\circ C$$
$$u_1 \in [3, 35]\tfrac{1}{h} \quad u_2 \in [0, 200]^\circ C.$$

Objective = maximize produced amount of B

$$J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(t) u_1(t) dt, \qquad \beta > 0$$

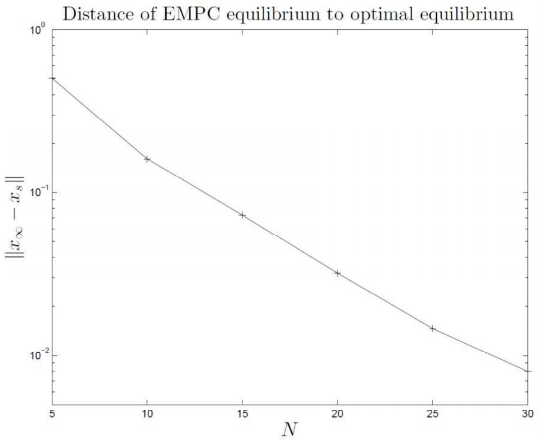
Rothfuß, R.; Rudolph, J. & Zeitz, M. Flatness based control of a nonlinear chemical reactor model. *Automatica*, **1996**, *32*, 1433-1439

Example – Van de Vusse reactor (revisited)



- $x_1 = \sigma_1 c_A, x_2 = \sigma_2 c_b, x_3 = \sigma_3 \vartheta$
- Discretized with Runge-Kutta 8(7), $N = 20$, sampling rate $\delta = 0.0033$

Example – Van de Vusse reactor (revisited)



- As predicted by the last theorem, for increasing horizon N, the closed-loop system converges to smaller neighborhoods of the turnpike x_s

Economic Model Predictive Control

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Workshop at the 56th IEEE Conference on Decision and Control
Melbourne, 10 December 2017

IV. EMPC without strict dissipativity

EMPC without strict dissipativity

In this section we discuss a selection of schemes which use relaxed terminal conditions or yield stability without imposing strict dissipativity

Outline

- Generalized terminal constraints
- Lyapunov-based approach
- Multi-objective approach

Generalized terminal constraints

It may happen that

- EMPC with equilibrium terminal constraints $x(t|N) = x_s$ is too restrictive / numerically infeasible
- the terminal cost V_f for EMPC with regional terminal constraints $x(t|N) \in \mathbb{X}_f$ is too difficult to compute

In these cases, other types of constraints may be useful

Idea: Require that $x(t|N)$ is an equilibrium, but not necessarily equal to x_s

[Fagiano/Teel '13, Müller/Angeli/Allgöwer '13, Ferramosca/Limon/Camacho '14]
(based on earlier ideas from stabilizing MPC)

Scheme with generalized terminal constraints

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t),u(k|t)) + \beta\ell(x(N|t),u(N|t))$$

subject to

$$\begin{aligned} x(k+1|t) &= f(x(k|t),u(k|t)), \quad k=0,\dots,N-1 \\ x(0|t) &= x(t) \\ (x(k|t),u(k|t))^\top &\in \mathbb{X}\times\mathbb{U}, \quad k=0,\dots,N \\ x(N|t) &= f(x(N|t),u(N|t)), \\ \ell(x(N|t),u(N|t)) &\le \kappa(t), \end{aligned}$$

where $\kappa(t+1)=\ell(x(N|t),u(N|t))$, $\kappa(0)$ "large", $\beta>0$

- Always end in an equilibrium that is at least as good as the previous one
- A large β provides incentive to select a good equilibrium

Properties

Theorem [Fagiano/Teel '13] Given $\varepsilon>0$, there exists $\beta(\varepsilon)>0$ such that

$$\ell(x^\star(N|t),u^\star(N|t)) \le \ell_{\min}(x(t)) + \varepsilon$$

where $\ell_{\min}(x(t))$ is the cost of the best equilibrium that is reachable from initial condition $x(t)$ in N steps

Problem: $\ell_{\min}$ may be significantly larger than $\ell(x_s,u_s)$

Properties

Using this β and the assumption that from each steady state (x,u) a better steady state (x',u'), i.e.,

$$\ell(x',u') \le \max\{\ell(x_s,u_s), \ell(x,u)-\varepsilon\}$$

can be reached in N steps, [Fagiano/Teel '13] propose an EMPC scheme which eventually reaches $\ell(x_s,u_s)$ up to ε

Problems:

- The scheme discards recent optimization results if the terminal equilibrium value does not improve
- The appropriate β may be difficult to find

The second point can be addressed by the adaptive choice

$$\beta(t+1) = B(\beta(t),x(t),\kappa(t)), \quad \beta(0)=\beta_0\ge 0$$

where β increases as long as the terminal equilibrium value can be improved [Müller/Angeli/Allgöwer '13f]

Discussion

Discussion of generalized equilibrium terminal constraints

- Averaged performance is bounded by "eventual" terminal equilibrium
- No transient performance estimates known (problem: influence of β)
- Asymptotic stability of the optimal steady state can be shown under additional (so far still rather restrictive) conditions, including strict dissipativity [Ferramosca/Limon/Camacho '14]
- Results can be extended to generalized regional terminal constraints [Müller/Angeli/Allgöwer '14] and to periodic constraints [Limon/Pereira/Muñoz de la Peña/Alamo/Grosso '14, Houska/Müller '17]

Lyapunov based EMPC

Lyapunov based EMPC combines the goals of stabilizing and economic MPC

- stabilize a given set Ω $(\Omega = \{x_s\}$ or a larger set)
- while at the same time minimizing an economic objective

The algorithmic ideas described in the next slides go back to [Heidarinejad/Liu/Christofides '12]

They rely on the knowledge of a stabilizing controller and a corresponding Lyapunov function for the system

Lyapunov function

Let $x_s \in \mathbb{X}$ be an equilibrium with open neighborhood O

Let $h : O \to \mathbb{U}$ a controller with $f(x, h(x)) \in O$ for all $x \in O$

$W : O \to \mathbb{R}$ is a Lyapunov function with respect to h if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for all $x \in O$ we have

$$\alpha_1(|x - x_s|) \le W(x) \le \alpha_2(|x - x_s|)$$

and

$$W(f(x, h(x))) \le W(x) - \alpha_3(|x - x_s|)$$

Note: decrease of W ensures asymptotic stability of any level set $\Omega := \{x \in \mathbb{R}^n \,|\, W(x) \le \rho\}$, $\rho \ge 0$, for $x^+ = f(x, h(x))$

Idea: impose decrease of W as additional constraint in the EMPC scheme, until Ω is reached

Setting

The set Ω to be stabilized is given as a level set of the Lyapunov function W, i.e.,

$$\Omega := \{x \in \mathbb{R}^n \,|\, W(x) \le \rho\}$$

for fixed $\rho \ge 0$

Note: $\rho = 0$ implies $\Omega = \{x_s\}$, i.e., stabilization of the optimal equilibrium is included as a special case

