

Nonlinear Model Predictive Control

PhD course, Università di Roma “Sapienza”, April 2013

Exercises

Exercise 1 (Lyapunov Functions) Consider the two-dimensional difference equation

$$x^+ = (1 - \|x\|) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x$$

with $x = (x_1, x_2)^\top \in \mathbb{R}^2$.

- Check that $V(x) = x_1^2 + x_2^2$ is a Lyapunov function for the equilibrium $x_* = 0$ on $S = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.
- Is V also a Lyapunov function on $S = \mathbb{R}^2$?
- Solve (a) and (b) for the difference equation

$$x^+ = \frac{1}{1 + \|x\|} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x.$$

Exercise 2 (Dynamic Programming) Consider the discrete time control system

$$x^+ = x + u$$

with $x \in \mathbb{X} = \mathbb{R}$ and $u \in \mathbb{U} = \mathbb{R}$ and the stage cost $\ell(x, u) = x^2 + u^2$.

- Compute by dynamic programming the optimal value function V_2 and the optimal feedback law μ_2 without stabilizing terminal constraints. (Hints for how to do this will be given in the explanation of the exercises in the course.)
- Check whether the resulting MPC closed loop is asymptotically stable.
- Compute a bound for the value $J_\infty^{cl}(x, \mu_2)$.
- Repeat (a)–(c) with the terminal constraint $x_{\mathbf{u}}(N) = 0$. Which feedback law μ_2 yields the better value $J_\infty^{cl}(x, \mu_2)$?

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Exercise 3 (MPC Computer Exercise)

- (a) Perform experiments with the file `double_integrator.m`, which implements an MPC controller without terminal constraints for the exact discrete time model of a sampled data double integrator. Get acquainted with the code and with the way constraints are encoded in MATLAB by changing parameters and constraints (an introduction will be provided in the course).

Note: The MATLAB files are coded directly in MATLAB without using Simulink.

- (b) Add equilibrium terminal constraints and compare the solutions to those without terminal constraints for different N .
- (c) Use the file as a template in order to implement the “car-and-mountains” example

$$x^+ = \begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} \sin(\vartheta(x) + u) \\ \cos(\vartheta(x) + u)/2 \end{pmatrix}$$

with

$$\vartheta(x) = \begin{cases} \arccos 2x_2, & x_1 \geq 0 \\ 2\pi - \arccos 2x_2, & x_1 < 0, \end{cases}$$

with equilibrium $x_* = (0, -1/2)$, stage cost $\ell(x, u) = \|x - x_*\|^2 + u^2$, control constraints $\mathbb{U} = [0, 0.2]$ and initial value $(0, 1/2)$. Determine the minimal stabilizing horizon by experiments for MPC without terminal constraints and with equilibrium terminal constraints.

Exercise 4 (Concept of MPC) After an NMPC course you explain the basic NMPC idea to a friend. After you finished your explanation, your friend asks:

“If I ride my bicycle and want to make a turn to the left, I first steer a little bit to the right to make my bicycle tilt to the left. Let us assume that this way of making a turn is optimal for a suitable finite horizon optimal control problem. This would mean that the optimal control sequence will initially steer to the right and later steer to the left. If we use this optimal control sequence in an NMPC algorithm, only the first control action will be implemented. As a consequence, we will always steer to the right, and we will make a turn to the right instead of a turn to the left. Does this mean that NMPC does not work for controlling my bicycle?”

What do you respond?

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Exercise 5 (Terminal constrained vs. unconstrained MPC) Consider again the control system

$$x^+ = x + u$$

with $x \in \mathbb{X} = \mathbb{R}$ and stage cost $\ell(x, u) = x^2 + u^2$, cf. Exercise 2, but now with $\mathbb{U} = [-1, 1]$.

- Consider the terminal constraint set $\mathbb{X}_0 = [-a, a]$ for some $a \geq 0$ (note that $a = 0$ corresponds to the equilibrium terminal constraint $x_{\mathbf{u}}(N) = x_* = 0$). Show that for each $N > 0$ and $a \geq 0$ there exists initial values $x_0 \in \mathbb{R}$ which are not contained in the feasible set \mathbb{X}_N .
- Check that without terminal constraints the MPC closed loop is asymptotically stable for $N = 2$ for arbitrary initial values $x_0 \in \mathbb{R}$. You can do this either by computing μ_2 and V_2 by dynamic programming and checking the conditions of the relaxed dynamic programming theorem or by implementing the closed loop in MATLAB (using the code from Exercise 3) and performing numerical experiments.