Lyapunov based EMPC scheme

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t))$$

subject to

$$\begin{aligned}
x(k+1|t) &= f(x(k|t), u(k|t)), \quad k = 0, \dots, N-1 \\
x(0|t) &= x(t) \\
(x(k|t), u(k|t))^\top &\in \mathbb{X} \times \mathbb{U}, \quad k = 0, \dots, N \\
W(x(1|t)) &\le W(f(x(t), h(x(t))) \quad \text{if } W(x(t)) > \rho \\
W(x(k|t)) &\le \rho, \quad k = 0, \dots, N \quad \text{if } W(x(t)) \le \rho
\end{aligned}$$

Idea: enforce decrease of W until Ω is reached, afterwards remain in Ω by ensuring $W(x(k|t)) \le \rho$

Properties

Theorem: The Lyapunov-based EMPC scheme has the following properties for all $x(0) \in O$ and $\tilde{\rho} = W(x(0))$

(i) The scheme is recursively feasible and $W(x(t)) \leq \max\{\rho, \tilde{\rho}\}$ for all $t \geq 0$

(ii) If $\rho > 0$ then there is $\tilde{t} > 0$ with $x(t) \in \Omega$ for all $t \geq \tilde{t}$

(iii) If $\rho = 0$ then $x(t) \to x_s$ as $t \to \infty$

Note: It is also possible to change ρ with time (already present in the original reference [Heidarinejad/Liu/Christofides '12])

Discussion

Discussion of Lyapunov-based EMPC

- Theorem does not require strict dissipativity
- No performance estimates known so far, except average performance in case $\rho = 0$
- Under strict dissipativity, other performance estimates could possibly be achieved (open question!)
- Many variants available, see the monograph [Ellis/Liu/Christofides, Economic Model Predictive Control, Springer '17]
- Main bottleneck: knowledge of W and h required for implementation

The next EMPC variant fixes the last problem

Multiobjective EMPC

Goal: make the closed loop trajectory converge to x_s while minimizing the economic cost

Lyapunov-based EMPC with $\rho = 0$ solves this problem

The main problem of Lyapunov-based EMPC is the required knowledge of a stabilizing controller h and a corresponding Lyapunov function W

Multiobjective EMPC [Zavala '15] avoids this problem by computing h and W via stabilizing MPC with terminal conditions

In each step, two optimal control problems — one with the economic objective and one with a stabilizing objective — are solved and suitably combined

We start by explaining the stabilizing problem

Multiobjective EMPC: stabilizing subproblem

$$\min_{u(\cdot|t)} J^{stab}(x(t), u(\cdot|t)) = \sum_{k=0}^{N-1} \ell^{stab}(x(k|t), u(k|t))$$

subject to

$$\begin{aligned} x(k+1|t) &= f(x(k|t), u(k|t)), \quad k = 0, \dots, N-1 \\ x(0|t) &= x(t) \\ (x(k|t), u(k|t))^\top &\in \mathbb{X} \times \mathbb{U}, \quad k = 0, \dots, N-1 \\ x(N|t) &= x_s \end{aligned}$$

with $\ell^{stab}(x_s, u_s) = 0$, $\ell^{stab}(x, u) > 0$ otherwise

($x(N|t) = x_s$ could be replaced by regional constraint + terminal cost)

Lyapunov function property

Define $V^{stab}(x(t)) = \inf_{u(\cdot|t)} J^{stab}(x(t), u(\cdot|t))$

Then, under standard assumptions on the stabilizing MPC scheme, there is $\alpha_4 \in \mathcal{K}_\infty$ such that for each admissible control sequence $\hat{u}$ the inequality

$$V^{stab}(f(x(t), \hat{u}(0))) \leq J^{stab}(x(t), \hat{u}) - \alpha_4(|x(t) - x_s|)$$

holds

Thus, for any $\sigma \in (0,1)$ there is an admissible control $\tilde{u}$ with

$$J^{stab}(f(x(t), \hat{u}(0)),\ \tilde{u}) \leq J^{stab}(x(t), \hat{u}) - (1-\sigma)\alpha_4(|x(t) - x_s|)$$

$\rightsquigarrow$ J^{stab} can serve as a Lyapunov function constraint in the economic subproblem of the EMPC scheme

Multiobjective EMPC: economic subproblem

$$\min_{u(\cdot|t)} \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t))$$

subject to

$$\begin{aligned} x(k+1|t) &= f(x(k|t), u(k|t)), \quad k = 0, \dots, N-1 \\ x(0|t) &= x(t) \\ (x(k|t), u(k|t))^\top &\in \mathbb{X} \times \mathbb{U}, \quad k = 0, \dots, N \\ J^{stab}(x(t), u(\cdot|t)) &\leq (1-\sigma)V^{stab}(x(t)) \\ &\quad + \sigma J^{stab}(x(t-1), u^\star(\cdot|t-1)), \quad t \geq 1 \end{aligned}$$

for $\sigma \in [0,1)$

$$\rightsquigarrow J^{stab}(x(t+1), u^\star(\cdot|t+1)) \leq J^{stab}(x(t), u^\star(\cdot|t)) - (1-\sigma)\alpha_4(|x(t) - x_s|)$$

$\rightsquigarrow$ σ determines the speed of convergence

Multiobjective EMPC: Example

We illustrate the role of σ by the chemical reactor without dissipativity

$$\begin{aligned} \dot{x}_1 &= 1 - r_1(x_1, x_3) - x_1 \\ \dot{x}_2 &= r_2(x_1, x_3) - x_2 \\ \dot{x}_3 &= u - x_3 \end{aligned}$$

with

$$r_1(x_1, x_3) = 10^4 x_1^2 e^{-\frac{1}{x_3}} + 400 x_1 e^{-\frac{0.55}{x_3}}, \quad r_2(x_1, x_3) = 10^4 x_1^2 e^{-\frac{1}{x_3}}$$

$x_1 =$ concentration of source material R

$x_2 =$ concentration of desired product P_1

$x_3 =$ dimensionless temperature of the mixture in the reactor

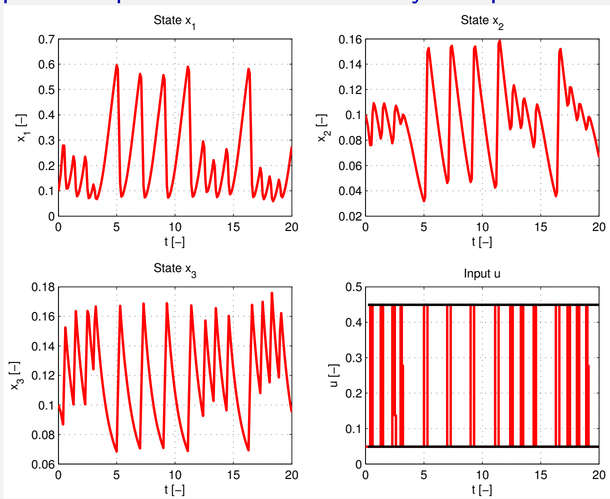
$u \mathrel{\hat{=}}$ heat flux through the cooling jacket

Constraints: $x_i \geq 0, i = 1,2,3$ and $u \in [0.049, 0.449]$

Objective: maximize P_1, i.e. the integral over $L(x,u) = -x_2$

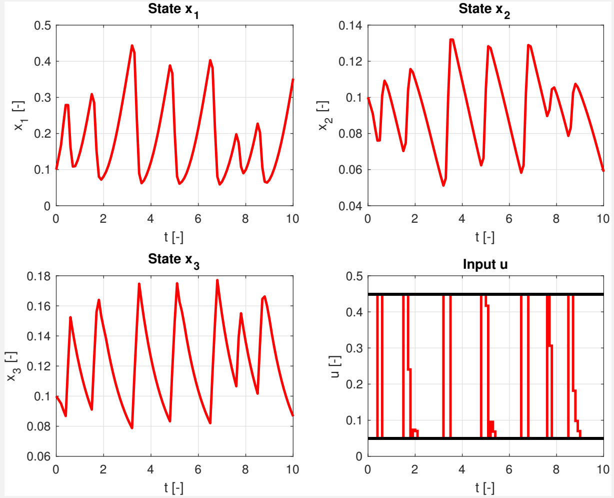
Multiobjective EMPC: Example

As seen before: the optimal trajectories are not constant $\rightsquigarrow$ no optimal equilibrium $\rightsquigarrow$ not strictly dissipative



Standard EMPC

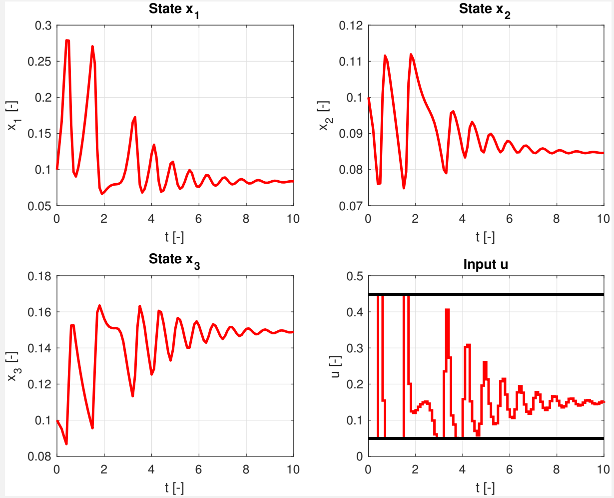
Multiobjective EMPC: Example



Multiobjective EMPC with $\sigma = 0.99$

In all simulations $\ell^{stab}(x,u) = |x - x_s|^2 + |u - u_s|^2$

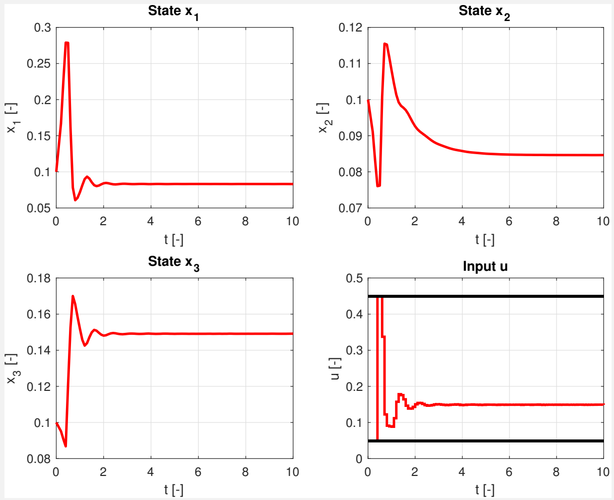
Multiobjective EMPC: Example



Multiobjective EMPC with $\sigma = 0.9$

In all simulations $\ell^{stab}(x,u) = |x - x_s|^2 + |u - u_s|^2$

Multiobjective EMPC: Example



Multiobjective EMPC with $\sigma = 0.5$

In all simulations $\ell^{stab}(x,u) = |x - x_s|^2 + |u - u_s|^2$

Multiobjective EMPC: Properties

Theorem: Consider the Multiobjective EMPC scheme under the usual stability assumptions for MPC with terminal constraints. Then for all $x(0) \in \mathbb{X}$ the EMPC closed loop solution $x(t)$ converges to x_s as $t \to \infty$

Idea of proof: The constraints enforce the inequality

$$J^{stab}(x(t+1), u^\star(\cdot|t+1)) \le J^{stab}(x(t), u^\star(\cdot|t)) - (1-\sigma)\alpha_4(|x(t) - x_s|)$$

yielding $J^{stab}(x(t), u^\star(\cdot|t)) \to 0$ as $t \to \infty$ and thus $x(t) \to x_s$

Note: Asymptotic stability may not hold! This is due to the fact that there is no upper bound on $J^{stab}(x(0), u(\cdot|0))$. Thus, the open loop optimal trajectory may move far away from x^s for $x(0) \approx x_s$; in fact even for $x(0) = x_s$

Multiobjective EMPC: Discussion

Discussion of Multiobjective MPC

- Theorem does not require strict dissipativity
- Average performance guaranteed, but no transient performance estimates known
- Under convexity assumptions, the (finite horizon) solution can be interpreted as a Pareto optimum
- Main drawback: two optimization problems need to be solved in each time step

Summary

We compare the EMPC-variants discussed so far with respect to the following characteristics

- Asymptotic stability
- Average performance
- Transient performance

as well as

- Assumptions on the problem
- Ingredients of the algorithm (functions, sets), other than system dynamics f and stage cost ℓ

EMPC with | without terminal conditions

- Asymptotic stability — yes | yes (practical)
- Average performance — yes | yes (with error term)
- Transient performance — yes | yes (with T-dep. error)
- Assumptions on the problem
  - — optimal operation at steady state | strict dissipativity (for average performance)
  - — strict dissipativity (for asymptotic stability and transient performance)
- Ingredients of the algorithm
  - — optimal steady state | none
  - — terminal constraint set and cost
- Remarks
  - — potentially small feasible set | recursive feasibility only for suff. large N

EMPC with generalized terminal conditions

- Asymptotic stability — yes
- Average performance — yes (with error term)
- Transient performance — no
- Assumptions on the problem
  - — reachability of optimal steady state (for average performance)
  - — strict dissipativity and other technical assumptions (for asymptotic stability)
- Ingredients of the algorithm
  - — none
- Remarks
  - — influence of β on transient performance unclear

Lyapunov-based | Multiobjective EMPC

- Asymptotic stability — yes | only convergence
- Average performance — yes | yes
- Transient performance — unknown | unknown
- Assumptions on the problem
  — optimal operation at steady state | optimal operation at steady state
- Ingredients of the algorithm
  — optimal steady state | optimal steady state
  — controller with Lyapunov function | terminal constraint set and cost
- Remarks
  — requires knowledge of Lyapunov function | requires solution of two optimal control problems

Remarks and Conclusion

- All considered schemes guarantee (approximate) averaged optimality under mild conditions on the problem structure
- In the absense of an optimal steady state, the advantage of EMPC over stabilizing MPC lies in its ability to find better solutions than the equilibrium (e.g., periodic ones)
- In the presence of an optimal steady state, average optimality is a rather weak optimality concept, which is moreover also satisfied by stabilizing MPC
- In this case, the advantage of EMPC lies in the transient performance. This has been confirmed in many simulations, but rigorously proved only for basic schemes
- So far, rigorous transient performance estimates have only been achieved under strict dissipativity. Is this property really necessary. . . ?