Exercise 6 (Checking exponential controllability)

- Consider a general discrete time control system $x^+ = f(x, u)$ and assume that the system is exponentially controllable to some equilibrium $x_* \in \mathbb{X}$ with $f(x_*, 0) = x_*$. This means that there exists constants $K > 0$ and $\eta \in (0, 1)$ such that for each $x_0 \in \mathbb{X}$ there exists an admissible \mathbf{u} such that the inequality

$$\|x_{\mathbf{u}}(k) - x_*\| \leq K\eta^k \|x_0 - x_*\|$$

holds.

Show that for the stage cost $\ell(x, u) = \|x - x_*\|^2$ the exponential controllability condition

$$\ell(x_{\mathbf{u}}(k), \mathbf{u}(k)) \leq C\sigma^k \ell^*(x_0)$$

holds for suitable $C > 0$ and $\sigma \in (0, 1)$.

- Does the exponential controllability condition also hold for $\ell(x, u) = \|x - x_*\|^2 + \lambda\|u\|^2$ for $\lambda > 0$? If no, can you find an additional condition which guarantees this property?

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Exercise 7 (MPC Computer Exercise) Consider once more the control system from Exercise 2, i.e.,

$$x^+ = x + u$$

with $x \in \mathbb{X} = \mathbb{R}$ and $u \in \mathbb{U} = \mathbb{R}$, but now with stage cost $\ell(x, u) = x^2 + |u|$. Implement the MPC closed loop in MATLAB and check whether you can find a horizon N such that the closed loop is asymptotically stable at $x_* = 0$. Can you give an explanation for your observations?

Exercise 8 (MPC Computer Exercise)

- (a) Write a MATLAB code simulating an MPC controller for the inverted pendulum on a cart

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= g \sin(x_1) - kx_2 + u \cos(x_1) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u\end{aligned}$$

with $g = 9.81$ and $k = 0.1$. Use the sampling time $T = 0.15$, the control constraints $\mathbb{U} = [-10, 10]$ and the stage cost

$$\ell(x, u) = 10x_1^2 + x_2^2 + x_3^2 + x_4^2 + 0.01u^2.$$

Determine an $N \in \mathbb{N}$ for which the closed loop is asymptotically stable by running simulations for different initial values. Implement different state constraints of your choice and observe the effect on the closed loop behavior.

- (b) Reduce the control constraints to $\mathbb{U} = [-3, 3]$ (removing all state constraints) and check whether the closed loop is still stable.

- (c) Change the stage cost to

$$\ell(x, u) = 20(1 - \cos x_1) + x_2^2 + x_3^2 + x_4^2 + 0.01u^2$$

and observe what happens.

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Exercise 9 (Optimal equilibria) Consider the two economic MPC examples from the course:

(1) $X = U = \mathbb{R}$, $\mathbb{X} = [-2, 2]$, $\mathbb{U} = [-3, 3]$

$$x^+ = 2x + u, \quad \ell(x, u) = u^2.$$

(2) $X = U = \mathbb{R}$, $\mathbb{X} = [0.1, 10]$, $\mathbb{U} = [0.1, 5]$

$$x^+ = u, \quad \ell(x, u) = -\ln(5x^{0.34} - u).$$

For each of the examples do the following:

(a) Compute the sets

$$E := \{(x, u) \in \mathbb{X} \times \mathbb{U} \mid f(x, u) = x\}$$

all equilibria of the equations.

(b) Determine the equilibrium $(x^e, u^e) \in E$ with the minimal value

$$\ell(x^e, u^e) = \min\{\ell(x, u) \mid (x, u) \in E\}.$$

(c) Check whether the equilibrium from (b) also minimizes ℓ over $\mathbb{X} \times \mathbb{U}$, i.e., whether $\ell(x^e, u^e) = \min\{\ell(x, u) \mid (x, u) \in \mathbb{X} \times \mathbb{U}\}$ holds. If yes, is (x^e, u^e) the unique minimizer over $\mathbb{X} \times \mathbb{U}$?

Exercise 10 (MPC Computer Exercise) Implement the economic MPC problems from Exercise 9 in the MATLAB MPC code and perform simulations.

(a) In the simulations with terminal constraints, verify that the solutions indeed satisfy the convergence

$$\lim_{k \rightarrow \infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k))) = \ell(x^e, u^e).$$

(b) In the simulations with terminal constraints, determine the limit

$$\lim_{k \rightarrow \infty} \ell(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k)))$$

by numerical simulations and compute its distance to $\ell(x^e, u^e)$ depending on N . Verify that the distance indeed decreases exponentially as N grows.