Literature for Part IV

Ellis, M., M. Liu, and P. Christofides, 2017, *Economic Model Predictive Control: Theory, Formulations and Chemical Process Applications*, Springer, Berlin

Fagiano, L. and A. R. Teel, 2013, *Generalized terminal state constraint for model predictive control*, Automatica 49(9), 2622–2631

Ferramosca, A., D. Limon, and E. F. Camacho, 2014, *Economic MPC for a Changing Economic Criterion for Linear System*, IEEE Transactions on Automatic Control 59(10), 2657–2667

Heidarinejad, M., J. Liu, and P. D. Christofides, 2012, *Economic model predictive control of nonlinear process systems using Lyapunov techniques*, AIChE Journal 58(3), 855–870

Houska, B. and M. A. Müller, 2017, *Cost-to-travel functions: a new perspective on optimal and model predictive control*, Systems & Control Letters 106, 79–86

Literature for Part IV (cont.)

Limon, D., M. Pereira, D. Muñoz de la Peña, T. Alamo, and J. M. Grosso, 2014, *Single-layer economic model predictive control for periodic operation*, Journal of Process Control 24(8), 1207–1224

Müller, M. A., D. Angeli, and F. Allgöwer, 2013, *Economic model predictive control with self-tuning terminal cost*, European Journal of Control 19(5), 408–416

Müller, M. A., D. Angeli, and F. Allgöwer, 2014, *On the performance of economic model predictive control with self-tuning terminal cost*, Journal of Process Control, 24(8), 1179–1186

Zavala, V. M., 2015, *A multiobjective optimization perspective on the stability of economic MPC*, in: Proceedings of ADCHEM 2015, IFAC Papers OnLine 48, 974–980

V. Advanced topics and open problems

Advanced topics and open problems

In this section we discuss a selection of schemes which go beyond the previous setting. Particularly, we consider discounted optimal control problems and problems which do not exhibit an optimal equilibrium

Outline:

- Discounted optimal control problems
- Optimal control problems with periodic optimal solutions
- Time-varying optimal control problems
- Uncertain Systems (Matthias)

Discounted optimal control problems

Discounted optimal control problems are of the form

$$\min_{u \in \mathcal{U}} \sum_{k=0}^{N-1} \beta^k \ell(x(k), u(k))$$

with $N \in \mathbb{N}$ or $N = \infty$, with discount factor $\beta \in (0,1)$

For discounted optimal control, the averaged optimality does not make sense, because for bounded ℓ

$$\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \beta^k \ell(x(k), u(k)) = 0$$

$\rightsquigarrow$ transient optimality is of interest

Transient performance theorem

Consider discounted EMPC without terminal conditions

Theorem [Grüne/Semmler/Stieler '15] If the discounted optimal control problem has the turnpike property and the optimal value function is continuous at x_s uniformly in β, then there is $\delta \in \mathcal{L}$ with

$$J_\infty^{cl}(x_0, \mu_N) \leq V_\infty(x_0) + \frac{\delta(N)}{1-\beta}$$

Note: The β-dependence of the error term is the counterpart of the T-dependence in the non-discounted case

It is unknown whether this result also holds (or even improves) with suitable terminal conditions

Relation to dissipativity

Dissipativity concepts have been developed for discounted problems as well [Grüne/Kellett/Weller '16, Grüne/Müller CDC '17]

The discounted strict dissipativity inequality reads

$$\beta\lambda(f(x,u)) \leq \lambda(x) + \ell(x,u) - \ell(x_s,u_s) - \alpha(\|x - x_s\|)$$

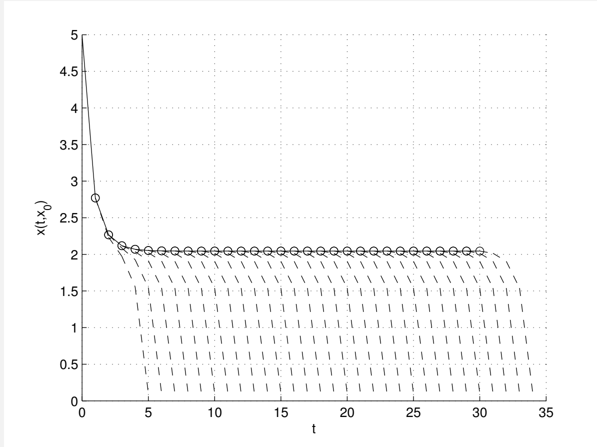
But: In general, discounted strict dissipativity only implies the turnpike property for $\beta \approx 1$ [Gaitsgory/Grüne/Höger/Kellett/Weller '17]

Discounted problems: example

We consider a classical economic growth model [Brock/Mirman '72]

$$x(t+1) = u(t), \quad \ell(x,u) = -\ln(Ax^\alpha - u)$$

Trajectories for $A = 5$, $\alpha = 0.34$, $x_0 = 5$, $\beta = 0.95$

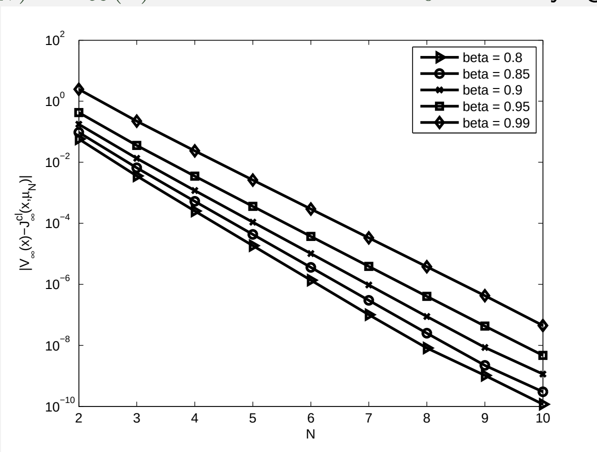


Discounted problems: example

We consider a classical economic growth model [Brock/Mirman '72]

$$x(t+1) = u(t), \quad \ell(x,u) = -\ln(Ax^\alpha - u)$$

$J^{cl}_\infty(5,\mu_N) - V_\infty(x)$, $A = 5$, $\alpha = 0.34$, $x_0 = 5$, varying N and β



Problems with time varying optimal operation

Our final two schemes concern problems without optimal operation at steady states

Instead, the system is optimally operated at periodic or more general time varying solutions

Here we distinguish two cases:

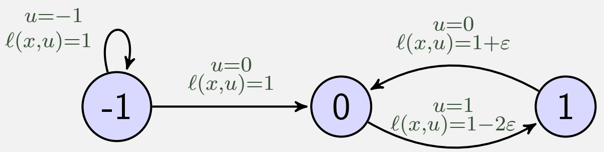
- Periodic optimal solutions generated by time invariant dynamics f and cost ℓ
- Time varying (possibly periodic) solutions generated by time varying dynamics f and/or cost ℓ

We start with the first situation

Periodic optimal trajectories

We first consider a simple example showing that periodic trajectories may be optimal even if f and ℓ are time invariant

We choose $\mathbb{X} = \mathbb{U} = \{-1, 0, 1\}$ and dynamics and cost indicated in the following figure



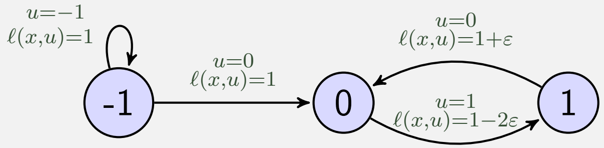
The average cost of the steady state $x = -1$ is 1

The average cost of the periodic orbit $(0, 1, 0, 1, 0, 1, \ldots)$ is $1 - \varepsilon$

$\rightsquigarrow$ the system is optimally operated at the periodic orbit

Will MPC "find" this orbit when starting in $x = -1$?

EMPC and periodic orbits



We start in $x = -1$

If the horizon N is odd, the trajectory

$$(-1, -1, 0, 1, 0, 1, \ldots, 0, 1)$$

is optimal $\rightsquigarrow$ the closed loop system will stay in -1 forever

Conclusion: MPC does not necessarily find optimal periodic orbits, even if N is arbitrarily large

EMPC and periodic orbits

Remedy: In order to find an optimal p-periodic orbit $(\hat{x}_0, \ldots, \hat{x}_{p-1})$, EMPC can be modified in two ways:

- impose periodic terminal constraints, e.g., $x(t|N) = \hat{x}_{t_p}$ with $t_p = t \mod p$ (regional constraints also possible)
  [Angeli/Amrit/Rawlings '09ff, Zanon/Grüne/Diehl '17]
- use the periodic optimization horizon $N_t = N - t_p$
  [Müller/Grüne '16]

Note: The second approach without terminal conditions needs no information about the periodic orbit except its period, but — similar to the steady state case — yields weaker results

Periodic strict dissipativity

The formal results rely on a periodic variant of strict dissipativity

$$\lambda_{k+1}(f(x,u)) \le \lambda_k(x) + \ell(x,u) - \ell(\hat{x}_k, \hat{u}_k) - \sigma(x,u)$$

for $k = 0, \ldots, p-1$, where $\lambda_p = \lambda_0$

or on a strict dissipativity condition for the stacked system

$$x^p = \begin{bmatrix} x_0 \\ \vdots \\ x_{p-1} \end{bmatrix}, \quad u^p = \begin{bmatrix} u_0 \\ \vdots \\ u_{p-1} \end{bmatrix}, \quad f^p(x^p, u^p) := \begin{bmatrix} f(x_{p-1}, u_0) \\ f(f(x_{p-1}, u_0), u_1) \\ \vdots \end{bmatrix}$$

(the relation between these two conditions is still waiting to be explored)

Properties of periodic EMPC scheme

Theorem: (a) Under the periodic strict dissipativity condition and suitable technical conditions (continuity), the optimal periodic orbit is asymptotically stable for the EMPC scheme with periodic terminal constraints and averaged optimality holds.

(the precise asymptotic stability property in (a) depends on the form of the function σ in the periodic strict dissipativity condition)

(b) Under the stacked strict dissipativity condition and suitable technical conditions (continuity), the closed loop of the EMPC scheme with periodic optimization horizon converges to the optimal periodic orbit and approximate averaged optimality holds

(in (b), asymptotic stability does not hold in general. This is due to a strange feature of the periodic turnpike property)

EMPC for time varying problems

Consider a problem with time varying dynamics and stage cost

$$x(k+1) = f(k, x(k), u(k)), \quad \ell(k, x, u)$$

Obviously, the extension of the EMPC scheme is straightforward, at least without terminal conditions

However, carrying over the previous results is nontrivial:

- what is the time varying counterpart of the optimal equilibrium / periodic orbit?
- which kind of approximate infinite horizon optimal performance can be expected?

We start by studying a simple example

Example problem

Prototype problem: Keep the temparature in a room in a desired range with mimimal energy consumption for heating and cooling

Very simple 1d model:

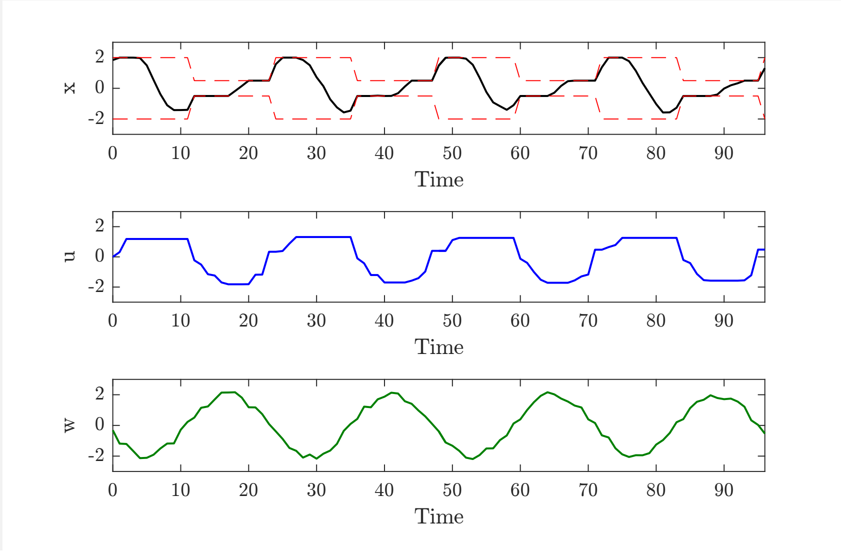
$$x(n+1) = \underbrace{x(n)}_{\text{inside temperature}} + \underbrace{u(n)}_{\text{heating/cooling}} + \underbrace{w(n)}_{\text{outside temperature}}$$

with stage cost

$$\ell(x,u) = u^2$$

and time varying $w(n)$ and desired temperature range $\mathbb{X}(n)$

Example: optimal trajectory



Optimality concept

In which infinite horizon sense can we expect that this trajectory is (near) optimal? Clearly,

$$\text{“minimize”}_{u} \quad J_\infty(x,u) = \sum_{n=0}^{\infty} \ell(x_u(n), u(n))$$

is not meaningful, because the sum will not converge

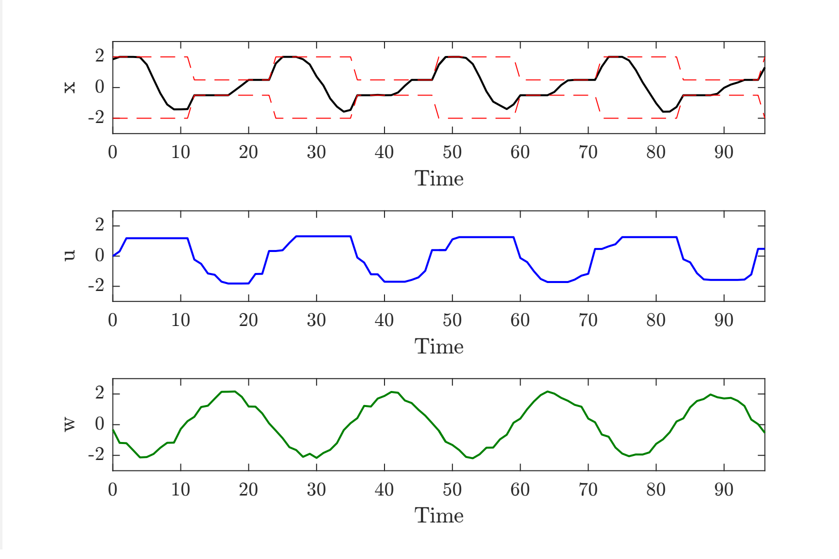
Remedy: Overtaking Optimality [Gale '67]

A trajectory $x^\star$ with control $u^\star$ is called overtaking optimal if

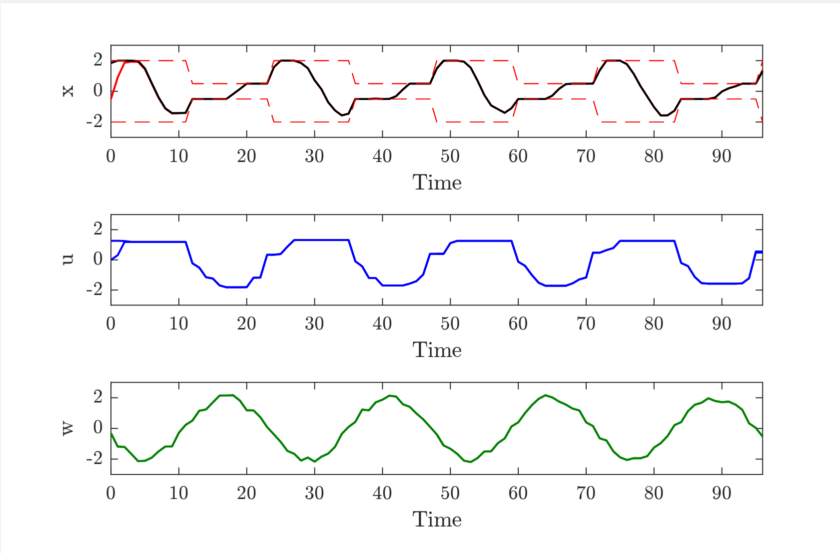
$$\limsup_{K\to\infty} \left(\sum_{n=0}^{K-1} \ell(n, x^\star(n), u^\star(n)) - \sum_{n=0}^{K-1} \ell(n, x_u(n), u(n)) \right) \le 0$$

holds for all admissible trajectory-control pairs (x_u, u) with $x_u(0) = x^\star(0)$

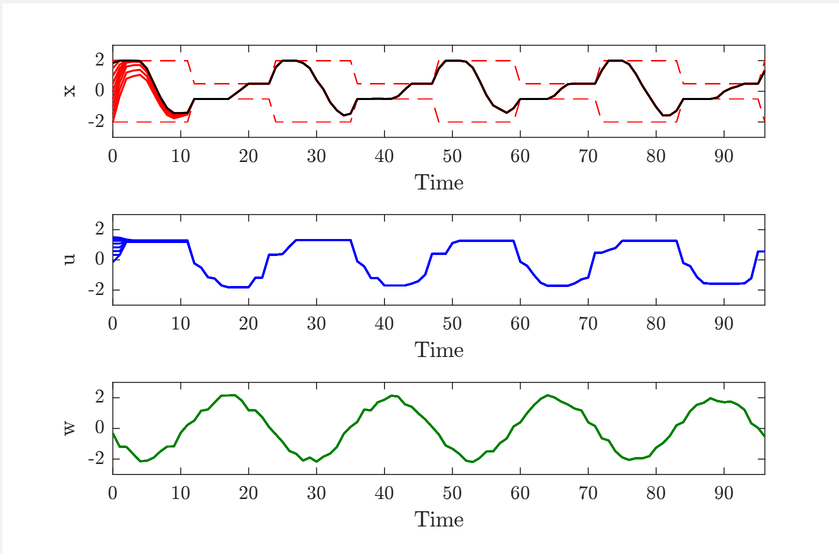
MPC closed loop



MPC closed loop for different initial value



MPC closed loop for different initial values



A generalized optimal equilibrium

Obviously, the closed loop trajectories converge to the black limit trajectory. How can we characterize it?

Idea: generalize the definition of optimal operation at a steady state to overtaking optimality:

We say that the system is optimally operated at a trajectory $\hat{x}$ with control $\hat{u}$ if

$$\limsup_{T\to\infty}\left(\sum_{n=0}^{T-1}\ell(n,\hat{x}(n),\hat{u}(n)) - \sum_{n=0}^{T-1}\ell(n,x_u(n),u(n))\right)\le 0$$

holds for all admissible trajectory-control pairs (x_u, u)

Note: this is similar to the definition of overtaking optimality, but now $x_u(0)\neq\hat{x}(0)$ is allowed

Main Result

Theorem: [Grüne/Pirkelmann CDC '17] Assume that a time varying turnpike property and a continuity property hold. Then there exists an error term $\delta(N)\to 0$ as $N\to\infty$ with

$$\limsup_{T\to\infty}\Bigg(\sum_{n=0}^{T-1}\ell(n,x_{\mu_N}(n),\mu_N(x_{\mu_N}(n))) - \sum_{n=0}^{T-1}\ell(n,x_u(n),u(n)) - T\delta(N)\Bigg)\le 0$$

for all admissible (x_u,u) with $x_u(0)=x_{\mu_N}(0)$

In other words: the MPC closed loop trajectory on $\{0,\dots,T\}$ is the initial piece of an overtaking optimal trajectory — up to the error $T\delta(N)$

Note: The factor "T" in the error term usually vanishes when looking at the relative error

Discussion of Main Result

- the time varying turnpike property can be ensured by a time varying strict dissipativity property
- this strict dissipativity property, in turn, always holds under suitable convexity assumptions (like in the steady state case, but more technical)
- the continuity property can be ensured by a controllability assumption (also in the periodic results before)
- probably the most important feature of the time varying case: in the steady state and in the periodic case, the optimal limit trajectories can be computed beforehand

  In the time varying case there is in general no easy way for this

  Hence, the fact that EMPC finds this trajectory "automatically" is of utmost importance

Literature for Part V

Angeli, D., R. Amrit, and J. B. Rawlings, 2009, *Receding horizon cost optimization for overly constrained nonlinear plants*, in: Proceedings of the 48th IEEE CDC, 7972–7977

Brock, W. A. and L. Mirman, 1972, *Optimal economic growth and uncertainty: the discounted case*, Journal of Economic Theory 4, 479–513

Gaitsgory, V., L. Grüne, M. Höger, C. M. Kellett, and S. R. Weller, *Stabilization of strictly dissipative discrete time systems with discounted optimal control*, submitted

Gale, D., 1967, *On optimal development in a multi-sector economy*, Review of Economiv Studies 34(1), 1–18

Grüne, L., C. M. Kellett, and S. R. Weller, 2016, *On a discounted notion of strict dissipativity*, in: Proceedings of NOLCOS 2016, IFAC-PapersOnLine 49(18), 247–252

Literature for Part V (cont.)

Grüne, L. and S. Pirkelmann, 2017, *Closed-loop performance analysis for economic model predictive control of time-varying systems*, in: Proceedings of the 56th IEEE CDC, to appear

Grüne, L., W. Semmler, and M. Stieler, 2015, *Using nonlinear model predictive control for dynamic decision problems in economics*, J. of Economic Dynamics and Control 60, 112–133

Müller, M. A. and L. Grüne, 2017, *On the relation between dissipativity and discounted dissipativity*, in: Proceedings of the 56th IEEE CDC, to appear

Müller, M. A. and L. Grüne, 2016, *Economic model predictive control without terminal constraints for optimal periodic behavior*, Automatica 70, 128–139

Zanon, M., L. Grüne, and M. Diehl, 2017, *Periodic optimal control, dissipativity and MPC*, IEEE Transactions on Automatic Control 62(6), 2943–2949

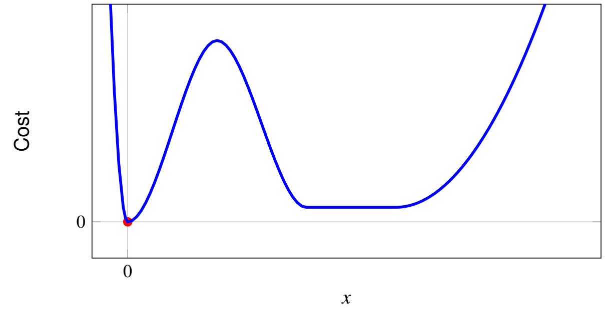
Economic model predictive control: state of the art and open problems

Timm Faulwasser, Lars Grüne, Matthias A. Müller

Pre-conference workshop CDC 2017

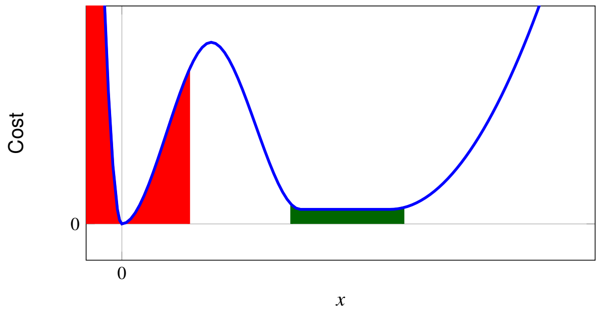
Economic MPC for uncertain systems

- System subject to disturbances/uncertainties: $x(t+1) = f\big(x(t), u(t), w(t)\big)$
- Motivating example:



Economic MPC for uncertain systems

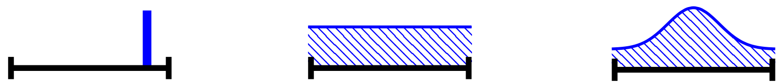
- System subject to disturbances/uncertainties: $x(t+1) = f\big(x(t), u(t), w(t)\big)$
- Motivating example:



Conclusion: just transferring robust stabilizing MPC approaches to economic setting might result in bad performance!

Economic MPC for uncertain systems

- How can we incorporate/leverage the disturbance in the MPC setup?



Min-max approach
- Use worst case

Averaging approach
- Use averaging over all states

Stochastic approach
- Use stochastic information

- Which closed-loop guarantees can be given?

Economic MPC for uncertain systems

Nominal System

$z(t+1) = f(z(t), v(t), 0)$

Error

$e(t) = x(t) - z(t)$

Robust control invariant (RCI) set

$$e(t) \in \Omega \quad \Rightarrow \quad e(t+1) \in \Omega, \quad \forall w(t) \in \mathbb{W}$$

Input parametrization

Use parametrization for the real input

$$u(t) = \varphi\big(v(t), x(t), z(t)\big)$$

($v(t)$ input to the nominal system) to determine RCI set Ω

Economic MPC for uncertain systems

Nominal System

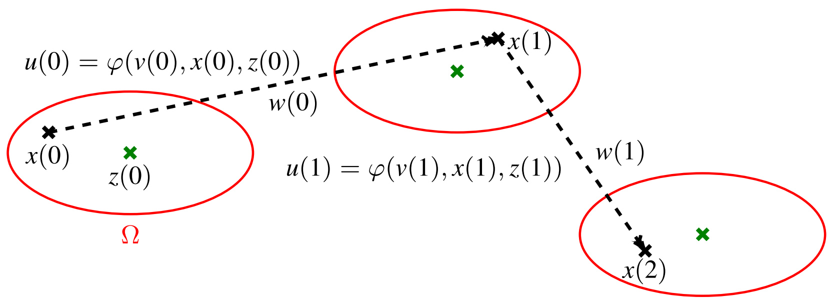
$z(t+1) = f(z(t), v(t), 0)$

Error

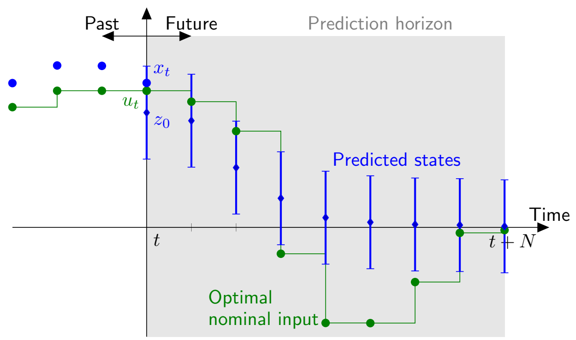
$e(t) = x(t) - z(t)$

Robust control invariant (RCI) set

$$e(t) \in \Omega \quad \Rightarrow \quad e(t+1) \in \Omega, \quad \forall w(t) \in \mathbb{W}$$

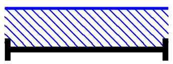


Economic MPC for uncertain systems



Idea:

- Take all possible states within invariant set into account
- Two different approaches:



Economic MPC for uncertain systems

Idea min-max robust economic MPC

- Consider the worst case within the RCI set
- Use modified stage cost function

$$\ell^{\max}(z, v) = \max_{\omega \in \Omega} \ell\big(z + \omega, \varphi(v, z + \omega, z)\big)$$

Main features:

- All possible real states considered $x(k|t) \in \{z(k|t)\} \oplus \Omega$
- Take real input into account $u(k|t) = \varphi\big(v(k|t), x(k|t), z(k|t)\big)$

  $\Longrightarrow$ Cost of input to stay in RCI set

Economic MPC for uncertain systems

Optimization problem

$$\min_{z(0|t),v(t)} \sum_{k=0}^{N-1} \ell^{\max}\big(z(k|t),v(k|t)\big) + \overline{V}_f\big(z(N|t)\big)$$

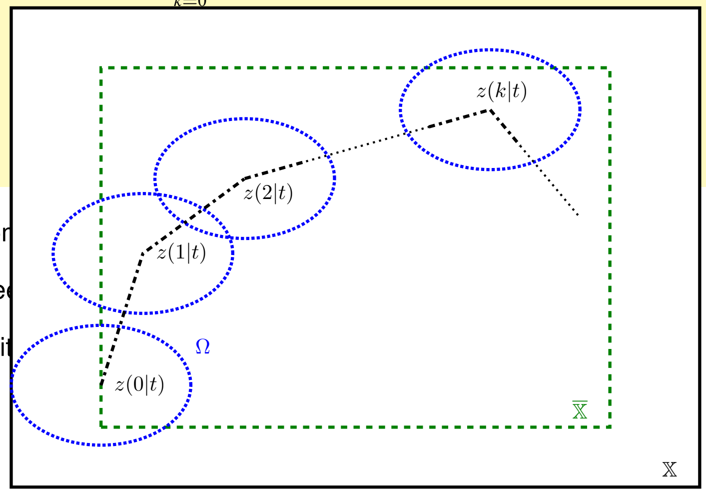
$$\begin{aligned}
\text{s.t. } & z(k+1|t) = f(z(k|t),v(k|t),0),\\
& x(t) \in \{z(0|t)\} \oplus \Omega,\\
& \big(z(k|t),v(k|t)\big)^\top \in \overline{\mathbb{X}} \times \overline{\mathbb{U}}, \quad k = 0,\dots,N-1,\\
& z(N|t) \in \overline{\mathbb{X}}_f
\end{aligned}$$

- Nominal dynamics only
- Free nominal initial state
- Suitably tightened constraint sets $\overline{\mathbb{X}}, \overline{\mathbb{U}}, \overline{\mathbb{X}}_f$

Economic MPC for uncertain systems

Optimization problem

$$\min_{z(0|t),v(t)} \sum_{k=0}^{N-1} \ell^{\max}\big(z(k|t),v(k|t)\big) + \overline{V}_f\big(z(N|t)\big)$$



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- Sui

Economic MPC for uncertain systems

Theorem [Bayer, Müller, Allgöwer '16]

Under standard assumptions (terminal region/cost, suitable constraint tightening) and given initial feasibility, we have

- recursive feasibility,
- closed-loop constraint satisfaction,
- infinite horizon averaged performance

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ell\big(x(t),u(t)\big) \le \ell^{\max}\big(z_s^{\mathsf{m}}, v_s^{\mathsf{m}}\big).$$

- Optimal steady-state: $(z_s^{\mathsf{m}}, v_s^{\mathsf{m}}) = \underset{z=f(z,v,0),(z,v)\in\overline{\mathbb{Z}}}{\arg\min} \ \ell^{\max}(z,v)$
- Performance result for the *real* closed-loop system
- Bound usually quite conservative

Economic MPC for uncertain systems

Idea averaging-based robust economic MPC

- Instead of worst case, consider average over RCI set Ω

- Use modified stage cost function

$$\ell^{\text{int}}(z,v) = \int_\Omega \ell\big(z+\omega, \varphi(v,z+\omega,z)\big)\mathrm{d}\omega$$

Economic MPC for uncertain systems

Theorem [Bayer, Müller, Allgöwer '14]

Under standard assumptions (terminal region/cost, suitable constraint tightening) and given initial feasibility, we have

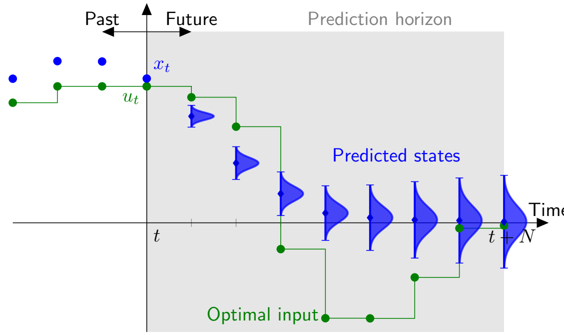
- recursive feasibility,
- closed-loop constraint satisfaction,
- infinite horizon averaged performance

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ell^{\text{int}}(z^*(0|t), v^*(0|t)) \le \ell^{\text{int}}(z_s^{\text{a}}, v_s^{\text{a}}).$$

- Optimal steady state: $(z_s^{\text{a}}, v_s^{\text{a}}) = \underset{z=f(z,v,0),(z,v)\in\overline{\mathbb{Z}}}{\arg\min} \ell^{\text{int}}(z,v)$
- Interpretation: Average performance result for the real closed loop, averaged over all possible disturbances

Economic MPC for uncertain systems

Can we improve performance using stochastic information?



Idea:

- Predict evolution of probabilities
- Consider expected value in the optimization problem

Economic MPC for uncertain systems

- Use stage cost

$$\ell_k^{\text{int}}(z(k|t), v(k|t)) := \underbrace{\int_{\Omega_k} \ell(z(k|t)+e, Ke+v(k|t))\rho_{\Omega_k}(e)de}_{=\mathbb{E}\{\ell(x(k|t),u(k|t))|x(t)\}}$$

Theorem [Bayer, Lorenzen, Müller, Allgöwer '16]

Under standard assumptions (terminal region/cost, suitable constraint tightening) and given initial feasibility, we have

- recursive feasibility,
- closed-loop constraint satisfaction,
- infinite horizon averaged performance

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\{\ell(x(t), v^*(0|t))|x(0)\} \le \ell_\infty^{\text{int}}(z_s, v_s)$$

- $\ell_\infty^{\text{int}}(z_s, v_s)$ represents the expected average cost at the optimal steady-state

Economic MPC for uncertain systems

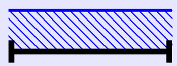
Conclusions

- Just transferring approaches from robust stabilizing MPC is not enough
- Different approaches to incorporate disturbances in economic MPC

Min-max approach

⊕ Accounts for worst case

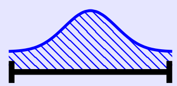
⊖ Typically quite conservative



Averaging approach

⊕ Usually better than min-max

⊖ Poor approx. of real distribution



Stochastic approach

⊕ Real distribution

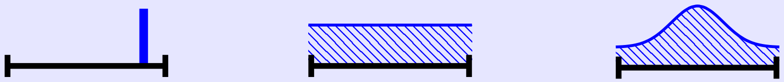
⊖ More complex

⊖ Results only available for linear case

Economic MPC for uncertain systems

Conclusions

- Just transferring approaches from robust stabilizing MPC is not enough
- Different approaches to incorporate disturbances in economic MPC



- Guaranteed average performance bounds for all approaches
- The more information taken into account, the better the performance
- Picture much less complete than in nominal case: transient performance, using no terminal constraints, classification of optimal operating conditions, etc.

Literature for Part VI

F. A. Bayer, M. Lorenzen, M. A. Müller and F. Allgöwer, Robust economic Model Predictive Control using stochastic information, Automatica 74, pp. 151 - 161, 2016.

F. A. Bayer, M. A. Müller and F. Allgöwer, Tube-based robust economic model predictive control, Journal of Process Control 24(8), pp. 1237 - 1246, 2014.

F. A. Bayer, M. A. Müller and F. Allgöwer, Min-max Economic Model Predictive Control Approaches with Guaranteed Performance, IEEE CDC, pp. 3210 - 3215, 2016.

Summary and wrap up

- Economic MPC: model predictive control using general performance criterion
- Various different EMPC schemes available with different advantages and disadvantages
- Basic case of optimal steady-state operation by now fairly well understood, closed-loop performance and convergence guarantees available
- Extensions to various settings (periodic optimal behavior, discounted problems, time-varying problems, uncertain systems, . . .), but still many open questions

Further information

T. Faulwasser, L. Grüne, & M. A. Müller. Economic Nonlinear Model Predictive Control: Stability, Optimality and Performance. Foundations and Trends in Systems and Control, **2018**.

Thanks for your attention! **Questions?**